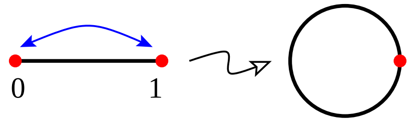
Topology of surfaces

https://math.okstate.edu/people/segerman/

Theorem. **(oriented case)** *Any closed connected orientable surface is homeomorphic to a sphere with g handles.*

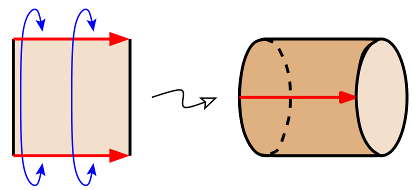
Gluing surfaces.

Circle:

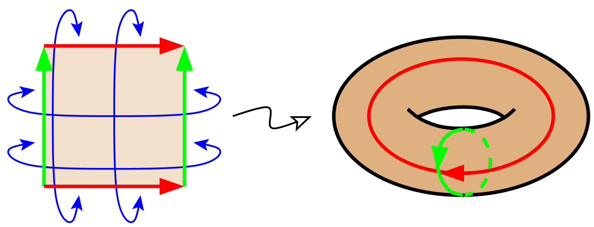


$= S^1$

Cylinder:

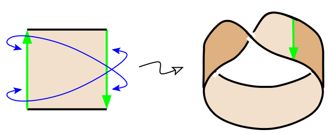


Torus:

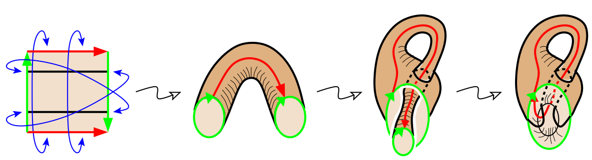


$= \mathbb{T}^2$

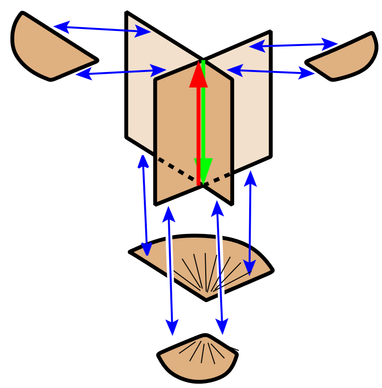
Möbius band:



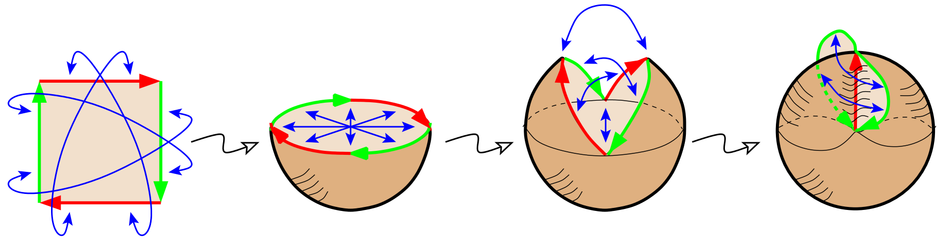
Klein bottle:

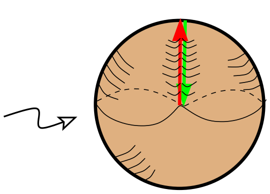


$= \mathbb{B}^2$



Projective plane:





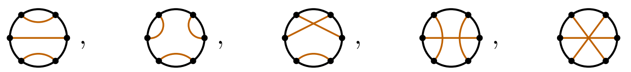
$= \mathbb{R}P^2$

Theorem. *Any connected surface is homeomorphic to a sphere with g handles and k holes or a sphere with μ Möbius bands (cross-caps) and k holes.*

