Racks, Quandles, and Their Relation to (Classical, Virtual, and Multi-Virtual) Knot Theory

Jeremy Case

The Ohio State University

Racks

Definition

A *rack* R is a set with two binary operations $*$ and $/$ satisfying: $\forall a, b, c \in R$,

- (left self distribution) $a * (b * c) = (a * b) * (a * c)$
- (right self distribution) $(b/c)/a = (b/a)/(c/a)$
- $(a * b)/a = b$
- $a * (b/a) = b$

Equivalently a rack may be defined as a set R with a binary operation $*$ satisfying left self distribution and such that $\forall a, b \in R$ there exists a unique c such that $a = c * b$

Quandles

Definition

A *quandle* Q is an idempotent rack, that is a rack such that $\forall a \in Q \; a * a = a$ (or equivalently $a/a = a$).

Free Quandles

Definition

Given a set S, the set of quandle words of S (which we will denote $QW(S)$) is the set defined recursively by,
$a \in S \implies a \in QW(S)$ and $a, b \in S \implies a * b \in S$ and $a/b \in S$.

An element of $QW(S)$ is a finite string of elements of S, quandle operations, and parenthesis s.t. they can be evaluated properly

Definition

For any set S the *free quandle* of S (which we will denote $FQ(S)$) is an equivalence class of elements of $QW(S)$ under the relations,

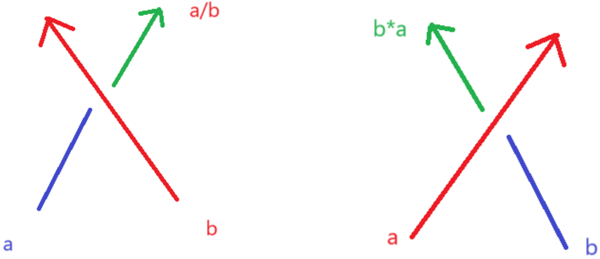
- $x * x \sim x$
- $(y * x)/y \sim x$
- $x * (y/x) \sim x$
- $x * (y * z) \sim (x * y) * (x * z)$
- $(y/z) * x \sim (y/x)/(z/x)$

Quandle Presentations

$< \{g_i\}_{i \in I} | \{r_j\}_{j \in J} >$ will denote the free quandle generated by $\{g_i\}_{i \in I}$ modulo the equivalence relations $\{r_j\}_{j \in J}$.
For example the quandle $< a, b | a * b = a, b * a = b >$ is the trivial quandle of 2 elements.

The Fundamental Quandle of a Classical Link

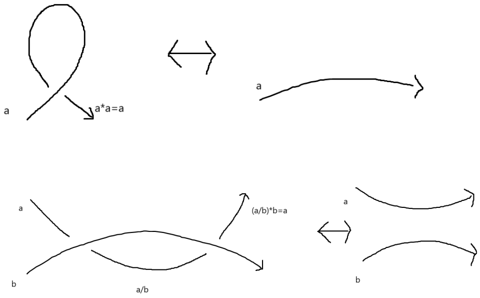
For any classical link diagram L, we will construct the *fundamental qaundle* of L as the quandle $< A|R >$ where A is the set of arcs of L and R is the set of relations associated with each crossing as follows:



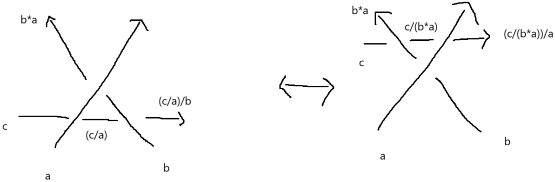
The Fundamental Quandle of a Classical Link

Theorem

The fundamental quandle is (up to isomorphism) a knot invariant



The Fundamental Quandle of a Classical Link



The Fundamental Quandle of a Classical Link

Letting $x = (c/a)/b$ and $y = (c/(b * a))/a$,
we have $c = (x * b) * a = (x * a) * (b * a)$ and $c = (y * a) * (b * a)$.
If follows from the the condition ($\forall e, f \in R$ there exists a unique g such that $e = g * f$) that $(x * a) = (y * a) \implies x = y$.

Quandle Isomorphisms

A *homomorphism* of quandles Q and Q' is a function $\phi : Q \to Q'$ such that $\forall a, b \in Q$, $\phi(a * b) = \phi(a) * \phi(b)$
An *isomorphism* of quandles Q and Q' is a bijective homomorphism of Q and Q'. Two quandles are said to be *isomorphic* if there exists an isomorphism between them.

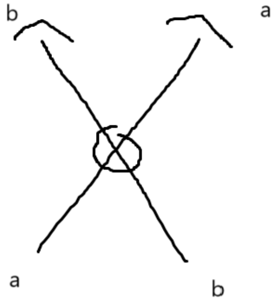
Completeness of the Fundamental Quandle

Theorem

For non-split links the fundamental quandle is a complete (up to mirror image) invariant. That is to say that for any two non split link diagrams L and L' if the fundamental quandle of L is isomorphic to the fundamental quandle of L' then L is isomorphic to either L' or it's mirror image.

Quandles for Virtual Knots

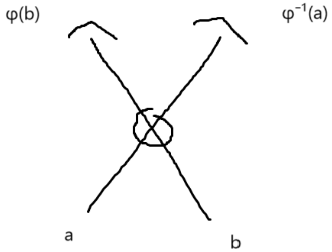
The fundamental quandle of a virtual link is computed in the same way as a classical link with the addition of the following relation at any virtual crossing.



The fundamental quandle is a virtual knot invariant but is not a complete virtual knot invariant.

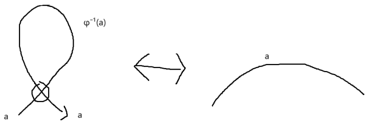
Quandle Colorings

An *autommorphism* of a quandle Q is an isomorphism $\phi : Q \to Q$. For any quandle Q, automorphism ϕ of Q, and virtual link diagram D, we will let $\text{Col}(D, Q, \phi)$ be the number of ways of assigning elements of Q to the arcs of D folowing the classical crossing relation and the virtual crossing relation below.



Quandle Colorings

Col(D, Q, ϕ) is a virtual link invariant.



Quandle Colorings

In the case of multivirtual links with n virtual crossing types we may choose $\phi_1, \cdots, \phi_n$ pairwise commuting automorphisms of Q associating each with one type of virtual crossing. At any virtual crossing the same crossing rule is applied as for virtual crossings using this automorphism.

Theorem

The number of quandle colorings for any fixed quandle Q and pairwise commuting automorphisms $\phi_1, \cdots, \phi_n$ is a multivirtual knot invariant.

References

L.H. Kauffman
Multi-virtual knot theory.
Journal of Knot Theory and it's Ramifications, 2025.

Louis H. Kauffman and Sujoy Mukherjee and Petr Vojtěchovský
Algebraic invariants of multi-virtual links.
arXiv, 2025.

Miller, Jac
Quandle Invariants of Knots and Links
https://www.sas.rochester.edu/mth/undergraduate/honorspaperspdfs