Recurrence Relationships in Quasi-Tree Subgraphs

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Definitions

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Given a bouquet B, we can create its associated framed chord diagram $F(B)$. First, consider the boundary of the vertex as the circle of the diagram. Draw a chord between the ends of each edge of B and assign it 0 iff its ribbon loop in B is orientable (untwisted).

Definitions

Given a framed chord diagram $D = \{\{a_i, b_i\} \mid 1 \leq i \leq n\}$, we can construct its intersection matrix $A(D)$. For each chord $\{a_i, b_i\}$, arbitrarily choose a pair: (a_i, b_i) or (b_i, a_i). Each matrix entry $A_{i,i}$ is 0 if its chord $\{a_i, b_i\}$ is assigned 0, and 1 otherwise. For $i < j$, entry $A_{i,j}$ is 1 if the corresponding chords intersect with cyclic order a_i, a_j, b_i, b_j, -1 if they intersect with cyclic order a_i, b_j, b_i, a_j, and 0 if the chords do not intersect. For $i < j$, $A_{i,j} = -A_{j,i}$.

Matrix Quasi-Tree Theorems

Theorem 1

Given an orientable bouquet B with n edges,

$$\kappa(B) = det(I_n + A(F(B))$$

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Theorem 2

Given a bouquet B with n edges and exactly one non-orientable loop e_1,

$$\kappa(B) = det(I_n + A(F(B))$$

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Notation

Fibonacci Numbers

$f_n = f_{n-1} + f_{n-2}$
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Relation

$f_n + f_{n-2} = \ell_{n-1}$

Special Bouquet Sequences

$\mathbb{F}_n$

$\forall n \geq 0$, let $\mathbb{F}_n$ denote the bouquet with the signed rotation $(1, 2, 1, 3, 2, 4, 3, ..., i, i-1, i+1, i, ...n-1, n-2, n, n-1, n)$ whose chord diagram $D(\mathbb{F}_n)$ has pairs $\{(1,3), (2,5), (4,7), ..., (2n-4, 2n-1), (2n-2, 2n)\}$

Theorem

$\kappa(\mathbb{F}_n) = f_{n+1}$

$\mathbb{F}'_n$

$\forall n \geq 2$, let $\mathbb{F}'_n$ denote the bouquet with the signed rotation $(1, 2, 3, 2, 1, 4, 3, 5, 4, ..., i, i-1, i+1, i, ..., n-1, n-2, n, n-1, n)$ whose chord diagram $D(\mathbb{F}'_n)$ has pairs $\{(1,5), (2,4), (3,7), (6,9), ..., (2n-4, 2n-1), (2n-2, 2n)\}$

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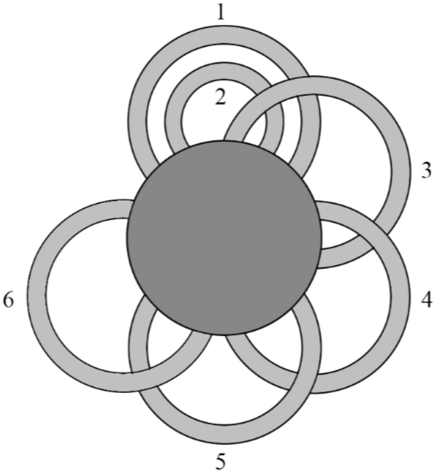

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Proof

For $n = 2$, $\kappa(\mathbb{F}'_2) = 1 = \ell_1$ (the subgraph with no edges). Now, let $n \geq 3$.

$$I_n + A(F(\mathbb{F}'_n)) = \begin{pmatrix} 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ -1 & -1 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}$$

, and by the Matrix Quasi-Tree Theorem, $\kappa(\mathbb{F}'_n) = det(I_n + A(F(\mathbb{F}'_n)))$.

Proof

Proof Cont.

For $n = 2$, $\kappa(\mathbb{F}_2') = 1 = \ell_1$ (the subgraph with no edges). Now, let $n \geq 3$.

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, and by the Matrix Quasi-Tree Theorem, $\kappa(\mathbb{F}'_n) = det(I_n + A(F(\mathbb{F}'_n)))$.

Let $M := I_n + A(F(\mathbb{F}'_n))$.

