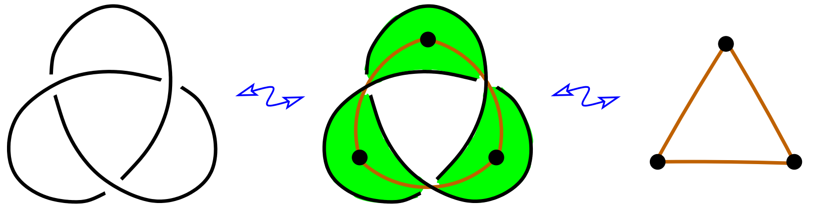
Ribbon graphs

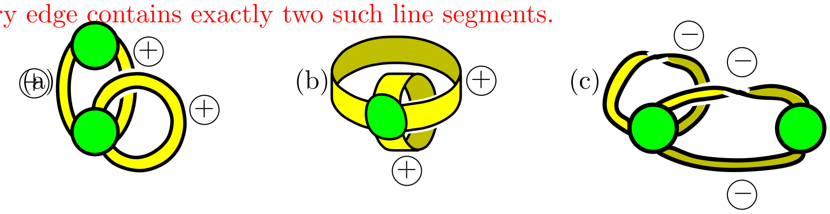
Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.*



The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; to virtual links in [ChVo, Ch]; and to the arrow polynomial in [BBC].

Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



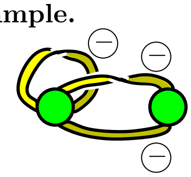
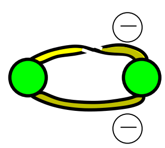
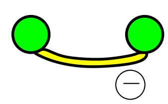
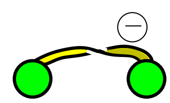
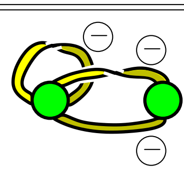
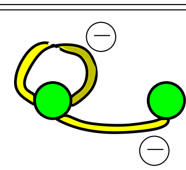
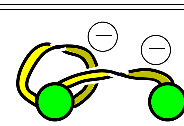
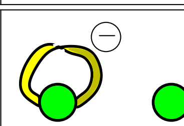
The Bollobás-Riordan polynomial

Reference: B. Bollobás and O. Riordan [BR].

$$BR_G(\{x_e, y_e\}, X, Y, Z) \; := \sum_{F\subseteq G} \Big(\prod_{e\in F} x_e\Big)\Big(\prod_{e\notin F} y_e\Big) X^{r(G)-r(F)} Y^{n(F)} Z^{k(F)-\mathrm{bc}(F)+n(F)}$$

For signed graphs, we set $\begin{cases} x_+ = 1, & x_- = (X/Y)^{1/2}, \\ y_+ = 1, & y_- = (Y/X)^{1/2}. \end{cases}$

Example.

				
(k,r,n,bc) term of R_G	(1,1,1,1) $(XY)^{1/2}Z$	(1,1,0,1) $(Y/X)^{1/2}$	(1,1,0,1) $(Y/X)^{1/2}$	(2,0,0,2) $Y^{3/2}/X^{1/2}$
$BR_G(X,Y,Z) \; = \; (Y/X)^{1/2}(XZ+2+Y$ $+X^2Z+2XZ+XYZ)$				
	(1,1,2,2) $X^{3/2}Y^{1/2}Z$	(1,1,1,1) $X^{1/2}Y^{1/2}Z$	(1,1,1,1) $X^{1/2}Y^{1/2}Z$	(2,0,1,2) $X^{1/2}Y^{3/2}Z$

Properties.

$BR_G = x_e BR_{G/e} + y_e BR_{G-e}$ if e is ordinary, that is neither a bridge nor a loop,

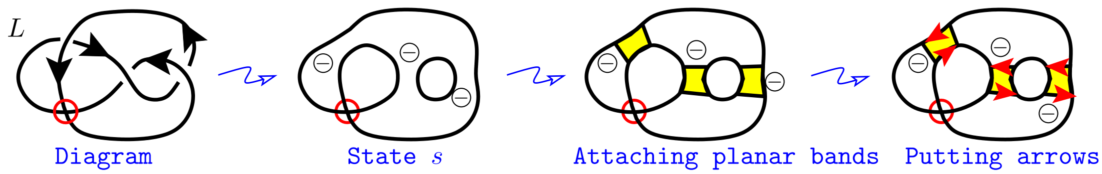
$BR_G = (x_e + Xy_e) BR_{G/e}$ if e is a bridge.

