Hyperbolic Equations

Transport Equation

$$u_t + u_x = 0 \quad , \quad -\infty < x < \infty, \quad 0 \leq t < \infty$$

with initial condition $u(x,0) = u_0(x)$.

The exact solution to this equation is $u(x,t) = u_0(x-t)$, that is, a solution that is transported to the right at a constant speed (which is 1 in this case).

Characteristics

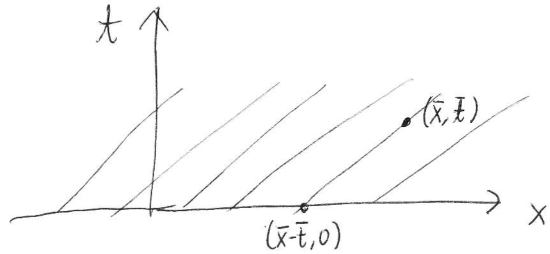
The 'wave' travels along 'characteristics', which are curves/lines on the x-t plane where the solution stays constant or increase/decrease in a way that is tractable to the initial condition.

For transport equation, let suppose the characteristics are $(x(t), t)$, namely, the curve is parametrized by t. If we monitor the change of solution u on characteristics, that is $\frac{d}{dt} u(x(t), t) = u_x \cdot x'(t) + u_t$.

Since u is a solution to the PDE, $u_x + u_t = 0$, if we take $x'(t) = 1$, then $\frac{d}{dt} u(x(t), t) = 0$.

This is saying that along the characteristics $x(t) = t + c$ the solution stays constant.

A plot of characteristics looks like this:



Therefore, to determine the solution u at $(\bar{x},\bar{t})$, one only needs to trace back the chacteristics and take the same value as the intersection of characteristics and initial condition $(t=0)$. In this case, $u(\bar{x},\bar{t}) = u(\bar{x}-\bar{t},0) = u_0(\bar{x}-\bar{t})$.

<u>Exercise</u>: Consider $u_t + u_x = t^2$, $-\infty < x < \infty$, $0 \le t < \infty$, with initial condition $u_0(x)$. Determine the solution $u(x,t)$ in terms of the initial condition.

By looking at the characteristics, we can observe that the information comes from that left and goes to the right. This information is very important in determining the legitimate boundary condition for problems with finite domain, and it is essential for the design of computational schemes.

To start the numerical method, let consider the transport equation with periodic boundary conditions, i.e.

$$u_t + u_x = 0, \quad 0 \le x \le 1, \quad 0 \le t < \infty, \quad \cdots\cdots (*)$$

and $u(0,t) = u(1,t)$, $\forall\, t \in [0,\infty)$.

Properties of Eq. 1

① $\int_0^1 u\,dx$ is constant.

② $\int_0^1 u^2\,dx$ is constant.

Proof: ② $u_t + u_x = 0$, multiply the whole equation by u, and we obtain $uu_t + uu_x = 0$. Integrate over $[0,1]$, in x,

$$\int_0^1 uu_t\,dx + \int_0^1 uu_x = 0 \Rightarrow \frac{d}{dt}\int_0^1 u^2\,dx + u^2\Big|_0^1 = 0.$$

By the periodicity, $u^2(1,t) = u^2(0,t)$ for all t.

Therefore, $\frac{d}{dt}\int_0^1 u^2\,dx = 0$. $\square$

Exercise: Prove ①.

Numerical Methods

If we consider a fully discrete method, a naive discretization is to use forward Euler in time, and central difference in space. Let denote the spatial discretization points to be $\{x_j\}$, $j = 0, 1, \cdots, M$, where $x_0 = 0$, $x_M = 1$ and $x_j - x_{j-1} = \Delta x$ for $j = 1, \cdots, M$, namely, a uniform discretization with mesh size Δx.

We denote the time steps by $\{t^n\}_{n=0}^{n=N}$, and the numerical solution that approximates $u(x_j, t^n)$ by u_j^n, $t^{n+1} - t^n = \Delta t$

The naive scheme thus can be written by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\frac{u_{j+1}^n - u_{j-1}^n}{\Delta x} \quad \cdots\cdots ②$$

<u>Exercise</u> Implement the above scheme. Describe its behavior.
Take $u_0(x) = \sin 2\pi x$.

Let's look at another method. Consider the 'upwind method'

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\frac{u_j^n - u_{j-1}^n}{\Delta x}$$

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right) \quad \cdots\cdots \quad ③$$

Intuitively, this method uses the biased stencil $]u_{j-1}, u_j[$, which is using more the information from the left. That employs the property of the equation that the information comes from the left, and that is why it is called 'upwind'.

Another method that is stable, as the upwind method, is Lax-Friedrichs method:

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right). \quad \cdots\cdots \quad ④$$

While some of the methods are stable and some of them are not, how can we know, through analysis, if they are stable or unstable? This can be done through "stability analysis".

<u>Von Neumann Analysis</u>

Suppose the solution of the numerical scheme is $u_j^n = g^n e^{ikx_j}$, if we take $\mathbf{u} = e^{ik\mathbf{x}}$ as the initial condition (note that $\mathbf{u}$ and $\mathbf{x}$ are vector notation for $\{u_j\}_{j=0}^M$ and $\{x_j\}_{j=0}^M$).

