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Piatetski-Shapiro's Work on Converse Theorems

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ABSTRACT. Converse theorems were a central feature of Piatetski-Shapiro's work on automorphic L-functions, from his first paper on the subject in 1971 to the last applications to functoriality in 2011. Converse theorems give criteria, in terms of L-functions, for a global representation of GL_n to be automorphic; if one views the representation as parametrizing an Euler product, they give analytic criteria for an Euler product to be modular. The converse theorems for GL_n all involve controlling the properties of these L-functions when twisted by cusp forms on smaller GL_m. The most basic ones require twisting by (essentially) all cuspidal representations of smaller rank groups, either rank up to $n-1$ or up to $n-2$. These are primarily spectral in nature and are those that have had the most applications. There are also those that significantly restrict the ramification of the twisting representations. These have a significant algebraic or arithmetic component (generation of congruence subgroups) but as yet have no applications that I am aware of. Then there are also the so-called "local converse theorems". We will survey what is known, what is expected, and how these have been used.

The first "converse theorem" that I am aware of is due to Hamburger. In a series of papers from 1921–1922 [**27**] Hamburger showed that the Riemann zeta function was completely characterized by its analytic properties, particularly the functional equation. Inspired by the work of Hamburger, Hecke wanted to characterize the Dedekind zeta functions of number fields in the same way. His approach was to prove his "converse theorem", characterizing the Dirichlet series coming from holomorphic modular forms of full level by their analytic properties [**29**], and to then use the structure of the space of modular forms. He was eventually successful for imaginary quadratic fields. His student Maaß introduced Maaß wave forms and proved a converse theorem for them [**49**] and was able to characterize the Dedekind zeta functions of real quadratic fields. Note that the theorems of Hecke and Maaß required only a single functional equation.

The connection of converse theorems with the theme of modularity we owe to Weil. Weil extended Hecke's theorem to holomorphic forms with level [**71**]. As a meta-application of his result, he was able to make precise a conjecture of Taniyama on the modularity of the L-functions attached to elliptic curves. With his converse theorem, Weil was able to specify the level of the associated modular form and make

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precise the ε–factor that should appear in the functional equation. In recognition of this, for many years this modularity conjecture bore the name "the Taniyama–Weil conjecture", until it was proven by Wiles, after which it became the "modularity conjecture" or "modularity theorem". In contrast to the theorems of Hecke and Maaß, Weil's converse theorem required a family of functional equations for the L-functions twisted by suitable Dirichlet characters.

In the middle of the twentieth century there was a paradigm shift in the theory of L-functions and modular forms. In 1950 Tate gave an adelic approach to the theory of the Dedekind zeta function and Dirichlet L-functions [**66**]. In the 1960's Gelfand and Piatetski-Shapiro connected the theory of modular forms of Hecke to the theory of representations of algebraic groups and the theory of automorphic representations was born. These were representations of the adelic points of algebraic groups. The work of Hecke on L-functions of modular forms, including the converse theorem as formulated by Weil, was recast in the language of automorphic representations of GL_2 by Jacquet and Langlands in 1970 [**32**]. Godement and Jacquet generalized Tate's thesis to GL_n in 1972 [**25**].

Piatetski-Shapiro's first papers on automorphic L-functions appeared in the proceedings of the 1971 Budapest conference on Lie Groups and their Representations. In this proceedings we find two papers that represent the two main themes of P-S's work for the rest of his career. The first was entitled "On the Weil-Jacquet-Langlands Theorem" in which P-S gave a converse theorem for GL_2 with a very restricted number of twists. We discuss this in Section 4 below. The second was entitled "Euler subgroups" and was his first attempt at formalizing a theory of Eulerian integral representations of L-functions.

Integral representations and converse theorems for L-functions were a constant theme in P-S's work. Much of what we know about converse theorems for GL_n was already developed by P-S before he left the Soviet Union. These ideas can be found in the two Maryland preprints the he wrote upon his arrival in the US in 1975/76 [**56, 57**]. These were developed in the context of global fields of characteristic $p > 0$. The development of these theorems for number fields would have to wait on the work of and with Jacquet and Shalika during their fruitful collaboration from 1975–1983. It is primarily the number field formulations that we deal with below.

