Multivirtual Quandles Invariants

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Review, What is a Rack?

Definition

A *rack* R is a set with two binary operations $*$ and $/$ satisfying: $\forall a, b, c \in R$,

- (left self distribution) $a * (b * c) = (a * b) * (a * c)$
- (right self distribution) $(b/c)/a = (b/a)/(c/a)$
- $(a * b)/a = b$
- $a * (b/a) = b$

Equivalently a rack may be defined as a set R with a binary operation $*$ satisfying left self distribution and such that $\forall a, b \in R$ there exists a unique c such that $a = c * b$

Review, What is a Quandle?

Definition

A *quandle* Q is an idempotent rack, that is a rack such that $\forall a \in Q \; a * a = a$ (or equivalently $a/a = a$).

Quandle 2-Cocycles

For any Quandle $(Q, *)$ and group $(G, \cdot)$ we call a mapping $\phi : Q \times Q \to G$ a quandle 2-cocycle with values in G if $\forall a, b, c \in Q$.

1. $\phi(a, a) = 0$
2. $\phi(a, b) + \phi(a * c, b * c) = \phi(a, c) + \phi(a * b, c)$

Operator Quandle 2-Cocycles

Given an operator quandle $(Q, *, A_T)$ (A_T here is the commuting list of automorphisms of $(Q, *)$) and a group $(G, \cdot)$, we call a quandle 2-cocycle with values in G of $(Q, *)$ $\phi : Q \times Q \to G$ a quandle operator 2-cocycle if

$$\phi(a, b) = \phi(\alpha(a), \alpha(b))$$

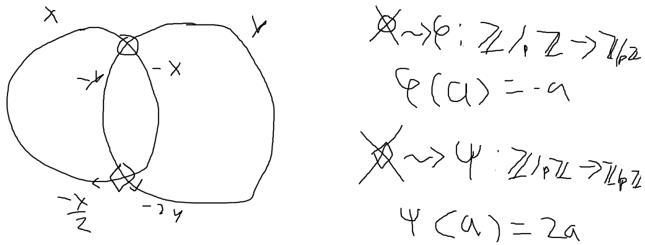
$\forall a, b \in Q$ and $\alpha \in A_T$.

Operator Quandle 2-Cocycles

Given any operator Quandle $(Q, *, A_T)$, group $(G, \cdot)$, multivirtual link diagram D, and quandle 2-cocycle ϕ, for any quandle operator coloring C of D by $(Q, *, A_T)$ and any crossing c, let (a, b) be the arcs going into the crossing c and $\epsilon = \pm 1$ denote the sign of the crossing.
Then,
$Coc((Q, *, A_T), G) := \sum_C \prod_c \phi(a, b)^\epsilon$
is a multi-virtual knot invariant where C is taken over all colorings, c is taken over all classical crossings, and the sum is defined by considering elements of G as elements of the group ring $\mathbb{Z}G$.

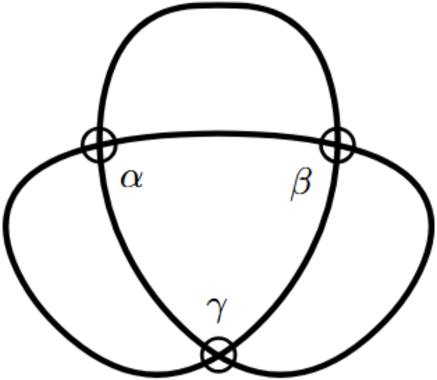
The Problems With Quandle Type Operators

Although these quandle type invariants can distinguish multivirtual link diagrams with no classical crossings, none of them are useful in distinguishing multivirtual knots with no classical crossings.



The Problems With Quandle Type Operators

It is not known wether the folloeing multivirtual knot diagram is equivalent to the unknot:



The Problems With Quandle Type Operators

Because the operator quandle coloring of any arc of a multivirtual knot diagram with no classical crossings only depends on the coloring of the previous, any quandle coloring is uniquely determined by a choice of element on any chosen starting arc and from the commutativity condition of the automorphisms at each crossing we have that all initial choices result in a valid coloring.

References

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