Knots and Graphs

Working Group [Summer 2025]

MATH 4193, class number 14974

Instructor: ***Sergei Chmutov***

RESEARCH PROJECTS

Project 1. ***Multi-Virtual Knot Theory.*** (Jeremy Case, Ethan Lu. TA: Luke Wiljanen.)

In 1999 L.Kauffman [Ka] introduced the theory of virtual knots. Since then many concepts and theorems of classical knot theory were generalized to the virtual knots. In particular, the classical Thistlethwaite Theorem relating the Jones polynomial of a link with the Tutte polynomial of an appropriated graph was generalized for virtual links in [Ch09]. Recently L.Kauffman [Ka1] introduced the multi-virtual knot theory where virtual crossings appear with extra indices and outlined further generalizations of various knot theoretic concepts. The goal of this project is to try to find a multi-virtual generalization of the Thistlethwaite Theorem.

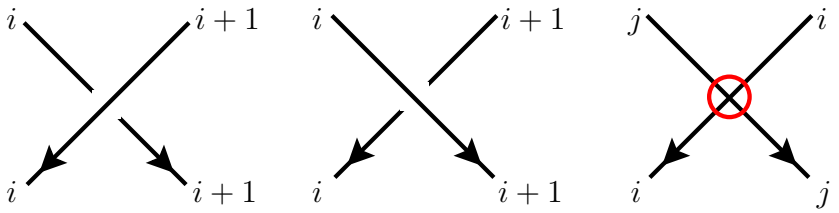
A general standard excellent introduction to knot theory is [Ad].

Project 2. ***Numerology of quasi-trees.*** (Yile Huang, Logan Keck, Rohan Mawalkar. TA: Jake Huryn)

A *quasi-tree* a ribbon graph with exactly one boundary component. This notion was introduced in [CKS]. Recently the number of quasi-tree subgraph of a given ribbon graph was related to some remarkable sequences of numbers like the Fibonacci and Lucas numbers. We plan to start with the paper [DJY] and try to generalize their results and find the relations with other remarkable sequences of numbers.

Project 3. ***Arrow polynomial of almost classical virtual links.*** (Levi Keck, Jason Tu. TA: Dennis Sweeney)

Arrow polynomial is one of the most general invariant of virtual links [DK]. There is a class of virtual links, almost classical virtual links, which behave very similarly to classical links in many aspects. For example, the arrow polynomial for them does not depend on the specific arrow variables [Kam, NNST]. An *almost classical link* is a link which admits a diagram with an *Alexander numbering* which associate an integer to each arc of the diagram according to the rule:



We are planning to find a direct proof of the theorem [Kam, NNST] and to try to generalize it. Another relevant paper is [De].

References

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[De] Q. Deng, *One conjecture on cut points of virtual links and the arrow polynomial of twisted links*, J. Knot Theory Ramifications, **31**(10)(2022) #2250066 Preprint `arXiv:2103.12283v2 [math.GT]`.

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[Kam] N. Kamada, *A multivariable polynomial invariant of virtual links and cut systems*, Topology and its Applications**301** (2021) #107518.

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[NNST] T. Nakamura, Y. Nakanishi, S. Satoh, Y. Tomiyama, *Twin groups of virtual 2-bridge knots and almost classical knots*, J. Knot Theory Ramifications, **21**(10) (2012) #120095.