Recurrence Relationships in Ribbon Graphs

Without Matrices

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Review

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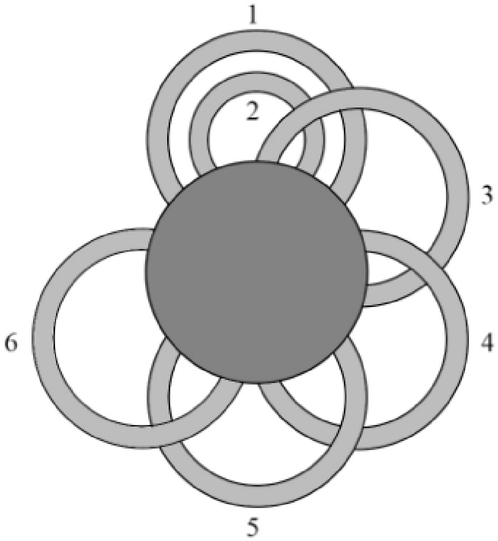
We are particularly interested in bouquets, ribbon graphs with exactly one vertex (so all edges are loops).

Bouquet Example



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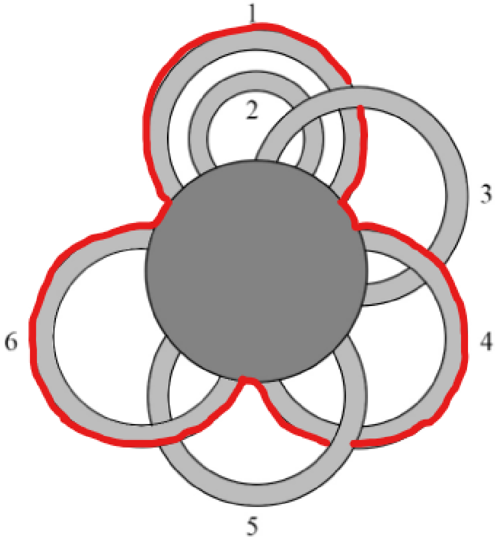

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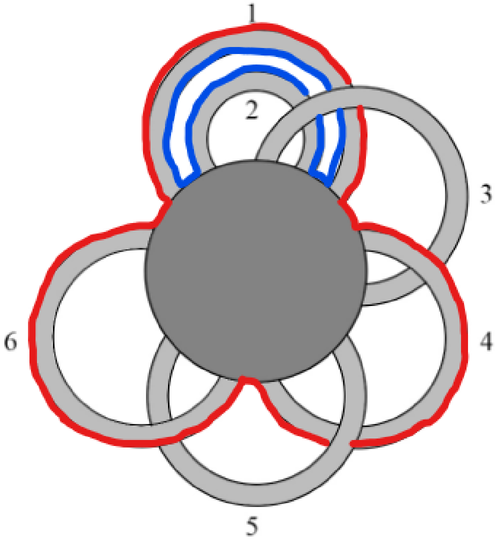

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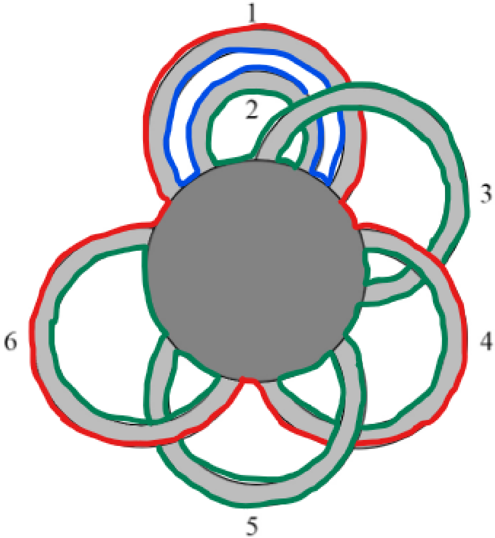

Subgraph Counting

Definition

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For some ribbon graph G, let $\kappa(G)$ equal the number of spanning subgraphs of G that are quasi-trees.

