Generalization of mod m Alexander numbering with Arrow polynomial variables

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Overview

Core Definitions

mod *m* Alexander numbering: A generalization of mod *m* Alexander numbering. Given an oriented virtual link diagram, assigning an integer to every semi-arc (section of the link between two classical crossings) in the pattern of the Figure 1 along its orientation(When passing cut points, the number also need to increase by 1). After finish numbering along the orientation, if the start number *a* and end number *b* holds $a - b \equiv 0 \pmod{m}$, then we say this diagram admits mod m Alexander numbering and shows almost-classicality.

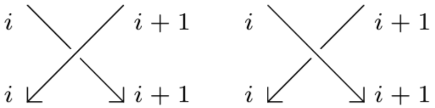


Figure 1: Alexander Numbering

State Expansion: choose an A- or B-oriented smoothing at every classical crossing. When there is no induced orientation, each of the resulting pieces is given a pole which points inwards towards the location where the crossing previously existed. The result is a collection of oriented loops with poles(cusps).

Figure 3: Creation of Poles in State Expansion

Loop index $\imath$: For a loop L of state S, an index $\imath$ is assigned by moving all poles past virtual crossings onto a small semi-arc before canceling all adjacent poles on the same side of a loop. The index is then given by $\imath(L) = \frac{\#\text{poles}}{2}$.

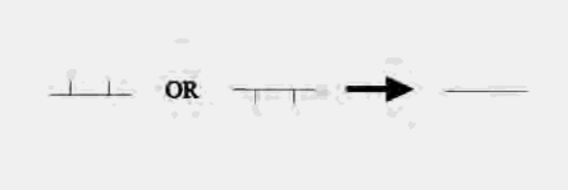


Figure 4: Rules for Pole Cancellation

Arrow Polynomial R_D: After one evaluates all states of a virtual link D and finds the index $\imath$ for each, the normalized Dye-Kauffman arrow polynomial R_D is given by

$$R_D(A; K_1, K_2, ...) = (-A^3)^{-w(d)} \sum_{S} A^{\alpha(S)-\beta(S)}(-A^2 - A^{-2})^{\delta(S)-1} \prod_{L \in S} K_{\imath(L)}$$

where $\alpha(S)$ is the number of A-splittings performed to reach state S, $\beta(S)$ is the number of B-splittings performed to reach state S, and $\delta(S)$ is the number of loops in S.

Gauss diagram: Flatten each component of D into a circle that records the traversal order of the diagram. For every classical crossing place two points on the circle (first encounter and second encounter along the orientation) and join them with an oriented chord. Label each chord with the crossing sign $(+/-)$.

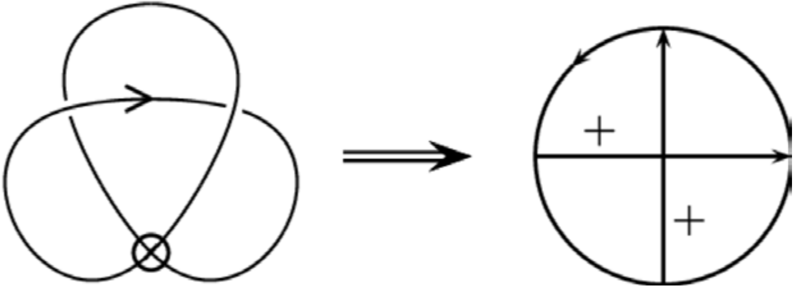


Figure 5: Gauss diagram for a virtual trefoil

Cut System

Cut points & Semi-arc: A cut point is a tiny triangle placed on a semi-arc; its arrow shows the local direction in which the Alexander label must increase by 1. we say an arrow point *coherent* if its orientation matches the direction of that semi-arc

A set of oriented cut points that allows an Alexander numbering is called a cut system.