$\kappa(\mathbb{F}'_n) = det(M) = det(M[1,1]) - det(M[3,1]) =$

$\kappa(\mathbb{F}_{n-1}) - (-1)m_{1,3}m_{2,2}\kappa(\mathbb{F}_{n-3}) = \kappa(\mathbb{F}_{n-1}) + \kappa(\mathbb{F}_{n-3}) = f_n + f_{n-2} = \ell_{n-1}$

Theorem

Definition

Let $\mathbb{F}_n'^1$ denote the bouquet of $\mathbb{F}_n'$ with the first edge twisted (non-orientable). Its signed rotation is $(-1, 2, 3, 2, 1, 4, 3, 5, 4, ..., i, i-1, i+1, i, ..., n-1, n-2, n, n-1, n)$ whose chord diagram $D(\mathbb{F}_n')$ has pairs $\{(1,5), (2,4), (3,7), (6,9), ..., (2n-4, 2n-1), (2n-2, 2n)\}$ and chord (1, 5) is assigned 1 while all others 0.

Theorem

$\forall n \geq 2,\ \kappa(\mathbb{F}_n'^1) = f_n + \ell_{n-1}$

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Other Theorems

Now, we'll try to understand these recurrence relations without the use of matrices.

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Lemma

Let $B(G) =$ the number of boundary components of G for any ribbon graph G. For disjoint ribbon graphs P and Q, where $\vee$ is the disjoint union operator, $B(P \vee Q) = B(P) + B(Q) - 1$

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Lemma

Let $B(G) =$ the number of boundary components of G for any ribbon graph G. For disjoint ribbon graphs P and Q, where $\vee$ is the disjoint union operator, $B(P \vee Q) = B(P) + B(Q) - 1$

Consequently,

Theorem 1

For disjoint ribbon graphs P and Q, where $\vee$ is the disjoint union operator,

$$\kappa(P \vee Q) = \kappa(P)\kappa(Q)$$

Subgraphs with More Boundary Components

Let $G = (V, E)$ be a ribbon graph, and let n be a positive integer.

Definition

$\mathcal{F}_n(G) = \{F \subseteq E \mid (V, F) \text{ has } n \text{ boundary components}\}$

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Definition

$f_n(G)$ is the cardinality of $\mathcal{F}_n(G)$

Put simply, $f_n(G)$ counts the number of spanning subgraphs of G with exactly n boundary components (and, in particular, $f_1(G) = \kappa(G)$).

Generalization of Theorem 1

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Theorem 2

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$$f_n(P \vee Q) = \sum_{i=1}^{n} f_i(P) f_{n+1-i}(Q)$$

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Given that graph P has k boundary components and that graph Q has j boundary components, we can conclude that $P \vee Q$ has $k + j - 1$ boundary components.

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So, each subgraph of $P \vee Q$ with n boundary components must have i boundary components "contributed" by P, and $n + 1 - i$ boundary components "contributed" from Q for some i.

First Partial Duality Theorem

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Theorem 3

Let G be a ribbon graph and $e \in E(G)$. Then, for any n,

$$\begin{aligned} f_n(G) &= f_n(G \backslash e) + f_n(G/e) \\ &= f_n(G \backslash e) + f_n(G^{\delta(e)} \backslash e) \end{aligned}$$

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We can find an obvious 1-1 correspondence between subgraphs of G that do not contain e and subgraphs of $G \backslash e$.

There is a slightly less obvious 1-1 correspondence between subgraphs of G that *do* contain e and subgraphs of G/e.