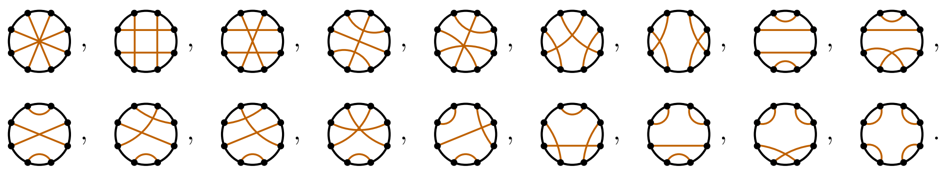
Chord diagrams

$\underline{n = 2}$

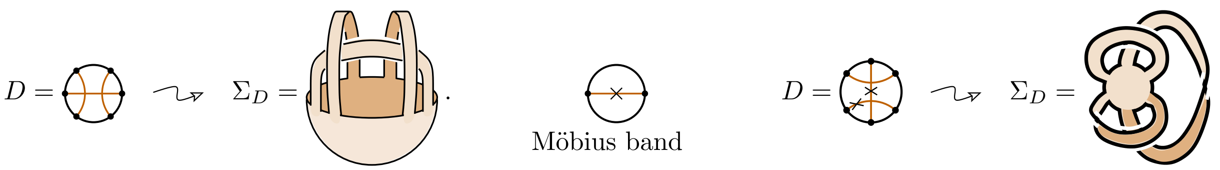
$\underline{n = 3}$



$\underline{n = 4}$



Construction of a surface (with boundary) from a chord diagram.

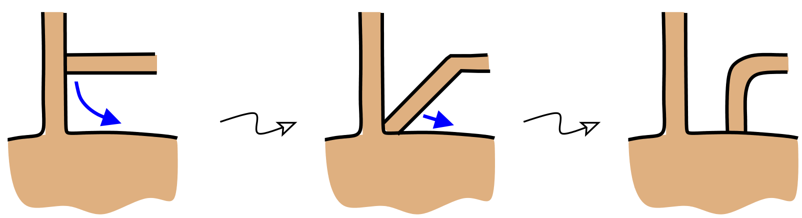


Theorem. *Any compact connected surface with boundary can be represented by a (possibly crossed) chord diagram.*

Two-term relation (2T):

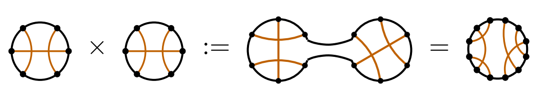
Theorem. *The surfaces represented by two chord diagrams in the 2T relation are homeomorphic.*

Proof.

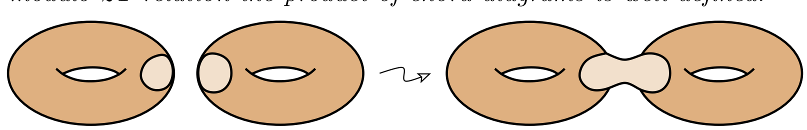


$\square$

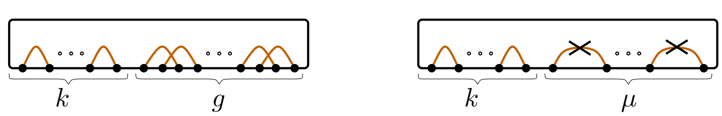
Product of chord diagrams.



Lemma. *modulo 2T relation the product of chord diagrams is well-defined.*



Caravans of camels.



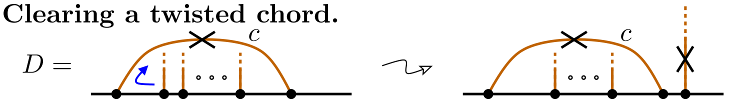
oriented caravan of
k "one-humped camels"
and g "two-humped camels"

non-oriented caravan of
k "one-humped camels"
and μ "one-humped twisted camels"

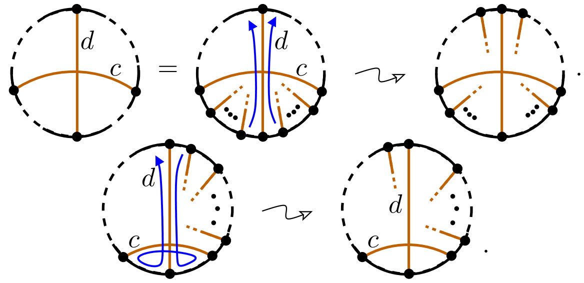
Classification Theorem.