Signed Rotation

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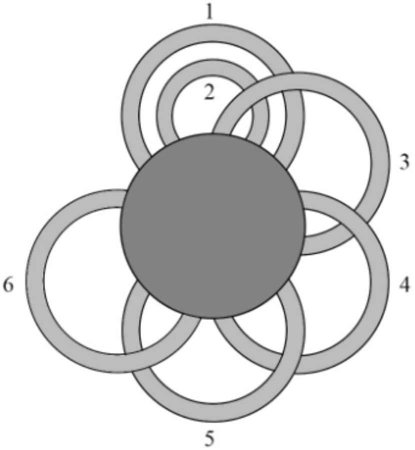
To find a bouquet's signed rotation:

- Number each edge 1 through n (where the graph has n edges).
- Pick a point along the vertex.
- Go clockwise along the vertex, and each time you hit an edge, put its number at the end of the signed rotation.
- If an edge is twisted, multiply **one** of its edges in the signed rotation by -1.

Signed Rotation Example

For example, the following graph could be written as:

$$(1, 2, 3, 2, 1, 4, 3, 5, 4, 6, 5, 6)$$



Known Recurrence Relations

Let f_n denote the n-th Fibonacci Number $(1, 1, 2, 3, 5, 8, 13, 21, \ldots)$
Let ℓ_n denote the n-th Lucas Number $(1, 3, 4, 7, 11, 18, 29, 47, \ldots)$

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With $\kappa(\mathbb{F}_n) = f_{n+1}$, $\kappa(\mathbb{F}'_n) = \ell_{n-1}$, $\kappa(\mathbb{F}^1_n) = f_{n+2}$, and $\kappa(\mathbb{F}'^1_n) = f_n + \ell_{n-1}$.

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It is *not* a coincidence that these sequences all have the same ending and the same recurrence relation.

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New Theorems

Now, we'll try to understand these recurrence relations without the use of matrices.

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Lemma

Let $B(G) =$ the number of boundary components of G for any ribbon graph G. For disjoint ribbon graphs P and Q, where $\vee$ is the one-vertex-join operator, $B(P \vee Q) = B(P) + B(Q) - 1$

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Let $B(G) =$ the number of boundary components of G for any ribbon graph G. For disjoint ribbon graphs P and Q, where $\vee$ is the one-vertex-join operator, $B(P \vee Q) = B(P) + B(Q) - 1$

Consequently,

Theorem 1

For disjoint ribbon graphs P and Q, where $\vee$ is the one-vertex-join operator,

$$\kappa(P \vee Q) = \kappa(P)\kappa(Q)$$

Subgraphs with More Boundary Components

Let $G = (V, E)$ be a ribbon graph, and let n be a positive integer.

Definition

$\mathcal{F}_n(G) = \{F \subseteq E \mid (V, F) \text{ has } n \text{ boundary components}\}$

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Definition

$f_n(G)$ is the cardinality of $\mathcal{F}_n(G)$

Put simply, $f_n(G)$ counts the number of spanning subgraphs of G with exactly n boundary components (and, in particular, $f_1(G) = \kappa(G)$).

Generalization of Theorem 1

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For disjoint ribbon graphs P and Q, where $\vee$ is the one-vertex-join operator,

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So, each subgraph of $P \vee Q$ with n boundary components must have i boundary components "contributed" by P, and $n + 1 - i$ boundary components "contributed" from Q for some i.

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Theorem 3

Let G be a ribbon graph and $e \in E(G)$. Then, for any n,

$$\begin{aligned} f_n(G) &= f_n(G \backslash e) + f_n(G/e) \\ &= f_n(G \backslash e) + f_n(G^{\delta(e)} \backslash e) \end{aligned}$$

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We can find an obvious 1-1 correspondence between subgraphs of G that do not contain e and subgraphs of $G \backslash e$.
There is a slightly less obvious 1-1 correspondence between subgraphs of G that *do* contain e and subgraphs of G/e.