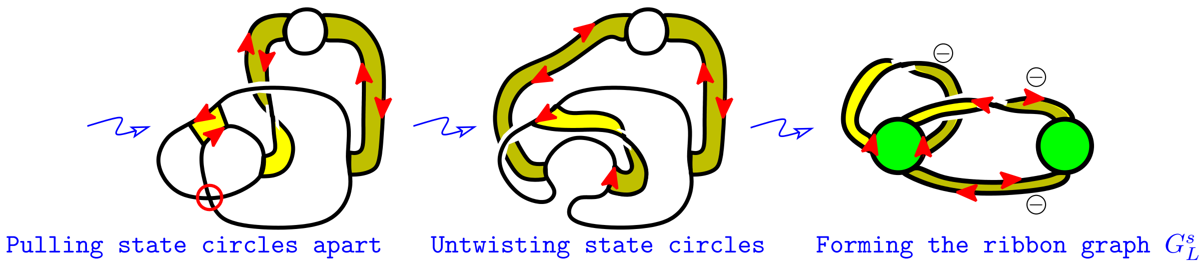
$BR_{G_1\sqcup G_2} = BR_{G_1\cdot G_2} = BR_{G_1}\cdot BR_{G_2}$

Theorem [Ch].
Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s, and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A,B,d) = A^e \left(X^k Y^v Z^{v+1} BR_{G_L^s}(X,Y,Z) \Big|_{X=\frac{Ad}{B},\ Y=\frac{Bd}{A},\ Z=\frac{1}{d}} \right) .$$

Construction of a ribbon graph from a virtual link diagram





$$\begin{aligned}
[L](A,B,d) &= A^3 \left(XY^2Z^3(Y/X)^{1/2}(XZ+2+Y+X^2Z+2XZ+XYZ) \Big|_{X=\frac{Ad}{B},\ Y=\frac{Bd}{A},\ Z=\frac{1}{d}} \right) \\
&= A^3 \cdot \tfrac{B}{A} \cdot \tfrac{B}{A} \left(\tfrac{A}{B} + 2 + \tfrac{Bd}{A} + \tfrac{A^2d}{B^2} + 2\tfrac{A}{B} + d \right) \\
&= 3A^2B + 2AB^2 + B^3d + A^3d + AB^2d
\end{aligned}$$

$$\begin{aligned}
J_L(t) &= (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}) \\
&= -t^{-3/4}\Big(3t^{-1/4} + 2t^{1/4} + t^{3/4}(-t^{1/2} - t^{-1/2}) + t^{-3/4}(-t^{1/2} - t^{-1/2}) + t^{1/4}(-t^{1/2} - t^{-1/2})\Big) \\
&= -t^{-3/4}\Big(3t^{-1/4} + 2t^{1/4} - t^{5/4} - t^{1/4} - t^{-1/4} - t^{-5/4} - t^{3/4} - t^{-1/4}\Big) \\
&= -t^{-3/4}\Big(-t^{5/4} - t^{3/4} + t^{1/4} + t^{-1/4} - t^{-5/4}\Big) = t^{1/2} + 1 - t^{-1/2} - t^{-1} + t^{-2}
\end{aligned}$$

References

[BR] B. Bollobás and O. Riordan, *A polynomial of graphs on surfaces*, Math. Ann. **323** (2002) 81–96.

[BBC] R. Bradford, C. Butler, S. Chmutov, *Arrow ribbon graphs*, Journal of Knot Theory and its Ramifications, **21**(13) (2012) # 1240002 (16 pages).

[Ch] S. Chmutov, *Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial*, Journal of Combinatorial Theory, Ser. B **99**(3) (2009) 617–638; preprint `arXiv:math.CO/0711.3490`.

[ChVo] S. Chmutov, J. Voltz, *Thistlethwaite's theorem for virtual links*. Journal of Knot Theory and Its Ramifications, **17**(10) (2008) 1189–1198; preprint `arXiv:math.GT/0704.1310`.

[DFKLS] O. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. Stoltzfus, *The Jones polynomial and graphs on surfaces*, Journal of Combinatorial Theory, Ser.B **98** (2008) 384–399; preprint `math.GT/0605571`.

[Ka1] L. H. Kauffman, *New invariants in knot theory*, Amer. Math. Monthly **95** (1988) 195–242.

[Ka2] L. H. Kauffman, *A Tutte polynomial for signed graphs*, Discrete Appl. Math. **25** (1989) 105–127.