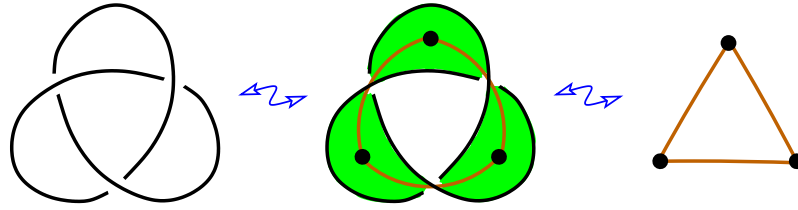


Ribbon graphs

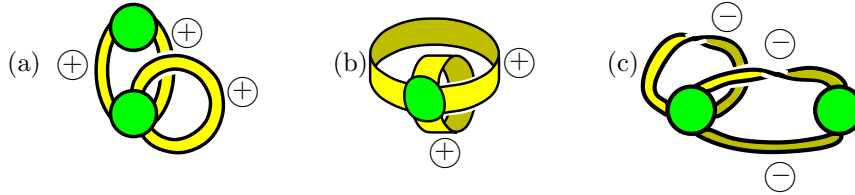
Thistlethwaite's Theorem [Ka1] *Up to a sign and multiplication by a power of t the Jones polynomial $J_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_\Gamma(-t, -t^{-1})$.*



The theorem was generalized to non-alternating links using signed graphs in [Ka2] and using the Bollobás-Riordan polynomial for ribbon graphs in [DFKLS]; to virtual links in [ChVo, Ch].

Definition. A *ribbon graph* G is a surface (possibly non-orientable) with boundary, represented as the union of two sets of closed topological discs called *vertices* $V(G)$ and *edges* $E(G)$, satisfying the following conditions:

- these vertices and edges intersect by disjoint line segments;
- each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- every edge contains exactly two such line segments.



The Bollobás-Riordan polynomial

Reference: B. Bollobás and O. Riordan [BR].

$$BR_G(\{x_e, y_e\}, X, Y, Z) := \sum_{F \subseteq G} \left(\prod_{e \in F} x_e \right) \left(\prod_{e \notin F} y_e \right) X^{r(G)-r(F)} Y^{n(F)} Z^{k(F)-bc(F)+n(F)}$$

For signed graphs, we set

$$\begin{cases} x_+ = 1, & x_- = (X/Y)^{1/2}, \\ y_+ = 1, & y_- = (Y/X)^{1/2}. \end{cases}$$

Example.

(k, r, n, bc) term of R_G	$(1, 1, 1, 1)$ $(XY)^{1/2}Z$	$(1, 1, 0, 1)$ $(Y/X)^{1/2}$	$(1, 1, 0, 1)$ $(Y/X)^{1/2}$	$(2, 0, 0, 2)$ $Y^{3/2}/X^{1/2}$
$BR_G(X, Y, Z) = (Y/X)^{1/2}(XZ + 2 + Y + X^2Z + 2XZ + XYZ)$				
	$(1, 1, 2, 2)$ $X^{3/2}Y^{1/2}Z$	$(1, 1, 1, 1)$ $X^{1/2}Y^{1/2}Z$	$(1, 1, 1, 1)$ $X^{1/2}Y^{1/2}Z$	$(2, 0, 1, 2)$ $X^{1/2}Y^{3/2}Z$

Properties.