P-S viewed the converse theorems as a vehicle for establishing Langlands' functoriality conjecture, which Langlands had formulated in 1969 [**44**]. As he was involved in the understanding of L-functions through integral representations, he often thought of how to formulate these theorems to be compatible with this theory. On the other hand he was well aware of and kept abreast of the work of Shahidi on the Langlands-Shahidi method of understanding L-functions through the Fourier coefficients of Eisenstein series [**65**]. He would push practitioners of both methods and others to think in terms of results that would be needed to apply the converse theorem to the problem of functoriality, such as the stability of local γ-factors under highly ramified twists. As it turned out, it was the Langlands-Shahidi method that came to fruition first and enabled P-S to realize his goal of establishing cases of functoriality via the converse theorem in a series of papers with Kim and Shahidi and I and then Shahidi and I in the 2000's [**16, 17, 22**]. In addition, his old converse theorems for function fields played an important role in Lafforgue's proof of the global Langlands correspondence in characteristic $p > 0$ [**43**].

While the applications to functoriality coming from the combination of the converse theorems with the Langlands-Shahidi method is probably coming to an end, at least with the current state of the converse theorems, the potential application of the converse theorems to other cases of functoriality is very much alive. There are only a few cases of functoriality that have been established by combining the converse theorem with the theory of integral representations, such as [**60**]. However, the converse theorem is very flexible and there is hope that we can even make it more flexible in the future. As the theory of L-functions via integral representations progresses, the converse theorems waits in the wings. As soon as we understand enough about twisted L-functions, the converse theorem will give us functoriality to GL_n.

I would like to thank the referee for helping me to improve the exposition of this article and for pointing out some mathematical inaccuracies in the original manuscript.

The principal papers of Piatetski-Shapiro on Converse Theorems

The following papers deal with the converse theorems per se, not with the myriad applications. They appear in chronological order.

1971. I.I. Piatetski-Shapiro, *On the Weil-Jacquet-Langlands theorem.* Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), pp. 583–595. Halsted, New York, 1975.
1975. I.I. Piatetski-Shapiro, *ConverseTheorem for* $GL(3)$. University of Maryland Lecture Note # 15, 1975.
1976. I.I. Piatetski-Shapiro, *Zeta-functions of* $GL(n)$. University of Maryland Preprint MD76-80-PS, 1976.
1979. H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on GL(3).* I & II. Ann. of Math. (2) **109** (1979), no. 1, 169–258.
1990. I.I. Piatetski-Shapiro, *The converse theorem for* $GL(n)$. Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989), 185–195. Israel Math. Conf. Proc., 3, Weizmann, Jerusalem, 1990.
1994. J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for* GL_n. Publ. Math. IHES **79** (1994), 157–214.
1996. J.W. Cogdell and I.I. Piatetski-Shapiro, *A converse theorem for* GL_4. Mathematical Research Letters **3** (1996), 1–10.
1999. J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for* GL_n, II. J. reine angew. Math. **507** (1999), 165–188.
2001. I.I. Piatetski-Shapiro, *Two Conjectures on L-functions.* Wolf Prize in Mathematics, Volume 2, 519–522. World Scientific Press, Singapore, 2001.

1. L-functions for $GL_n \times GL_m$ with $m < n$

Before we can discuss converse theorems we need to understand the integral representations they invert. For more details on what follows in this section and the subsequent sections, one can either consult the original papers referred to in the text or to one of the more detailed surveys [**13, 14**].

We will need the following subgroups of GL_n.

$$GL_n \supset P_n = \left\{ \begin{pmatrix} & & & * \\ & * & & * \\ & & & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\} \supset N_n = \left\{ \begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix} \right\}$$

$$\supset Y_m = \left\{ \begin{pmatrix} I_{m+1} & * \\ 0 & z \end{pmatrix} \mid z \in N_{n-m-1} \right\}$$

where P_n is the mirabolic subgroup of GL_n, which is the stabilizer of the vector $(0, \ldots, 0, 1)$ in affine space, N_n is the standard maximal unipotent subgroup.