Theorem. *Any (twisted) chord diagram is equivalent, modulo 2T relations, to a caravan.*

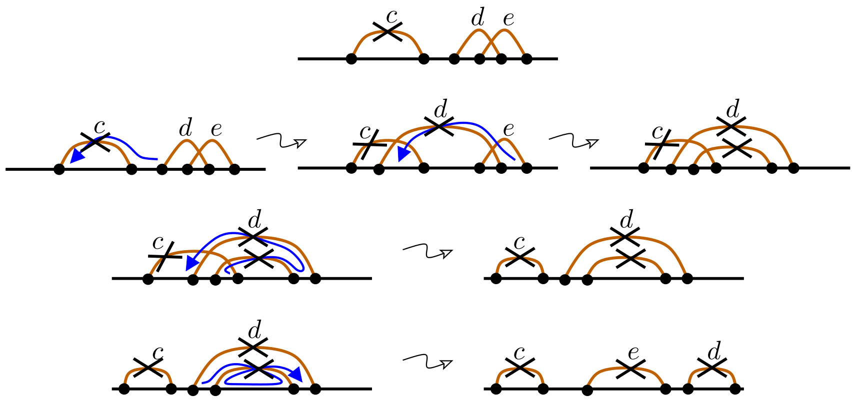
Proof. **Step1. Clearing a twisted chord.**



Step 2. Factoring two-humped camel.



Step 3. Twisted camel kills two-humped camel.

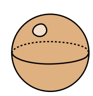


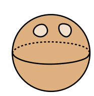
□

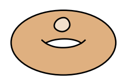
Theorem. *Any connected closed (without boundary) surface is homeomorphic to a sphere with g handles or a sphere with μ Möbius bands (cross-caps), that is either to the connected sum of g tori or the connected sum of μ copies of $\mathbb{R}P^2$.*

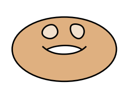
Exercises.

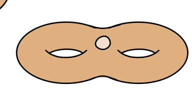
1. Draw the surfaces for all 18 chord diagrams with 4 chords.

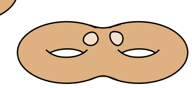
2. Represent a sphere with a hole by a chord diagram. 

3. Represent a sphere with two holes by a chord diagram. 

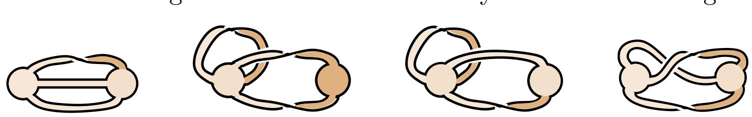
4. Represent a torus with a hole by a chord diagram. 

5. Represent a torus with two holes by a chord diagram. 

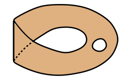
6. Represent a double torus with one hole by a chord diagram. 

7. Represent a double torus with two holes by a chord diagram. 

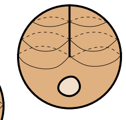
8. Represent the following non-orientable surfaces by twisted chord diagrams.



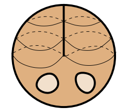
9. Represent a Möbius band with one hole by a (crossed) chord diagram.



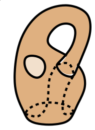
10. Represent a projective plane with a hole by a (crossed) chord diagram.



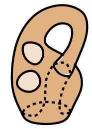
11. Represent a projective plane with two holes by a (crossed) chord diagram.



12. Represent a Klein bottle with a hole by a (crossed) chord diagram.

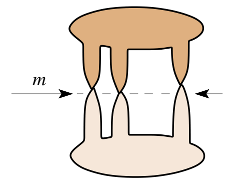


13. Represent a Klein bottle with two holes by a (crossed) chord diagram.



14. Prove the Lemma that modulo 2T relation the product of chord diagrams is well-defined.

15. Consider the surface on the right figure.
Is it oriented?
Find her genus and the number of holes.



16. Prove that the connected sum of a projective plane $\mathbb{R}P^2$ and a torus $\mathbb{T}^2$ is homeomorphic to the connected sum of a projective plane $\mathbb{R}P^2$ and a Klein bottle $\mathbb{B}^2$, and also to the connected sum of three projective planes

$$\mathbb{R}P^2 \# \mathbb{T}^2 \quad \cong \quad \mathbb{R}P^2 \# \mathbb{B}^2 \quad \cong \quad \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$$

(Hint: This is Step 3 of the proof of the Classification Theorem.)