$$BR_G = x_e BR_{G/e} + y_e BR_{G-e}$$

$$BR_G = (x_e + Xy_e) BR_{G/e}$$

$$BR_{G_1 \sqcup G_2} = BR_{G_1 \cdot G_2} = BR_{G_1} \cdot BR_{G_2}$$

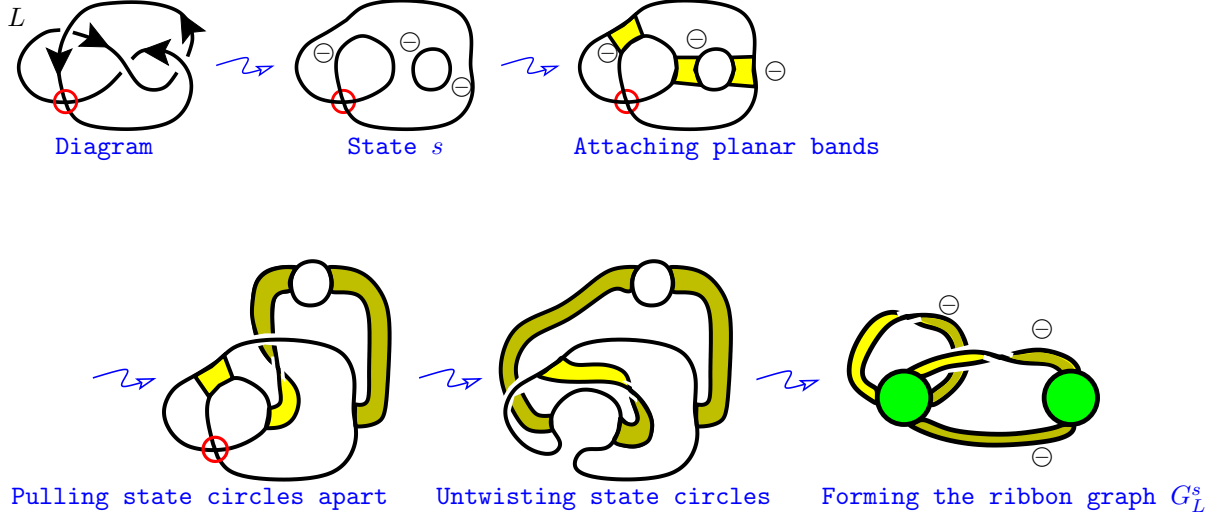
if e is ordinary, that is neither a bridge nor a loop,
if e is a bridge.

Theorem [Ch].

Let L be a virtual link diagram with e classical crossings, G_L^s be the signed ribbon graph corresponding to a state s , and $v := v(G_L^s)$, $k := k(G_L^s)$. Then $e = e(G_L^s)$ and

$$[L](A, B, d) = A^e \left(X^k Y^v Z^{v+1} B R_{G_L^s}(X, Y, Z) \right) \Big|_{X=\frac{Ad}{B}, Y=\frac{Bd}{A}, Z=\frac{1}{d}}.$$

Construction of a ribbon graph from a virtual link diagram



$$\begin{aligned} [L](A, B, d) &= A^3 \left(XY^2 Z^3 (Y/X)^{1/2} (XZ + 2 + Y + X^2 Z + 2XZ + XY Z) \right) \Big|_{X=\frac{Ad}{B}, Y=\frac{Bd}{A}, Z=\frac{1}{d}} \\ &= A^3 \cdot \frac{B}{A} \cdot \frac{B}{A} \left(\frac{A}{B} + 2 + \frac{Bd}{A} + \frac{A^2 d}{B^2} + 2\frac{A}{B} + d \right) \\ &= 3A^2 B + 2AB^2 + B^3 d + A^3 d + AB^2 d \\ J_L(t) &= (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}) \\ &= -t^{-3/4} \left(3t^{-1/4} + 2t^{1/4} + t^{3/4}(-t^{1/2} - t^{-1/2}) + t^{-3/4}(-t^{1/2} - t^{-1/2}) + t^{1/4}(-t^{1/2} - t^{-1/2}) \right) \\ &= -t^{-3/4} \left(3t^{-1/4} + 2t^{1/4} - t^{5/4} - t^{1/4} - t^{-1/4} - t^{-5/4} - t^{3/4} - t^{-1/4} \right) \\ &= -t^{-3/4} \left(-t^{5/4} - t^{3/4} + t^{1/4} + t^{-1/4} - t^{-5/4} \right) = t^{1/2} + 1 - t^{-1/2} - t^{-1} + t^{-2} \end{aligned}$$

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