Let k be a global field, $\mathbb{A}$ its ring of adeles, and let $\psi : k\backslash\mathbb{A} \to \mathbb{C}^1$ a non-trivial additive character. ψ then defines a character of N_n, and so its subgroup Y_m, by

$$\psi(n) = \psi\left(\sum_i n_{i,i+1}\right).$$

Let $\pi \simeq \otimes'\pi_v$ be a (irreducible) cuspidal automorphic representation of $GL_n(\mathbb{A})$ and $\pi' \simeq \otimes'\pi'_v$ a (irreducible) cuspidal representation of $GL_m(\mathbb{A})$. If $\varphi \in V_\pi$, for $p \in P_{m+1}(\mathbb{A})$ we set

$$\mathbb{P}_m\varphi(p) = |\det(p)|^{-\frac{n-m-1}{2}} \int_{Y_m(k)\backslash Y_m(\mathbb{A})} \varphi\left(y\begin{pmatrix} p & \\ & I_{n-m-1}\end{pmatrix}\right)\psi^{-1}(y)\, dy.$$

This integral is absolutely convergent and defines a rapidly decreasing cuspidal automorphic function on $P_{m+1}(\mathbb{A}) \subset GL_{m+1}(\mathbb{A})$.

Now, given $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ we set (for k a number field)

$$(1.1) \qquad I(s,\varphi,\varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}_m\varphi\begin{pmatrix} h & \\ & 1\end{pmatrix}\varphi'(h)|\det(h)|^{s-\frac{1}{2}}\, dh.$$

These global integrals have *nice* analytic properties, that is,

(1) they are absolutely convergent
(2) they are bounded in vertical strips of finite width
(3) they satisfy a functional equation

$$I(s,\varphi,\varphi') = \widetilde{I}(1-s,\varphi^\vee,\varphi'^{,\vee})$$

where $\varphi^\vee(g) = \varphi({}^tg^{-1})$.

If we substitute the Fourier expansion for $\mathbb{P}_m\varphi$ and unfold we obtain, for $Re(s) >> 0$,

$$(1.2) \quad I(s,\varphi,\varphi') = \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-\frac{n-m-1}{2}}\, dh$$

with $W_\varphi \in \mathcal{W}(\pi,\psi)$ and $W'_{\varphi'} \in \mathcal{W}(\pi',\psi^{-1})$ the associated Whittaker functions from the Fourier expansions. By the uniqueness of local, and hence global, Whittaker models, for factorizable φ and φ' this last integral is Eulerian and can be written

$$I(s,\varphi,\varphi') = \prod_v I_v(s,W_v,W'_v)$$

where the local integrals are the local version of the global integral in (1.1),

(1.3)

$$I_v(s,W_v,W'_v) = \int_{N_m(k_v)\backslash GL_m(k_v)} W_v\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W'_v(h)|\det(h)|^{s-\frac{n-m-1}{2}}\, dh.$$

At the finite places where the representations are unramified and W_v and W'_v are the normalized K_v–fixed vectors, the local integral computes the local L-function exactly, i.e.,

$$I_v(W_v^\circ, W_v'^{,\circ}) = L(s, \pi_v \times \pi'_v)$$

and for the remaining places S the ratio of the local integral to the local L-function is entire. So we have

$$I(s, \varphi, \varphi') = \left(\prod_{v \in S} \frac{I_v(s, W_v, W'_v)}{L(s, \pi_v \times \pi'_v)} \right) \cdot L(s, \pi \times \pi')$$

where the global L-function is given by the Euler product

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v).$$

If one now combines

- the nice analytic properties of the global integrals given above
- the unramified calculation for $v \notin S$ as quoted above
- the local theory of L-functions for $v \in S$

then one has the following theorem, essentially due to Jacquet, Piatetski-Shapiro, and Shalika [**20**].

Theorem 1.1. *Let k be a number field. Then $L(s, \pi \times \pi')$ is nice, that is*

(1) *it has an entire continuation to all of the complex plane*
(2) *this continuation is bounded in vertical strips of finite width*
(3) *it satisfies the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

In the case of a global function field of a curve over a finite field, the continuation and boundedness statements are replaced by rationality statements as functions of q^{-s} where q is the cardinality of the field of constants [**57**].