Substituting this into scheme (Eq. ②), we get

$$g^{n+1} e^{ikx_j} = g^n e^{ikx_j} - \frac{\Delta t}{2\Delta x} g^n \left(e^{ikx_{j+1}} - e^{ikx_{j-1}}\right).$$

By cancelling g^n and e^{ikx_j}, one obtains

$$g = 1 - \frac{\Delta t}{2\Delta x}\left(e^{ik\Delta x} - e^{-ik\Delta x}\right)$$

$$= 1 - \frac{\Delta t}{2\Delta x}\left(2i \sin k\Delta x\right) = 1 - i\frac{\Delta t}{\Delta x}\sin(k\Delta x). \quad \cdots ⑤$$

If we look at the form of u $(g^n e^{ikx_j})$, it can be seen easily that $\| u^n \| = \| g^n e^{ikx} \| = \| g^n \| \| e^{ikx} \| = |g|^n \| u(\cdot,0) \|$. So the magnitute of u^n changes along time, and is determined by the norm of g. If $|g| < 1$, the solution will decay to 0 $(\| u^n \| \to 0)$. If $|g| = 1$, the norm of u^n is constant, and if $|g| > 1$, $\| u^n \| \to \infty$, and the solution goes to infinity. The last case $(|g| > 1)$ corresponds to instability, in with the numerical solution blows up.

Therefore, the goal is to have $|g| \le 1$ in the designed scheme. In ⑤, $|g| = \left(1 + \left(\frac{\Delta t}{\Delta x}\sin k\Delta x\right)^2\right)^{1/2}$, so no matter how small $\frac{\Delta t}{\Delta x}$ is, $|g|$ is larger than 1. We can conclude that scheme in Eq. ② is unstable.

Let's now look at scheme in Eq ③. In upwind scheme,

$$g = 1 - \frac{\Delta t}{\Delta x}\left(-e^{-ik\Delta x} + 1\right) = 1 - \frac{\Delta t}{\Delta x}\left(1 - \cos k\Delta x + i \sin k\Delta x\right)$$

$$= 1 - \frac{\Delta t}{\Delta x}\left[1 - \left(1 - 2\sin^2\left(\tfrac{k}{2}\Delta x\right)\right) + i\, 2\sin\left(\tfrac{k}{2}\Delta x\right)\cos\left(\tfrac{k}{2}\Delta x\right)\right]$$

$$= \left(1 - \frac{\Delta t}{\Delta x} 2\sin^2\left(k\tfrac{\Delta x}{2}\right)\right) + i \cdot 2\sin\left(\tfrac{k\Delta x}{2}\right)\cos\left(\tfrac{k}{2}\Delta x\right)$$

$$= 1 - \frac{\Delta t}{\Delta x} 2\sin^2\xi + i\, 2\sin^2\xi\cos^2\xi \qquad \left(\xi = \frac{k\Delta x}{2}\right) \quad \cdots ⑥$$

Exercise Show g in Eq. ⑥ has a norm smaller (or equal) than 1 under certain constraint of $\Delta t/\Delta x$.

Exercise Perform Von Neumann analysis on Lax-Friedrichs method (Eq. ④).

Exercise Implement both upwind and Lax-Friedrichs methods, with initial condition $u(x,0) = \sin(2\pi x)$ on $x \in [0,1]$, with periodic BC. Overlay two solutions, and describe your observation.

Exercise Perform truncation error analysis on upwind and Lax-Friedrichs methods. Verify the order of accuracy obtained from your analysis by implementing both method as the previous exercise.

Von Neumann Analysis uses a Fourier analysis approach to analyze stability of the scheme. The stability can also be analyzed through looking at the 'discrete' energy that corresponds to $\int_0^1 u^2 dx$. Physically, one wants a numerical scheme to preserve or decrease energy instead of increasing the energy.

The discrete version of the energy can be written as

$$E^n = \sqrt{\sum_{j=0}^{M-1} (u_j^n)^2}.$$

Let's take upwind method as an example.

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n).$$

Multiply both sides by u_j^{n+1}, and we get

$$(u_j^{n+1})^2 = \left[u_j^n - \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)\right]\left[u_j^n - \frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)\right]$$

$$= (u_j^n)^2 - 2\frac{\Delta t}{\Delta x}\left((u_j^n)^2 - u_j^n u_{j-1}^n\right) + \left(\frac{\Delta t}{\Delta x}\right)^2 (u_j^n - u_{j-1}^n)^2$$

$$\left(\frac{\Delta t}{\Delta x} = \lambda\right)^{\to} = (u_j^n)^2 - 2\lambda (u_j^n)^2 + 2\lambda u_j^n u_{j-1}^n + \left(\frac{\Delta t}{\Delta x}\right)^2 \left((u_j^n)^2 - 2u_j^n u_{j-1}^n + (u_{j-1}^n)^2\right)$$

Sum the whole equation over j (take $\sum_{j=0}^{M-1}$)

$$\Rightarrow (E^{n+1})^2 = (E^n)^2 - 2\lambda (E^n)^2 + 2\lambda^2 (E^n)^2 + 2\lambda \sum_j u_j^n u_{j-1}^n - 2\lambda^2 \sum u_j^n u_{j-1}^n$$

$$= (E^n)^2 - 2\lambda(1-\lambda)(E^n)^2 + 2\lambda(1-\lambda) \sum_j u_j^n u_{j-1}^n$$

$$= (E^n)^2 - \underbrace{2\lambda(1-\lambda)}_{>0 \text{ if } \lambda<1} \left[(E^n)^2 - \sum_j u_j^n u_{j-1}^n\right]$$

Note that $(E^n)^2 - \sum_j u_j^n u_{j-1}^n > 0$ because $\sum_j |u_j^n u_{j-1}^n| \le \sum_j (u_j^n)^2 \sum_j (u_{j-1}^n)^2$
$= (E^n)^2$

So $(E^{n+1})^2 \le (E^n)^2 \Rightarrow E^{n+1} \le E^n$ (*decreasing energy*)

<u>Exercise</u> Implement upwind method and observe $\sum_j u_j^n$ (mass) and $\sum_j (u_j^n)^2$ (energy).

<u>Exercise</u> Perform the same analysis for Lax-Friedrichs method. Verify your analysis by implementing it.