The Relation between Ribbon-graphic Delta-matroid and Eulerian Delta-matroid

Yile Huang

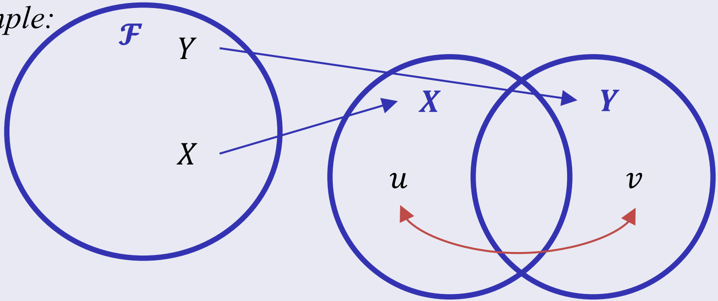
July 21, 2025

Delta-Matroid

Definition:
A *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ that satisfies the Symmetric Exchange Axiom, where $\mathcal{F} \subseteq \mathcal{P}(E)$

Symmetric Exchange Axiom:
For $X, Y \in \mathcal{F}$; for $\forall u \in X \triangle Y, \exists v \in X \triangle Y, \text{s.t.}\, X \triangle \{u, v\} \in \mathcal{F}$

Example:



Set System of Ribbon Graphs

Definition:
Let $G = (V, E)$ be a ribbon graph, where V is the set of vertices and E is the set of edges.

Definition:
Let $D_{\leq n}(G) = \left(E, \mathcal{F}_{\leq n}(G)\right)$, where E is the set of edges and $\mathcal{F}_{\leq n}(G) = \{A \subseteq E | f(A) \leq n + k(G)\}$;
Let $D_n(G) = \left(E, \mathcal{F}_n(G)\right)$, where E is the set of edges and $\mathcal{F}_n(G) = \{A \subseteq E | f(A) = n + k(G)\}$;

where $f(A)$ is the number of boundary components of the spanning subgraph formed by A, and $k(G)$ is the number of connected components of ribbon graph G.

$D_{\leq 0}(G) = D_0(G) \coloneqq D(G)$

Ribbon Graphic Delta-matroid

Theorem:
For ribbon graph G, the set system $D_{\leq n}(G) = \left(E, \mathcal{F}_{\leq n}(G)\right)$ is a delta-matroid

Sketch of proof:
For any $X, Y \in \mathcal{F}_{\leq n}(G)$, choose some $u \in X \triangle Y$;
Case 1: $X \triangle u \in \mathcal{F}_{\leq n}(G) \Rightarrow X \triangle \{u, u\} \in \mathcal{F}_{\leq n}(G)$, done;
Case 2: $X \triangle u \notin \mathcal{F}_{\leq n}(G) \Rightarrow X \triangle u \in \mathcal{F}_{n+1}(G)$, and we can always find $v \in X \triangle Y, \text{s.t.}\, f(X \triangle \{u, v\}) = f(X \triangle u) - 1$, done.
Thus, for $\forall u \in X \triangle Y, \exists v \in X \triangle Y, \text{s.t.}\, X \triangle \{u, v\} \in \mathcal{F}_{\leq n}(G)$

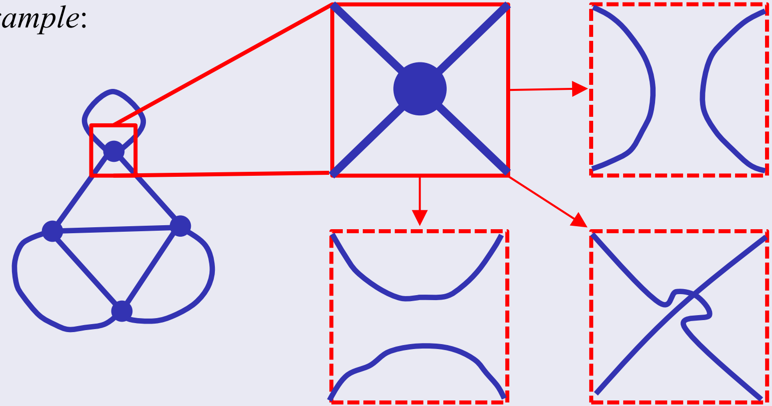
Q.E.D.

Definition: A delta-matroid can be obtained in the way of $D(G) = \left(E, \mathcal{F}_0(G)\right)$ is called a *ribbon-graphic delta-matroid*.

Transition

Definition:
Let $F = (V, E)$ be a 4-regular graph, a *transition* τ_v at a vertex $v \in V$ is a partition of the half-edges at v into two pairs.

Example:

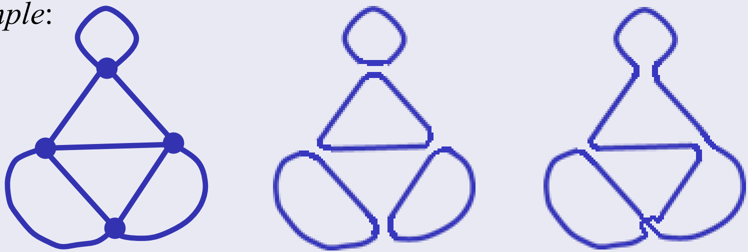


Transition System & Graph State

Definition:
Let $F = (V, E)$ be a 4-regular graph, a *transition system* $\tau = \{\tau_v | v \in V\}$ is the collection of transitions at all vertices of F.

Definition:
For F a 4-regular graph, its *graph state* of a transition system τ, $F(\tau)$, is the set of all the free loops after τ is applied to F. $|F(\tau)|$ is the number of free loops.