2. Converse Theorems

The converse theorem now inverts this process. We can pose the question as follows.

If $\pi = \otimes' \pi_v$ is an irreducible admissible representation of $GL_n(\mathbb{A})$, then, by the local theory alluded to above, π encodes an Euler product of degree n by

$$L(s, \pi) = \prod_v L(s, \pi_v).$$

We must assume, as a hypothesis of any converse theorem, that this Euler product converges in some right half-plane $Re(s) >> 0$. We also must require a modicum of automorphy for π, namely that the central character ω_π is already automorphic, that is, is an idele class character.

For a converse theorem we will need to consider twisted L-functions, following the paradigm of Weil, and so we need a twisting set $\mathcal{T}$. Let

$$\mathcal{T}(m) = \coprod_{1 \le d \le m} \{\pi' \mid \pi' \text{ is a cuspidal automorphic representation of } GL_d(\mathbb{A})\}$$

and take $\mathcal{T} \subset \mathcal{T}(m)$ for some m. Then we have a family of twisted L-functions as well

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v) \quad \text{for} \quad \pi' \in \mathcal{T}$$

which still converges for $Re(s) >> 0$. Note that while π is only irreducible admissible, all the twisting representations π' are automorphic, and even cuspidal.

Question: Suppose $L(s, \pi \times \pi')$ is *nice* for all $\pi' \in \mathcal{T}$, where as above *nice* means: entire continuation, bounded in vertical strips of the continuation, and satisfies the functional equation as in the above theorem. What can we conclude about π? Is π automorphic? Is π cuspidal? Is it automorphic up to a finite number of Euler factors, i.e., if we change π at a finite number of places?

Piatetski-Shapiro's primary work on converse theorems fall into three broad types.

1. Limiting the rank of the twists (spectral methods).
2. Limiting the ramification of the twists (generation of congruence subgroups).
3. Speculation on GL_1 twists.

He did consider other variants of the converse theorem, including converse theorems with poles and local converse theorems, which we will also discuss briefly.

3. Limiting the rank of the twists

The most basic converse theorem is the following. The notation and assumptions are as in the previous section.

Theorem 3.1. *Suppose $L(s, \pi \times \pi')$ are nice for all $\pi' \in \mathcal{T}(n-1)$. Then π is a cuspidal automorphic representation of $GL_n(\mathbb{A})$.*

This theorem is proven by a simple spectral inversion [**18**]. We can reduce to the case of generic π. Then for every vector $\xi \in V_\pi$ we can associate a function on the group, namely its Whittaker function $W_\xi(g) \in \mathcal{W}(\pi, \psi)$ for ψ a fixed non-trivial additive character of $k\backslash\mathbb{A}$. This is a function on $GL_n(\mathbb{A})$ which is left invariant under the rational points of the maximal unipotent subgroup $N_n(k)$. One first averages this as much as possible towards making it $GL_n(k)$ invariant. To this end one forms

$$U_\xi(g) = \sum_{p \in N_n(k)\backslash P_n(k)} W_\xi(pg) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right). \tag{3.1}$$

With a bit of work, one shows that this converges absolutely and uniformly for g in a compact subset, is left invariant under $P_n(k)$ and its restriction to $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$ is cuspidal and rapidly decreasing (modulo the center). Note that if $\xi = \varphi$ was a cusp form, this would be its Fourier expansion.

One can do the same with the opposite mirabolic, namely if

$$Q_n = Stab_{GL_n} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and we let α denote the permutation matrix $\alpha = \begin{pmatrix} & 1 \\ I_{n-1} & \end{pmatrix}$, then we can form

$$V_\xi(g) = \sum_{q \in N'_n(k) \backslash Q_n(k)} W_\xi(\alpha q g)$$

where $N'_n = \alpha^{-1} N_n \alpha$. This converges in the same way U_ξ does and is left invariant under $Q_n(k)$

Now neither U_ξ nor V_ξ are obviously automorphic, but since together $P_n(k)$ and $Q_n(k)$ generate $GL_n(k)$, it would suffice to show $U_\xi = V_\xi$ to obtain the automorphy of both, and this is what is done via spectral inversion.

