Seifert State Ribbon Graphs of Classical Links

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State Expansion

Recall the concept of state expansion, which involves choosing an A or B-splitting at every classical crossing.

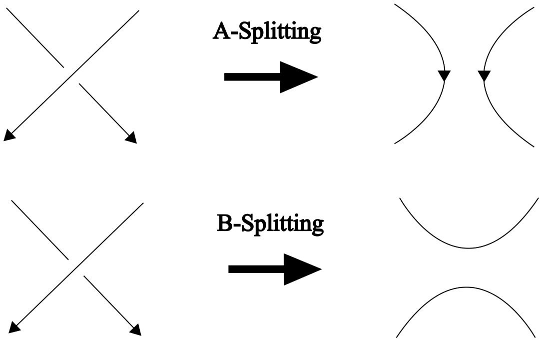

Figure: Review of State Expansion

Seifert State

For this presentation, we are concerned with a final state known as the Seifert state, which requires choosing a splitting at each crossing that induces an orientation on each of the resulting arcs.

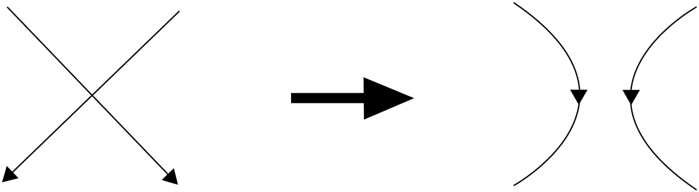

Figure: Splitting to Induce Orientation

Seifert State Ribbon Graph

At each expanded crossing, we can attach the two arcs with a ribbon to produce a ribbon graph. We can also put counterclockwise arrows on each free edge arc of the ribbon. These arrows are particularly relevant for the arrow Thistlethwaite theorem, but we will just use them to make observations.

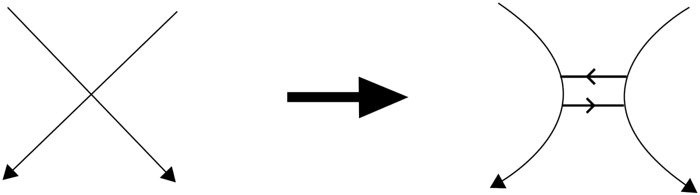

Figure: Creation of Seifert State Ribbon Graph

Leading Question

It is an interesting question to consider what types of ribbon graphs may correspond to classical links via this construction. To aid in this process, we will need to recall the concepts of winding numbers and Alexander numberings from a previous presentation.

Alexander Numbering

Recall the concept of an Alexander numbering, which requires every arc to be labeled with an integer so that locally each crossing looks like one of the following:

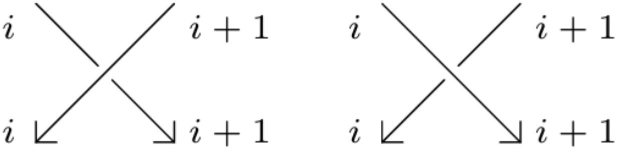

Figure: Alexander Numbering

Winding Number

Recall the concept of the winding number. The winding number of a curve around a point is the signed number of counterclockwise rotations the curve makes around the point (clockwise rotations are counted as negative). Importantly, each point in a region with no arcs has the same winding number and moving across an arc from right to left increases the winding number by 1.

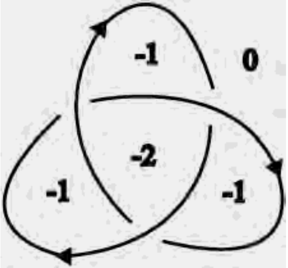

Figure: Trefoil with Winding Numbers

Induced Alexander Numbering

For any classical link, the winding numbers of the regions of the link induce an Alexander numbering on the arcs following the rules given in the diagram below:

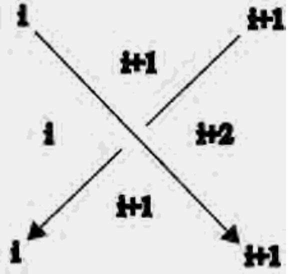

Figure: Induced Alexander Numbering

Expanded State

So, for any classical link, we can examine what happens with regards to the Alexander numbering as we expand to the Seifert state and construct a ribbon graph. We can see that the vertices of the graph inherit integer labels.