First Partial Duality Theorem cont'd

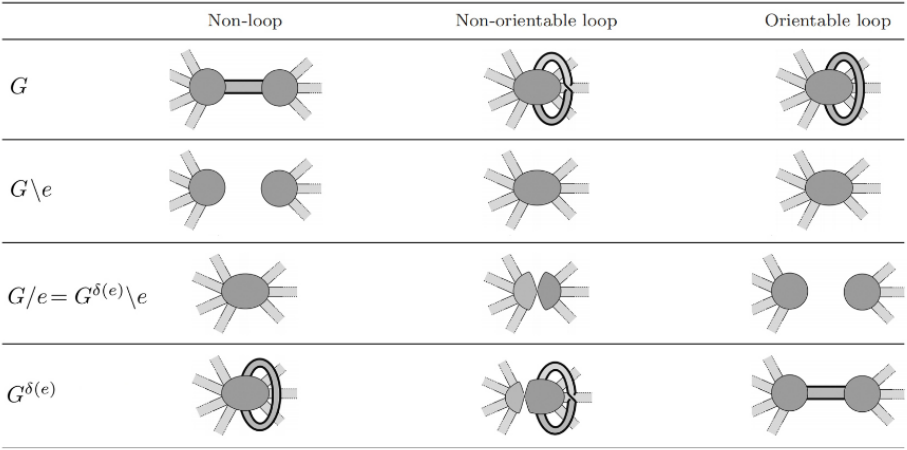

Figure: Table of Partial Duals

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Theorem 4

Let $G = (V, E)$ be a ribbon graph, $A \subseteq E$. Then, for any n,

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It is sufficient to verify this for a single edge. Let $e \in E$. Then,

$$\begin{aligned} f_n(G) &= f_n(G \backslash e) + f_n(G/e) \\ &= f_n\left((G^{\delta(e)})^{\delta(e)} \backslash e\right) + f_n(G^{\delta(e)} \backslash e) \\ &= f_n(G^{\delta(e)}) \end{aligned}$$

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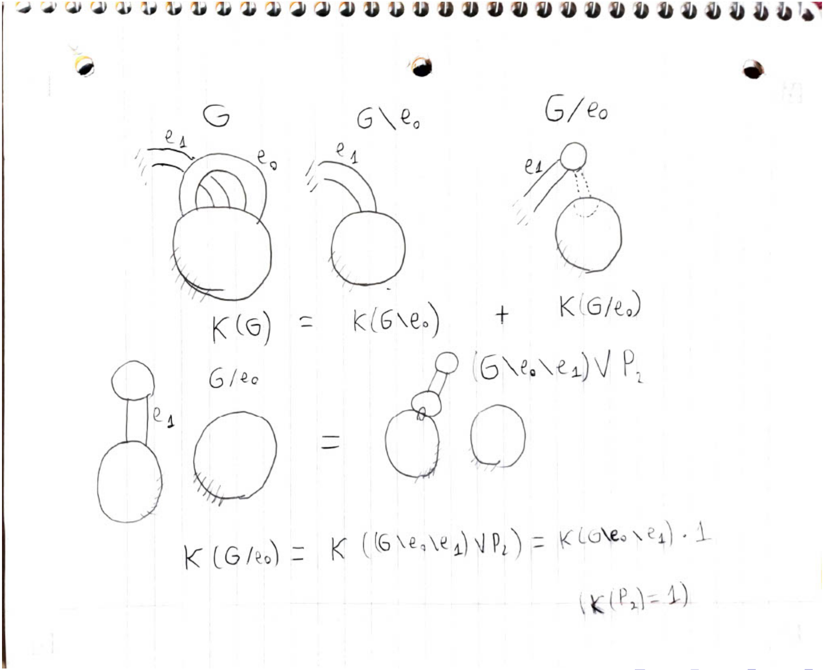
Fact