We take $(\pi', V_{\pi'})$ to be any irreducible subrepresentation of the space of automorphic forms on $GL_{n-1}(\mathbb{A})$. We first insert U_ξ into our global integral (1.1) to form $I(s, U_\xi, \varphi')$ for each $\varphi' \in V_{\pi'}$. This now converges for $Re(s) >> 0$, for all s if π' is cuspidal, and if we unfold it as above we still have

$$I(s, U_\xi, \varphi') = \prod_v I_v(s, W_v, W'_v).$$

If we first assume that π' is cuspidal, then we can write this as

$$I(s, U_\xi, \varphi') = \left(\prod_{v \in S} \frac{I_v(s, W_v, W'_v)}{L(s, \pi_v \times \pi'_v)} \right) L(s, \pi \times \pi') = e(s) L(s, \pi \times \pi')$$

with $e(s)$ entire. Since $L(s, \pi \times \pi')$ is assumed to be nice, this gives an entire continuation of $I(s, U_\xi, \varphi')$ which is bounded in vertical strips. When π' is not cuspidal, it can be written as a subrepresentation of a representation which is induced from cuspidal representations σ_i of $GL_{m_i}(\mathbb{A})$ for various $m_i < n-1$. In this case we have

$$I(s, U_\xi, \varphi') = e(s) \prod_i L(s, \pi \times \sigma_i)$$

and again using our assumption that all the $L(s, \pi \times \sigma_i)$ are nice we see that $I(s, U_\xi, \varphi')$ extends to an entire function of s, bounded in vertical strips.

One next repeats these steps with V_ξ in place of U_ξ. The main differences are that now the global integrals $I(s, V_\xi, \varphi')$ converge for $Re(s) << 0$, i.e., in a left half-plane, and that in this half-plane they unfold to

$$I(s, V_\xi, \varphi') = \prod_v I_v(1-s, \rho(w_{n,n-1}) \widetilde{W}_v, \widetilde{W}'_v)$$

which, in the case of π' cuspidal is

$$I(s, V_\xi, \varphi') = \tilde{e}(1-s) L(1-s, \tilde{\pi} \times \tilde{\pi}')$$

and otherwise is related to $\prod L(1-s, \tilde{\pi} \times \tilde{\sigma}_i)$. In either case, the assumption of the converse theorem gives that these integrals also extend to entire functions of s, bounded in vertical strips.

Finally, combining the global functional equations, as assumed in the converse theorem, and the local functional equations at those $v \in S$, we obtain the equality

$$I(s, U_\xi, \varphi') = I(s, V_\xi, \varphi')$$

for all φ' lying in irreducible subrepresentations of the space of automorphic forms on $GL_{n-1}(\mathbb{A})$. To conclude we first use Mellin inversion in the s variable, as formulated in Lemma 11.3.1 of [**32**], to obtain

$$\int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbb{A})} U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h)\ dh = \int_{SL_{n-1}(k)\backslash SL_{n-1}(\mathbb{A})} V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h)\ dh$$

and then using the weak form of Langlands spectral theory for SL_{n-1} [**45, 51**] we arrive at

$$U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$$

for $h \in SL_{n-1}(\mathbb{A})$. We can specialize this to $h = I_{n-1}$, which seems to loose information, but the apply it to $\pi(g)\xi$ for all $g \in GL_n(\mathbb{A})$ to obtain $U_\xi(g) = V_\xi(g)$ for all $g \in GL_n(\mathbb{A})$. As noted above this implies that U_ξ is automorphic on GL_n, and then the cuspidality follows from the expansion (3.1).

This is the basic outline for all converse theorems. A useful variant of this theorem is the following [**33, 16**].

