Theoretical Juggling

(and Knot Theory too)

Ethan Lu

The Ohio State University

July 25, 2025

Outline

Because I am only a theoretical juggler, the Juggling Lab software (available at https://jugglinglab.org/) will be used to demonstrate most of the juggling.

Formulation of Problem

Juggling is about tossing objects in some sort of rhythmic pattern, and mathematics also describes patterns. Is there some sort of mathematics in juggling?

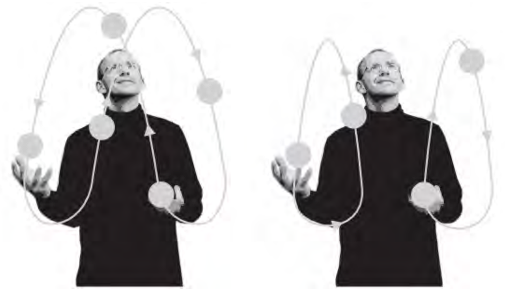


image from "The Mathematics of Juggling" by Polster.

Formulation of Problem

We make simplifying assumptions when modeling juggling, some of them are:

Formulation of Problem

We make simplifying assumptions when modeling juggling, some of them are:

- Balls are thrown at regular intervals and thrown instantly.

Formulation of Problem

We make simplifying assumptions when modeling juggling, some of them are:

- Balls are thrown at regular intervals and thrown instantly.
- The hands do not move.

Formulation of Problem

We make simplifying assumptions when modeling juggling, some of them are:

- Balls are thrown at regular intervals and thrown instantly.
- The hands do not move.
- Balls are thrown in a periodic pattern.

Formulation of Problem

We make simplifying assumptions when modeling juggling, some of them are:

- Balls are thrown at regular intervals and thrown instantly.
- The hands do not move.
- Balls are thrown in a periodic pattern.
- One hand throws on odd beats and the other hand throws on even beats.

Formulation of Problem

We make simplifying assumptions when modeling juggling, some of them are:

- Balls are thrown at regular intervals and thrown instantly.
- The hands do not move.
- Balls are thrown in a periodic pattern.
- One hand throws on odd beats and the other hand throws on even beats.

Formulation of Problem

Because throws occur at regular intervals ($t \in \mathbb{Z}$), the juggling pattern can be thought as a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(t)$ is equal to the time the ball thrown at time t is caught.

Formulation of Problem

Because throws occur at regular intervals ($t \in \mathbb{Z}$), the juggling pattern can be thought as a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(t)$ is equal to the time the ball thrown at time t is caught.

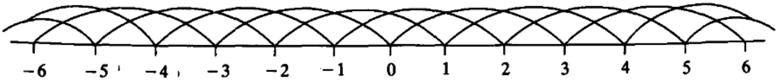
Example

For the 3-ball shower this is $f(t) = t + 3$

This function must be a bijection to correspond to an actual juggling pattern and is studied in (Buhler et al.).

Notation and Diagrams

This function can be visualized on the 3-ball cascade juggling pattern with the following diagram (which represents side view of the juggler when they are moving forward at a constant speed):



from

Notation and Diagrams

The juggling pattern can be represented more compactly by concatenating the numbers of beats in the throws over the whole period together. This is called siteswap notation.

Notation and Diagrams

The juggling pattern can be represented more compactly by concatenating the numbers of beats in the throws over the whole period together. This is called siteswap notation.

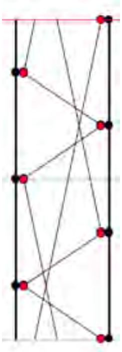
Example

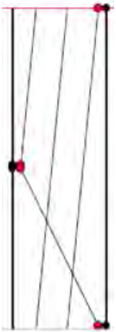
The 3-ball cascade has the siteswap notation of $\langle 3 \rangle$.
The fountain has siteswap notation $\langle 4 \rangle$.
A n-shower has a siteswap notation of $\langle (2n-1)1 \rangle$

The combinatorics of valid siteswap sequences is mentioned in (Maclauley).

Notation and Diagrams

Alternatively, we can use "ladder diagrams" that represent an overhead view of the juggler to encode information about the juggling pattern. The following are ladder diagrams for $\langle 711 \rangle$ and $\langle 51 \rangle$





From Juggling to Braids to Links

The ladder diagrams look very similar to braid diagrams.

The braid diagrams just have extra information about crossings.

From Juggling to Braids to Links

To figure out the crossings, we use information about the height of balls (if one ball is higher than the other, it over-crosses). This gives an interpretation of juggling as braiding.

From Juggling to Braids to Links

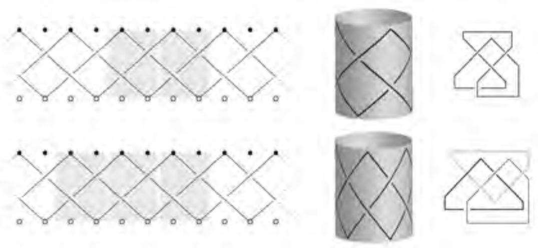
Whether one ball crosses over the other will depend only on the length of the throws in the ladder diagram (because by kinematics the trajectory is a parabola).

From Juggling to Braids to Links

Every braid can be realized through ladder diagrams along with the aformentioned crossing rule (Devadoss et al.). Which means by Markov's theorem, there is a juggling pattern for every link! Physically, the trajectory of the balls can be thought of as a link if the juggler walks in circles.

From Juggling to Braids to Links

Every braid can be realized through ladder diagrams along with the aformentioned crossing rule (Devadoss et al.). Which means by Markov's theorem, there is a juggling pattern for every link! Physically, the trajectory of the balls can be thought of as a link if the juggler walks in circles.



from (Devadoss et al.).

An Excercise for the Listener

Excercise: Pick your favorite knot/link from https://katlas.org/ (or Mathematica's KnotData[]), find it's braid representative and juggle pattern, then juggle it.

An Excercise for the Listener

Excercise: Pick your favorite knot/link from https://katlas.org/ (or Mathematica's KnotData[]), find it's braid representative and juggle pattern, then juggle it.

Open Problem: Find a knot/link with braid index $\geq$ 15 and juggle it.

An Excercise for the Listener

Excercise: Pick your favorite knot/link from https://katlas.org/ (or Mathematica's KnotData[]), find it's braid representative and juggle pattern, then juggle it.

Open Problem: Find a knot/link with braid index ≥ 15 and juggle it.

Disregarding finding a knot/link the juggling is thought to be impossible (https://www.youtube.com/watch?v=7RDfNn7crqE)

Yay!

Congratulations, you are now a theoretical juggler!

References I

- J. Buhler, D. Eisenbud, R. Graham, C. Wright
  Juggling Drops and Descents.
  The American Mathematical Monthly, 101:6, 507-519

- S. Devadoss, J. Mugno
  Juggling Braids and Links
  The Mathematical Intelligencer 29, 15–22 (2007),arXiv:math/0602476

- M. Macauley
  Braids and Juggling Patterns.
  https://www.math.clemson.edu/∼macaule/papers/seniorthesis.pdf