Let $G = (E, V)$ containing $e_0, e_1 \in E$ and $v \in V$ s.t. the signed rotation of v contains the sequence (e_0, e_1, e_0) and e_0 is a loop. Then

$$\kappa(G) = \kappa(G \backslash e_0) + \kappa(G \backslash e_0 \backslash e_1)$$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K(G) = K(G\e0) + K(G/e0)
G/e0
e1
=
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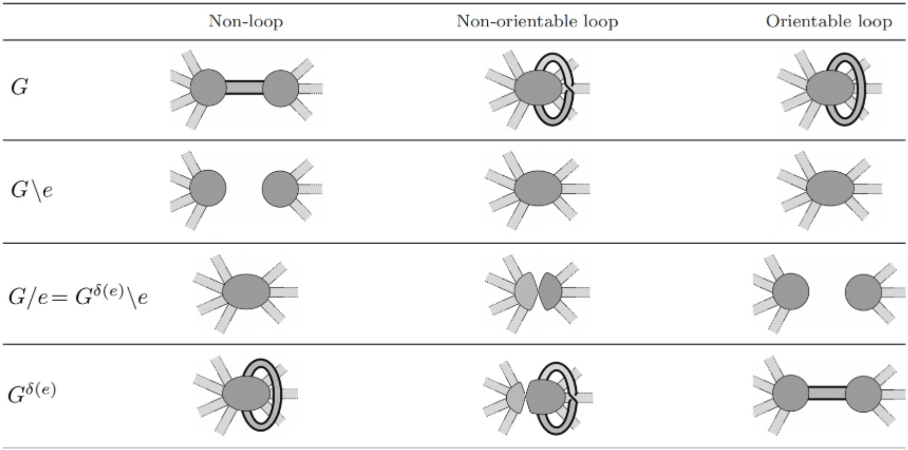

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Another Interesting Sequence

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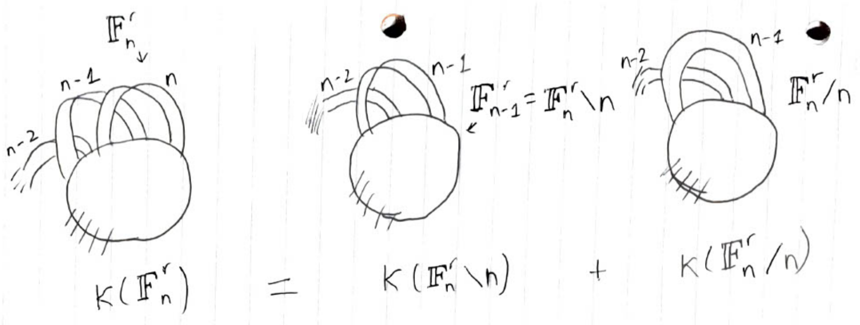
$$\kappa(\mathbb{F}_n^r) = \kappa(\mathbb{F}_{n-1}^r) + \kappa(\mathbb{F}_{n-2}^r) + \kappa(\mathbb{F}_{n-3}^r)$$

More generally, any sequence with such an ending will follow this recurrence relation.



Of course, $\mathbb{F}_n^r \backslash n = \mathbb{F}_{n-1}^r$. But also, by our previous result,

$$\begin{aligned}\kappa(\mathbb{F}_n^r/n) &= \kappa(\mathbb{F}_n^r/n\backslash(n-1)) + \kappa(\mathbb{F}_n^r/n\backslash(n-1)\backslash(n-2)) \\ &= \kappa(\mathbb{F}_n^r\backslash n\backslash(n-1)) + \kappa(\mathbb{F}_n^r\backslash n\backslash(n-1)\backslash(n-2)) \\ &= \kappa(\mathbb{F}_{n-2}^r) + \kappa(\mathbb{F}_{n-3}^r)\end{aligned}$$

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For example, the Fibonacci numbers have characteristic polynomial

$$P(x) = x^2 - x - 1$$

More Boundary Components

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Another interesting result arises from considering f_n where $n > 1$. Consider $\mathbb{F}_n$ (or any sequence with such an ending). Then the following turns out to be true:

Theorem

The characteristic polynomial of $f_m(\mathbb{F}_n)$ is the characteristic polynomial of $\kappa(\mathbb{F}_n)$ raised to the m-th power.