Theorem 3.2. *Let S be a finite set of finite places and consider the twisting set*

$$\mathcal{T}^S(n-1) = \{\pi' \in \mathcal{T}(n-1) \mid \pi'_v \text{ is unramified for all } v \in S\}$$

Suppose the $L(s, \pi \times \pi')$ are nice for all $\pi' \in \mathcal{T}^S(n-1)$. Then π is a quasi-automorphic representation of $GL_n(\mathbb{A})$, that is, there is an automorphic representation Π of $GL_n(\mathbb{A})$ such that $\pi_v \simeq \Pi_v$ for all $v \notin S$.

The proof of this theorem follows the outline above, so spectral inversion, combined with the weak approximation theorem. However, to compensate for the slight ramification restriction on the twists at the places in S, we need to use the theory of the conductor [**35**]. For $v \in S$, let $\xi_v^\circ \in V_{\pi_v}$ be the new vector (or the essential vector). Let

$$K_0(\mathfrak{p}_v^{n_v}) = \left\{ g_v \in GL_n(\mathfrak{o}_v) \mid g_v \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \dots & 0 & * \end{pmatrix} \pmod{\mathfrak{p}_v^{n_v}} \right\}$$

and

$$K_1(\mathfrak{p}_v^{n_v}) = \{ g \in K_0(\mathfrak{p}_v^{n_v}) \mid g_{n,n} = 1 \}.$$

When n_v is the local conductor of π_v as in [**35**], the space of vectors in V_{π_v} fixed by $K_1(\mathfrak{P}_v^{n_v})$ is one dimensional and we can take ξ_v° to be a non-zero vector in this span, usually normalized so that $W_{\xi_v^\circ}(e) = 1$. Since

$$K_1(\mathfrak{p}_v^{n_v}) \supset \begin{pmatrix} GL_{n-1}(\mathfrak{o}_v) & \\ & 1 \end{pmatrix},$$

the vector ξ_v° is unramified as far as $GL_{n-1}(k_v)$ is concerned. Let $K_{0,S}(\mathfrak{n}) = \prod_{v \in S} K_0(\mathfrak{p}_v^{n_v})$.

We now for U_ξ and V_ξ as above, but for ξ such that $\xi_v = \xi_v^\circ$ for all $v \in S$. Since the ξ_v° are unramified as far as $GL_{n-1}(k_v)$ is concerned, the twisting set in Theorem 3.2 is sufficient for applying Mellin inversion and spectral inversion to conclude

$$U_\xi(g) = V_\xi(g) \quad \text{for} \quad g \in G' = K_{0,S}(\mathfrak{n}) G^S$$

where, as usual, $G^S = \prod'_{v \notin S} GL_n(k_v)$. This function will be left invariant under the group generated by $P_n(k) \cap G'$ and $Q_n(k) \cap G'$, which is $GL_n(k) \cap G'$. So the map $\xi \mapsto U_\xi$ embeds $\pi^S = \otimes'_{v \notin S} \pi_v$ into a space of automorphic forms on G' with respect to $GL_n(k) \cap G'$. Since $GL_n(k)G' = GL_n(\mathbb{A})$ by weak approximation, then this space of automorphic forms will determine an automorphic representation Π of $GL_n(\mathbb{A})$ with the property that $\Pi_v \simeq \pi_v$ for $v \notin S$. This is the Π of the theorem.

Note that we lose control of the local factors of π for those places where we do not twist by every representation and we lose cuspidality. In fact, this loss of cuspidality is quite important for applications to functoriality, since it lets us obtain lifts of generic cuspidal representations of classical groups even when the image representation is not cuspidal on GL_n.

We also have versions of the above theorems when $n \geq 3$ and we allow twists only up to rank $n-2$ [**33, 19**].

THEOREM 3.3. *Let S be a finite set of finite places and consider the twisting set*

$$\mathcal{T}^S(n-2) = \{\pi' \in \mathcal{T}(n-2) \mid \pi'_v \text{ is unramified for all } v \in S\}$$

Suppose $L(s, \pi \times \pi')$ are nice for all $\pi' \in \mathcal{T}^S(n-2)$. If $S = \emptyset$ then π is cuspidal automorphic. If $S \neq \emptyset$, then π is a quasi-automorphic representation of $GL_n(\mathbb{A})$, that is, there is an automorphic representation Π of $GL_n(\mathbb{A})$ such that $\pi_v \simeq \Pi_v$ for all $v \notin S$.