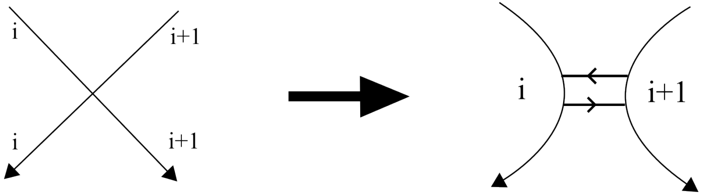

Figure: Induced Vertex Labeling

Observations

From this diagram, we can make two additional observations. First, only vertices with labels which are one apart may be adjacent (meaning the result will be bipartite). Second, the arrows on the ribbons always agree with the orientation of the smaller vertex and disagree with the orientation of the larger vertex.

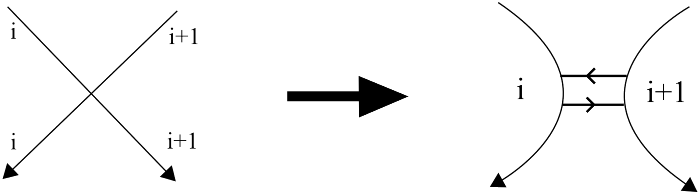

Figure: Induced Vertex Labeling

Example

In the below example, we see two different types of relationships between vertices. There are neighbors and parent/child relationships.

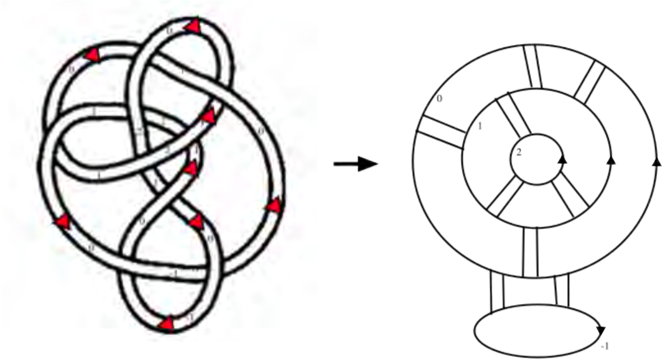

Figure: Example Seifert State Ribbon Graph

Alexander's Theorem

In 1923, Alexander proved the following:

Theorem

Every classical link can be represented as a closed braid.

Figure: Closed Braid

Consequence of Alexander's Theorem

Constructing the Seifert state of a closed braid and then forming a ribbon graph always gives a ribbon graph made of nested vertices, so by Alexander's Theorem, every classical link always has a Seifert state ribbon graph of this form.

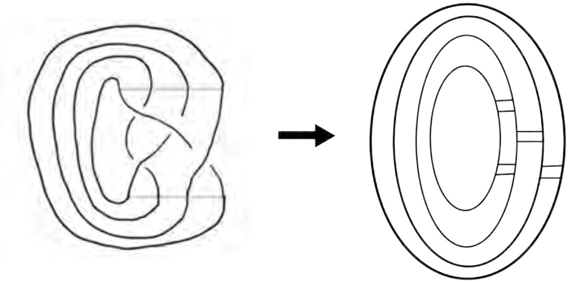

Figure: Seifert State Ribbon Graph of Nested Vertices

References

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- "The Rolfsen Knot Table." The Rolfsen Knot Table - Knot Atlas, katlas.org/wiki/The_Rolfsen_Knot_Table. Accessed 18 July 2025.
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References

- Bradford, R., Butler, C., & Chmutov, S. (2012). Arrow ribbon graphs. Journal of Knot Theory and Its Ramifications, 21(13), 1240002. https://doi.org/10.1142/S0218216512400020