Proof of Theorem

We will prove this theorem using induction. The base case is clear, as $f_1(\mathbb{F}_n) = \kappa(\mathbb{F}_n)$. Then, our goal is to prove that increasing the subscript on f by 1 will have the effect of multiplying the characteristic polynomial by $(x^2 - x - 1)$, the characteristic polynomial of $\kappa(\mathbb{F}_n)$.

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This does not extend directly to higher boundary component numbers because of Theorem 2.
Rather, we have $f_m(\mathbb{F}_n) = f_m(\mathbb{F}_{n-1}) + \sum_{i=1}^{m} f_i(\mathbb{F}_{n-2}) f_{m+1-i}(P_2)$.

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Luckily, $f_m(P_2)$ is non-zero for only $m = 1, 2$ (with $f_1(P_2) = f_2(P_2) = 1$), so the above becomes simply
$f_m(\mathbb{F}_n) = f_m(\mathbb{F}_{n-1}) + f_m(\mathbb{F}_{n-2}) + f_{m-1}(\mathbb{F}_{n-2})$.

Proof of Theorem, cont'd

It immediately follows that $f_{m-1}(\mathbb{F}_n) = f_m(\mathbb{F}_{n+2}) - f_m(\mathbb{F}_{n+1}) - f_m(\mathbb{F}_n)$.

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So, say we have the characteristic polynomial for $f_{m-1}(\mathbb{F}_n)$ in terms of x called $P_{m-1}(x)$. Then, by the above equality, we can get $P_m(x)$ by replacing each term ax^b of $P_{m-1}(x)$ with $a(x^{b+2} - x^{b+1} - x^b)$.

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This is, of course, the same as multiplying by $x^2 - x - 1$. So, $P_m(x) = (x^2 - x - 1)P_{m-1}(x)$.

Proof of Theorem, cont'd

It immediately follows that $f_{m-1}(\mathbb{F}_n) = f_m(\mathbb{F}_{n+2}) - f_m(\mathbb{F}_{n+1}) - f_m(\mathbb{F}_n)$.

So, say we have the characteristic polynomial for $f_{m-1}(\mathbb{F}_n)$ in terms of x called $P_{m-1}(x)$. Then, by the above equality, we can get $P_m(x)$ by replacing each term ax^b of $P_{m-1}(x)$ with $a(x^{b+2} - x^{b+1} - x^b)$.

This is, of course, the same as multiplying by $x^2 - x - 1$. So, $P_m(x) = (x^2 - x - 1)P_{m-1}(x)$.

So, the inductive step is proved, and the theorem is true.

Theorem Extended

This theorem is not only true for graphs with the ending of $\mathbb{F}_n$ though. It, in fact, applies to *every* sequence of graphs we've examined thus far.

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How much this theorem could be generalized is an interesting question. The following is false:

False Theorem

If G_n is a sequence of ribbon graphs such that $f_m(G_n)$ has a linear recurrence relation $\forall m \in \mathbb{Z}^+$, then the characteristic polynomial of $f_m(G_n)$ is equal to the characteristic polynomial of $\kappa(G_n)$ raised to the m-th power.

Counterexample

Consider $\mathbb{F}_n^{1,n}$, which is $\mathbb{F}_n$ with the first and last edges twisted, where we want to find the characteristic polynomial of $f_m(\mathbb{F}_n^{1,n})$.

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So, using we can substitute this into the recurrence relation for $f_m(\mathbb{F}_n^{1})$, which has the effect of multiplying the characteristic polynomial, and get that the characteristic polynomial of $f_m(\mathbb{F}_n^{1,n})$ is

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$(x^2 - x - 1)(x - 1) = (x^3 - 2x + 1)$, so

$$\kappa(\mathbb{F}_n^{1,n}) = 2\kappa(\mathbb{F}_{n-1}^{1,n-1}) - \kappa(\mathbb{F}_{n-3}^{1,n-3})$$

References

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