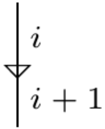


Figure 6: Cut Points

Cut-point moves

Moves I–III: Any two cut systems on the *same* diagram differ by a finite sequence of these three local replacements. Each move leaves “label +1 across the arrow" unchanged, so the existence of an Alexander numbering is preserved.

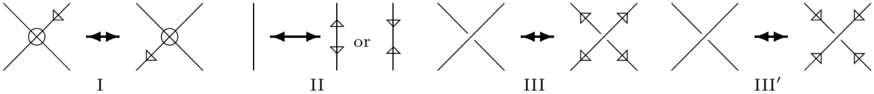
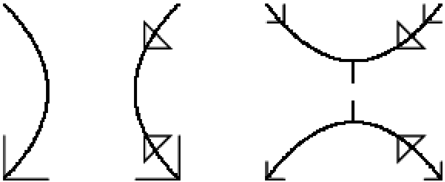


Figure 7: Oriented cut-point moves

Cut system in the Arrow Polynomial

A coherent pair of cut points is attached for a classical crossing. Each triangle marks a place where an Alexander label must jump +1.

After performing an A or B smoothing, those two cut points survive as an inward-pointing and an outward-pointing pole decorating the state loop. The arrow polynomial records the difference between "forward" and "backward" poles on each loop; that count is exactly the loop-index $\imath$ we have been talking about.



(i) (ii)

Introduction of The Topic

Goal for the project: Last time we generalize arrow polynomial from Alexander numbering to mod m Alexander numbering, which is stated in Kamada (2021) Prop. 9. He gives the *only-if* direction statement:

Theorem (Theorem 1)

Let D be a virtual link diagram presenting a mod m almost classical virtual link. Then, X_D is in $\mathbb{Z}[A^1, d_{\frac{m}{2}}, d_{\frac{2m}{2}}, ...]$ when m is even, and it is in $\mathbb{Z}[A^1, d_m, d_{2m}, ...]$ when m is odd. So is R_D.

Interpreted by our notations, it is

$$\text{mod-}m \text{ numbering} \implies \imath \equiv 0 \ (\text{mod } m \text{ or } m/2).$$

Now we are trying to see if this conditioning statement can be generalized to an equivalence.

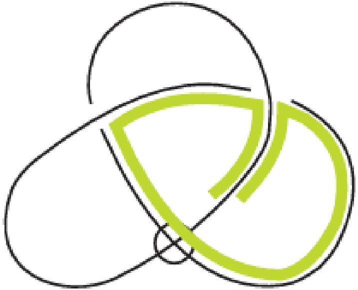
Why the Naïve Converse Fails

Theorem (Converse for Theorem 1)

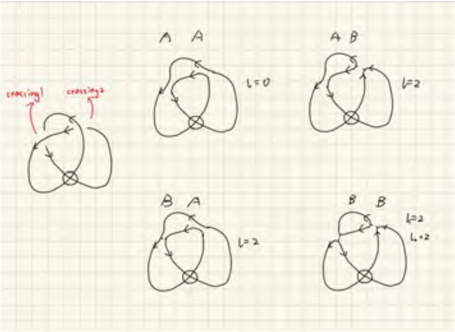
Let D be a virtual link diagram presenting a mod m almost classical virtual link. If x_D holds $\imath \equiv 0 \pmod{m}$, then it admits mod-m numbering.

This converse doesn't hold since $\imath$-index for every loop is a global indicator that represents net sum of cusps in the circle, and only cares if the numbering matches at the oriented cut point. In other words, $\imath$-index ignores numbering along semi-arcs. Numbering along the path can be modified by cut system, which is excluded by $\imath$-index, so we need to include cut system (also make statement sharper) to complete the generalization of Theorem 1.

Counter-example little change on a $m = 2$ virtual trefoil loop: Every virtual diagram automatically has $\imath \equiv 0 \pmod 1$, yet it is *not* checkerboard-colorable(as shown in the figure below), hence doesn't admit mod-2 Alexander numbering.