Example:



Allowable Transversal

Definition:
Let τ_v and $\tau_v{}'$ be exactly two transitions at each vertex chosen from its three choices, let $\tau = \{\text{either } \tau_v \text{ or } \tau_v{}' | v \in V\}$, called an *allowable transversal*, be a transition system picks up one of the chosen transitions at each vertex; then, the *set of allowable transversal* $\mathcal{T}$ is the collection of all such transition systems.

Theorem:
For F a 4-regular graph, let $T \in \mathcal{T}$ be an allowable transversal of F, then the set system

$$D(F, \mathcal{T}, T) = (T, \{\tau \cap T | \tau \in \mathcal{T} \text{ and } |F(\tau)| = k(F)\})$$

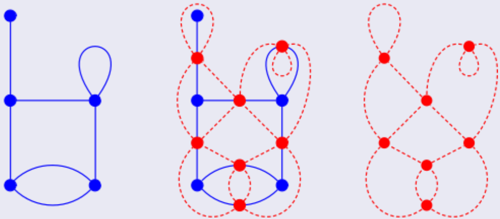
is a delta-matroid.

Definition: A delta-matroid can be obtained in the way of $D(F, \mathcal{T}, T)$ is called an *Eulerian delta-matroid*.

Medial graph

Definition:
Let G be a connected graph cellularly embedded in a surface. Its *medial graph*, G_m, is the embedded graph constructed by placing a vertex on each edge of G, and then drawing the edges of the medial graph by following the face boundaries of G (so each vertex of G_m is of degree 4). The medial graph of an isolated vertex is a free loop. The vertices of G_m are 4-valent and correspond to the edges of G.

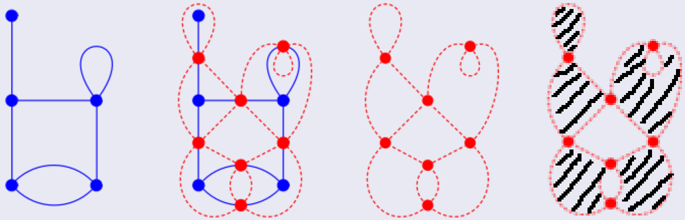
Example:



Canonical face 2-coloring

Definition:
Every medial graph has a canonical face 2-colouring given by colouring faces corresponding to a vertex of G black, and the remaining faces white.

Example:

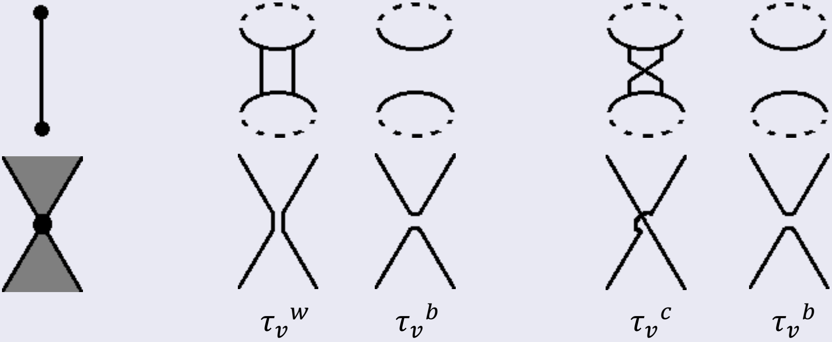


Definition: we call a vertex transition white if it pairs half-edges that share a white face, black if it pairs half-edges that share a black face, and crossing otherwise.

Ribbon-graphic delta-matroid and Eulerian delta-matroid

Theorem:
A delta-matroid D is Eulerian iff $D \cong D(G)$, for some ribbon graph G.

Sketch of proof:
(1) form Ribbon-graphic to Eulerian



Ribbon-graphic delta-matroid and Eulerian delta-matroid

Sketch of proof (continued):
Let set $NB = \{\tau_v{}^w \text{ or } \tau_v{}^c | v \in V\}$ be the set of all the non-black transition at each point; let G_m be the medial graph of G; let $\mathcal{T}_m$ be the set of allowable transversals, in which all the transition systems contained by its element only contains chosen transitions in the previous page. Then,
$D(G_m, \mathcal{T}_m, NB) = (NB, \{\tau \cap NB | \tau \in \mathcal{T}_m \text{ and } |G_m(\tau)| = k(G_m)\})$
is the Eulerian delta-matroid we want. And it is obvious that

$$D(G) = D(G_m, \mathcal{T}_m, NB).$$

(2) form Eulerian to Ribbon-graphic
By using the fact that ribbon graph can be obtained as a cycle family graph of F (The cycle family graphs of F are precisely the embedded graphs that have a medial graph isomorphic to F.), the second part is proved.

Q.E.D.

References

[1] C. Chun, I. Moffatt, S. D. Noble, R. Rueckriemen, Matroids, delta-matroids and embedded graphs, *J. Combin. Theory Ser. A*, 167 (2019) 7–59.

[2] A. Bouchet, Representability of Δ-matroids, in: Proceedings of the 6th Hungarian Colloquium of Combinatorics, in: Colloq. Math. Soc. János Bolyai, 1987, pp. 167–182.

[3] A. Bouchet, Greedy algorithm and symmetric matroids, Math. Program. 38 (1987) 147–159.

[4] A. Bouchet, Maps and delta-matroids, Discrete Math. 78 (1989) 59–71.

[5] J. Ellis-Monaghan, I. Moffatt, Twisted duality for embedded graphs, Trans. Amer. Math. Soc. 364 (2012) 1529–1569.

[6] Wikipedia. https://en.wikipedia.org/wiki/Medial_graph