First Partial Duality Theorem cont'd

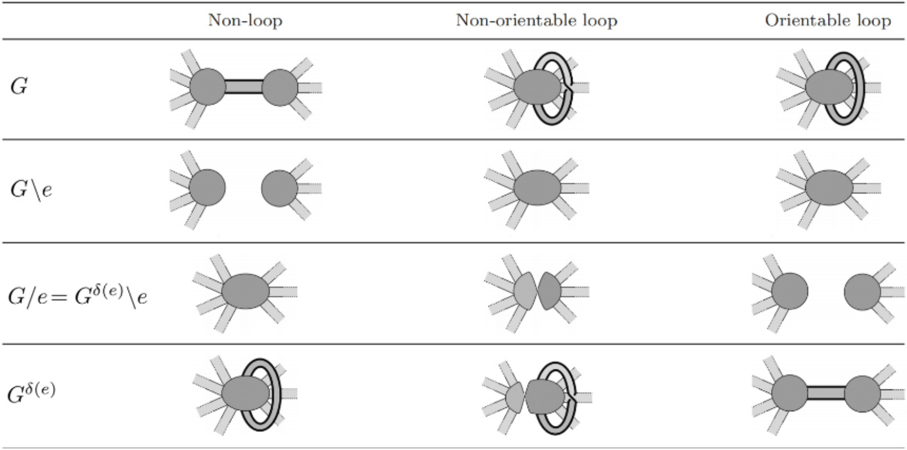

Figure: Table of Partial Duals

Second Partial Duality Theorem

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Theorem 4

Let $G = (V, E)$ be a ribbon graph, $A \subseteq E$. Then, for any n,

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Theorem 4

Let $G = (V, E)$ be a ribbon graph, $A \subseteq E$. Then, for any n,

$$f_n(G) = f_n(G^{\delta(A)})$$

It is sufficient to verify this for a single edge. Let $e \in E$. Then,

$$\begin{aligned} f_n(G) &= f_n(G \backslash e) + f_n(G/e) \\ &= f_n\left((G^{\delta(e)})^{\delta(e)} \backslash e\right) + f_n(G^{\delta(e)} \backslash e) \\ &= f_n(G^{\delta(e)}) \end{aligned}$$

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Sequences of Graphs

Let's remind ourselves of a few relevant sequences of ribbon graphs:

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$$\mathbb{F}_n = (1, 2, 1, 3, 2, 4, 3, \cdots, i, i-1, i+1, i, \cdots, n-1, n-2, n, n-1, n)$$
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Where we have:

$$\kappa(\mathbb{F}_n) = f_{n+1}$$
$$\kappa(\mathbb{F}'_n) = \ell_{n-1}$$
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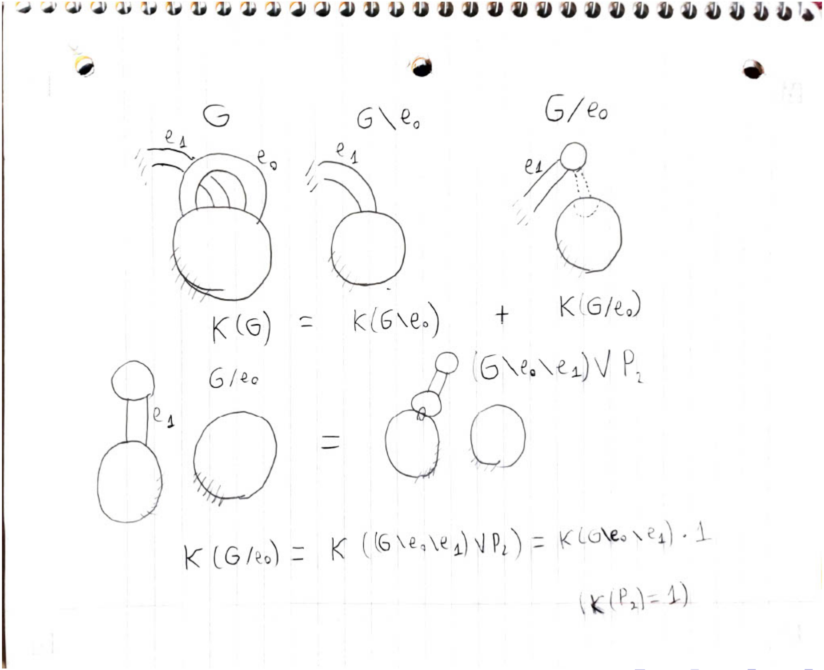
Fact

Let $G = (E, V)$ containing $e_0, e_1 \in E$ and $v \in V$ s.t. the signed rotation of v contains the sequence (e_0, e_1, e_0) and e_0 is a loop. Then

$$\kappa(G) = \kappa(G \backslash e_0) + \kappa(G \backslash e_0 \backslash e_1)$$


G
G\e₀
G/e₀
e₁
e₀
e₁
e₁
K(G) = K(G\e₀) + K(G/e₀)
G/e₀
e₁
(G\e₀\e₁)∨P₂
=
K(G/e₀) = K((G\e₀\e₁)∨P₂) = K(G\e₀\e₁) · 1
(K(P₂)=1)

