Matroids. J. Oxley [Ox], D. J. A. Welsh [Wel], H. Whitney [Wh].

Independent sets.

A matroid is a pair $M = (E, \mathcal{I})$ consisting of a finite set E and a nonempty collection $\mathcal{I}$ of its subsets, called *independent sets*, satisfying the axioms:

(**I1**) *Any subset on an independent set is independent.*

(**I2**) *If X and Y are independent and $|X| = |Y| + 1$, then there is an element $x \in X - Y$ such that $Y \cup x$ is independent.*

Circuits.

A matroid is a pair $M = (E, \mathcal{C})$ consisting of a finite set E and a nonempty collection $\mathcal{C}$ of its subsets, called *circuits*, satisfying the axioms:

(**C1**) *No proper subset of a circuit is a circuit.*

(**C2**) *If C_1 and C_2 are distinct circuits and $c \in C_1 \cap C_2$, then $(C_1 \cup C_2) - c$ contains a circuit.*

Bases.

A matroid is a pair $M = (E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms:

(**B1**) *No proper subset of a base is a base.*

(**B2**) *If B_1 and B_2 are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.*

Rank function.

A matroid is a pair $M = (E, r)$ consisting of a finite set E and a function r, *rank*, assigning a number to a subset of E and satisfying the axioms:

(**R1**) *The rank of an empty subset is zero.*

(**R2**) *For any subset X and any element $y \notin X$,*

$$r(X \cup \{y\}) = \begin{cases} r(X)\ , & \textit{or} \\ r(X) + 1\ . \end{cases}$$

(**R3**) *For any subset X and two elements y,z not in X, if $r(X \cup y) = r(X \cup z) = r(X)$, then $r(X \cup \{y, z\}) = r(X)$.*

Properties.

- For an independent set X, the rank is equal to its cardinality, $r(X) = |X|$.
- Circuits are minimal dependent subsets.
- Circuits are the subsets X with $r(X) = |X| - 1$.
- A base is a maximal independent set.
- Rank of a subset X is equal to the cardinality of the maximal independent subset of X.
- All bases have the same cardinality which is called the *rank of matroid*, $r(M)$.
- Rank of a subset X is equal to the cardinality of the maximal independent subset of X.

Examples.

1. The ***cycle matroid*** $\mathcal{C}(G)$ of a graph G. The underlying set E is the set of edges $E(G)$. A subset $X \subset E$ is independent if and only if it does not contain any cycle of G. A base consist of edges of a spanning forest of G. The rank function is given by $r(X) := v(G) - k(X)$, where $v(G)$ is the number of vertices of G and $k(X)$ is the number of connected components of the spanning subgraph of G consisting of all the vertices of G and edges of X.

2. The ***bond matroid*** $\mathcal{B}(G)$ of a graph G. The circuits of $\mathcal{B}(G)$ are the minimal edge cuts, also known as the *bonds* of G. These are minimal collections of the edges of G which, when removed from G, increase the number of connected components. The rank $r(X)$ is equal to the maximal number of edges deletion of which do not increase the number of connected components of the spanning subgraph wich edges from X.

3. The ***uniform matroid*** $U_{k,n}$ is a matroid on an n-element set E where all subsets of cardinality $\leqslant k$ are independent. For the complete graph K_3 with three vertices, $\mathcal{C}(K_3) = U_{2,3}$. The matroid $U_{2,4}$ is not *graphical*. That is there is no any graph G such that $\mathcal{C}(G) = U_{2,4}$. It is also not *cographical*. That is there is no any graph G such that $\mathcal{B}(G) = U_{2,4}$.

4. A finite set of vectors in a vector space over a filed $\mathcal{F}$ has a natural matroid structure which is called ***representable*** (over $\mathcal{F}$). We may think about the vectors as column vectors of a matrix. The rank function is the dimension of the subspace spanned by the subset of vectors, or the rank of the corresponding submatrix. The cycle matroid $\mathcal{C}(G)$ is representable (over $\mathcal{F}_2$). The correspondent matrix is the incidence matrix of G, i.e. the matrix whose (i,j)-th entry is 1 if and only if the i-th vertex is incident to the j-th edge. The uniform matroid $U_{2,4}$ is not representable over $\mathcal{F}_2$, but it is representable over $\mathcal{F}_3$.

Dual matroids. Given any matroid M, there is a dual matroid M^* with the same underlying set and with the rank function given by $r_{M^*}(H) := |H| + r_M(M \setminus H) - r(M)$. In particular $r(M) + r(M^*) = |M|$. Any base of M^* is a complement to a base of M. The bond matroid of a graph G is dual to the cycle matroid of G: $\mathcal{B}(G) := (\mathcal{C}(G))^*$.