(a) Möbius's checkboard discolorability



(b) $\imath$-index of Möbius

What to Prove

Our task is therefore to prove a stronger statement:

Theorem

If$\exists$ cut system C such that every semi-arc of D carries k m coherent cut points, then diagram D admits a mod-m Alexander numbering.

Why this works. As we found the reason that converse fails is that $\imath$-index doesn't check cut points, then we add a cut system can keep the diagram still invariant. If such a cut system exists, label one semi-arc by 0 and walk along the orientation; each time you cross a cut point, add 1 mod m. Because each semi-arc contains exactly $km(k \in \mathbb{R})$ identical arrows, its endpoints receive the same label, so the process is well-defined around the entire diagram.

Gauss–Diagram Route

Flatten each component of the virtual link into a circle and mark every classical crossing by a chord with an arrow and a sign. The coherent cut system becomes $k\,m$ identical little arrows drawn on each interval between consecutive chord–endpoints. Starting at an arbitrary basepoint, write 0 and walk once around the circle, adding $+1$ every time we cross one of those arrows. Because every interval contains exactly $k\,m$ arrows, the label returns to 0 at the end of the circuit, so the assignment is well-defined. The two endpoints of each chord now bear the labels required by the Alexander rule, giving the desired numbering directly inside the Gauss diagram.

Wirtinger–Form Route

Erase all virtual crossings and regard the diagram as a 4-valent graph with the usual Wirtinger presentation. Treat each coherent cut point as an oriented “step” that forces its generator to map to the successive generator by $+1$ in $\mathbb{Z}_m$. Because every semi-arc carries a whole multiple $k\,m$ of such steps, the total change along any closed loop in the Wirtinger graph is 0; hence the assignment factors through a well-defined homomorphism $\pi_1(D) \twoheadrightarrow \mathbb{Z}_m$. Pulling this homomorphism back to the arcs produces exactly a mod-m Alexander numbering. Thus the coherence condition can be verified or constructed purely in group-theoretic terms, without leaving the plane.

Worked Example

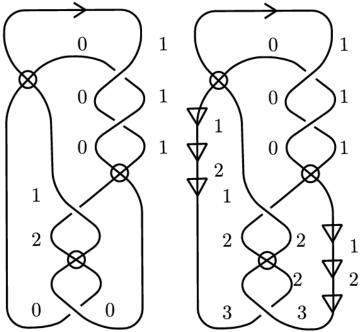


Figure 10: A worked example admits mod 3 Alexander numbering

$\imath$ test

The diagram has loop–indices $\imath_1 = \imath_2 = 0 \pmod 3$. So it satisfies Proposition 9. However, without extra structure, the original diagram doesn't admit mod 3 Alexander numbering.

The reason for the failure is that two long semi-arcs each contain a "+1,1" pair, so label changes cancel globally but clash locally.

In this way, we insert three identical arrows on every semi-arc so that each arrow points *with* the orientation of that arc. Now every semi-arc records exactly *three* $+1$ jumps.

Because $3 \equiv 0 \pmod 3$, the label at the far end of every semi-arc returns to its starting value, removing the previous contradiction.

Why the example holds for the equivalence

Walking from an arbitrary basepoint and adding $+1$ at each arrow produces a consistent labeling. In this way, this example holds for two things at once:

- Loop–index divisibility alone is *insufficient*.
- The extra "$k \cdot m$ coherent arrows on every semi-arc" condition is exactly what lets us integrate the labels and complete the equivalence theorem.

Key references

N. Kamada, *A multivariable polynomial invariant of virtual links and cut systems*, 2021.

A. Nakamura, Y. Nakanishi, A. Satoh, and S. Tomiyama, *Twin groups and their applications in virtual knot theory*, JKTR (2012).

W. Deng, *Arrow polynomial of twisted links*, JKTR (2022).

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