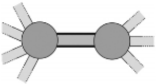
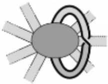
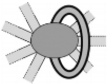
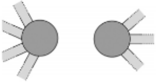
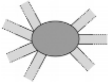
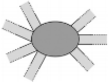
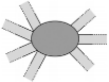
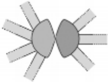
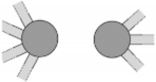
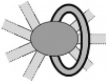
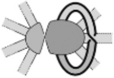
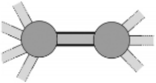
	Non-loop	Non-orientable loop	Orientable loop
G			
$G \backslash e$			
$G/e = G^{\delta(e)} \backslash e$			
$G^{\delta(e)}$			

Figure: Table of Partial Duals

Another Interesting Sequence

Consider the sequence,

$$\mathbb{F}_n^r = (1, 2, -1, 3, -2, 4, -3, \cdots i, 1{-}i, i{+}1, -i, \cdots, n{-}1, 2{-}n, n, 1{-}n, -n)$$

(which is just $\mathbb{F}_n$ with all ribbons twisted)

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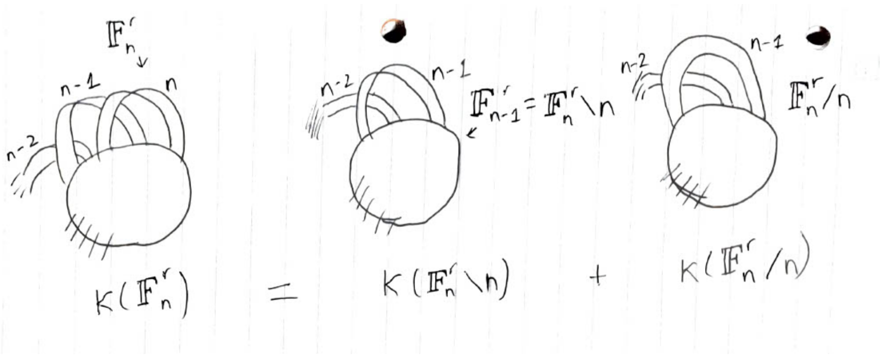
$$\kappa(\mathbb{F}_n^r) = \kappa(\mathbb{F}_{n-1}^r) + \kappa(\mathbb{F}_{n-2}^r) + \kappa(\mathbb{F}_{n-3}^r)$$

More generally, any sequence with such an ending will follow this recurrence relation.



Of course, $\mathbb{F}^r_n \backslash n = \mathbb{F}^r_{n-1}$. But also, by our previous result,

$$\begin{aligned}\kappa(\mathbb{F}^r_n/n) &= \kappa(\mathbb{F}^r_n/n\backslash(n-1)) + \kappa(\mathbb{F}^r_n/n\backslash(n-1)\backslash(n-2)) \\ &= \kappa(\mathbb{F}^r_n\backslash n\backslash(n-1)) + \kappa(\mathbb{F}^r_n\backslash n\backslash(n-1)\backslash(n-2)) \\ &= \kappa(\mathbb{F}^r_{n-2}) + \kappa(\mathbb{F}^r_{n-3})\end{aligned}$$

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$$f_2(\mathbb{F}_n) = 2f_2(\mathbb{F}_{n-1}) + f_2(\mathbb{F}_{n-2}) - 2f_2(\mathbb{F}_{n-3}) - f_2(\mathbb{F}_{n-4})$$

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$$f_2(\mathbb{F}_n) = 2f_2(\mathbb{F}_{n-1}) + f_2(\mathbb{F}_{n-2}) - 2f_2(\mathbb{F}_{n-3}) - f_2(\mathbb{F}_{n-4})$$

Interestingly, the characteristic polynomial of this recurrence relation is $x^4 - 2x^3 - x^2 + 2x + 1 = (x^2 - x - 1)^2$, where $x^2 - x - 1$ is the characteristic polynomial of $f_1(\mathbb{F}_n)$.

References

Q. Deng, X. Jin, Q. Yan, The number of quasi-trees of bouquets with exactly one non-orientable loop, Preprint arXiv:2406.11648v1 [math.CO]