The Whitney planarity criteria [Wh]**.** A graph G is planar if and only if its bond matroid $\mathcal{B}(G)$ is graphical. In this case, it will be the cycle matroid of the dual graph, $\mathcal{B}(G) = (\mathcal{C}(G))^* = \mathcal{C}(G^*)$.

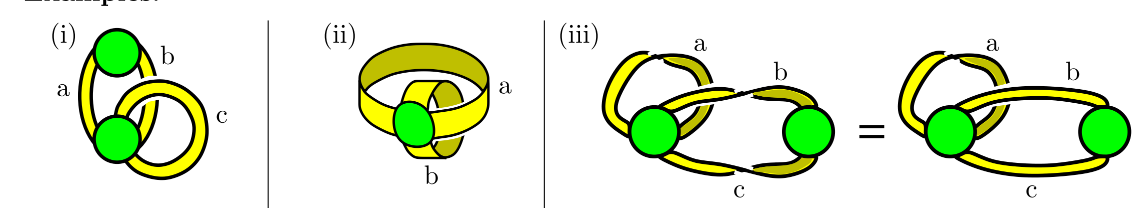
Tutte polynomial.
$$\boxed{T_M(x,y) \ := \sum_{X\subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{n(X)}}$$

<u>***Δ-matroids***</u> [Bouchet]

Matroids	Δ-matroids
A matroid is a pair $M = (E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of its subsets, called *bases*, satisfying the axioms: **(B1)** *No proper subset of a base is a base.* **(B2)** =(**Exchange axiom**) *If B_1 and B_2 are bases and $b_1 \in B_1 - B_2$, then there is an element $b_2 \in B_2 - B_1$ such that $(B_1 - b_1) \cup b_2$ is a base.*	*A Δ-matroid* is a pair $M = (E, \mathcal{F})$ consisting of a finite set E and a nonempty collection $\mathcal{F}$ of its subsets, called *feasible sets*, satisfying the **Symmetric Exchange axiom** *If F_1 and F_2 are two feasible sets and $f_1 \in F_1 \Delta F_2$, then there is an element $f_2 \in F_1 \Delta F_2$ such that $F_1 \Delta \{f_1, f_2\}$ is a feasible set.*

Definition. A *quasi-tree* is ribbon graph G with a single boundary component, $bc(G) = 1$.

Examples.



(i) Spanning quasi-trees: $\{a\}, \{b\}, \{a,b,c\}$

(ii) Spanning quasi-trees: $\emptyset, \{a,b\}$

(iii) Spanning quasi-trees: $\{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}$

Theorem. *Let $G = (V, E)$ be a ribbon graph. Then $D(G) := (E, \{$spanning quasi-trees$\})$ is a Δ-matroid.*

Minors in Δ-matroids

Let $D = (E, \mathcal{F})$ be a Δ-matroid and $e \in E$.
e is a *loop* iff $\forall F \in \mathcal{F}, e \notin F$. e is a *coloop* iff $\forall F \in \mathcal{F}, e \in F$.
If e is not a loop, $D/e := (E \setminus \{e\}, \{F \setminus \{e\} | F \in \mathcal{F}, e \in F\})$.
If e is not a coloop, $D \setminus e := (E \setminus \{e\}, \{F | F \in \mathcal{F}, F \subset E \setminus \{e\}\})$.

Twists of Δ-matroids. Let $D = (E, \mathcal{F})$ be a Δ-matroids and $A \subset E$.

$$D * A := (E, \{F \Delta A | F \in \mathcal{F}\}).$$

Dual Δ-matroid: $D^* := D * E$.

Matroids associated with a Δ-matroid

Let $D = (E, \mathcal{F})$ be a Δ-matroid.
$D_{min} := (E, \mathcal{F}_{min})$, where $\mathcal{F}_{min} := \{F \in \mathcal{F} | F$ is of minimal possible cardinality$\}$.
$D_{max} := (E, \mathcal{F}_{max})$, where $\mathcal{F}_{max} := \{F \in \mathcal{F} | F$ is of maximal possible cardinality$\}$.

Facts.

- D_{min} and D_{max} are usual matroids. *Width* $w(D) := r(D_{max}) - r(D_{min})$.
- $(D(G))_{min} = \mathcal{C}(G)$. $(D(G))_{max} = (\mathcal{C}(G^*))^*$.
- $D(G) = \mathcal{C}(G)$ iff G is a planar ribbon graph.

The twist polynomial of a delta-matroid $D = (E, \mathcal{F})$

is the generating function for the width of all twists of D,

$$^{\partial}w_D(z) := \sum_{A \subseteq E} z^{w(D*A)}$$

References

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