If we follow the proof of the basic converse theorem outlined above, the Mellin and spectral inversion will give us

$$\mathbb{P}_{n-2} U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \mathbb{P}_{n-2} V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$$

for $h \in SL_{n-2}(\mathbb{A})$. The projection $\mathbb{P}_{n-2}$ involves an integration against a character over the abelian group

$$Y_{n-2} = \left\{ \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \right\}$$

with $u \in \mathbb{A}^{n-1}$. To obtain automorphy we must use the natural invariance properties of $\mathbb{P}_{n-2} U_\xi$ and $\mathbb{P}_{n-2} V_\xi$ as well as a further Fourier inversion along the group Y_{n-2}. From the invariance of $\mathbb{P}_{n-2} U_\xi$ and $\mathbb{P}_{n-2} V_\xi$ one shows that most of the Fourier coefficients along Y_{n-2} are equal. P-S then utilized a very clever local construction to make the remaining Fourier coefficients equal to 0, and then worked around this local restriction via the weak approximation theorem. This then yields Theorem 3.3 in the case $S = \emptyset$. When S is not empty, one again uses the theory of the conductor and weak approximation as indicated above.

P-S brought these results with him from the Soviet Union, at least in the case of k a global function field. These can be found in his 1975/76 University of Maryland preprints [**56, 57**]. The proofs in the number fields case is similar, but we needed to wait on the work of Jacquet and Shalika on the local archimedean theory of integral representations of $GL_n \times GL_m$ [**37**].

One can try to prove a converse theorem using twists of rank up to m with $m < n-2$. The method of Mellin and spectral inversion works exactly as before and one arrives at

$$\mathbb{P}_m U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \mathbb{P}_m V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$$

for $h \in SL_{m-1}(\mathbb{A})$. However, the projection operator $\mathbb{P}_m$ now involves integration against a character over the *non-abelian* subgroup Y_m. To proceed one would have to work out an additional Fourier inversion for this non-abelian group. This is something we never figured out how to do (although see the work of Nien mentioned in Section 7 below).

The conjectured best result that one could hope to prove along these lines is the following conjecture attributed to Jacquet.

CONJECTURE 3.4 (Jacquet). Let S be a finite set of finite places and consider the twisting set

$$\mathcal{T}^S\left(\left[\frac{n}{2}\right]\right) = \left\{\pi' \in \mathcal{T}\left(\left[\frac{n}{2}\right]\right) \mid \pi'_v \text{ is unramified for all } v \in S\right\}$$

Suppose $L(s, \pi \times \pi')$ are nice for all $\pi' \in \mathcal{T}^S\left(\left[\frac{n}{2}\right]\right)$. If $S = \emptyset$ then π is cuspidal automorphic. If $S \neq \emptyset$, then π is a quasi-automorphic representation of $GL_n(\mathbb{A})$, that is, there is an automorphic representation Π of $GL_n(\mathbb{A})$ such that $\pi_v \simeq \Pi_v$ for all $v \notin S$.

The $\left[\frac{n}{2}\right]$ bound on the twists is natural since these are the smallest rank twists for which one can test for cuspidality using the entirity of the twisted L-functions.

Surprisingly, most of the applications of the converse theorem to functoriality can be obtained from the most basic converse theorem with $\mathcal{T} = \mathcal{T}^S(n-1)$. These include:

(i) The recursion of Deligne and Piatetski-Shapiro used by Lafforgue in his proof of the global Langlands conjecture for function fields [**47, 43**]
(ii) Functoriality for generic cuspidal representations of classical groups [**16, 17, 42, 39**] as well as the *GSpin* groups [**2**]
(iii) The work of Kim and Shahidi on the tensor lifting of $GL_2 \times GL_3$ to GL_6 [**40**] and the exterior square lifting from GL_4 to GL_6 [**38**] yielding automorphy of the Sym^3 and Sym^4 liftings for GL_2 and the best bounds towards the general Ramanujan conjecture for GL_2 [**41**].

The version of the converse theorem with $\mathcal{T} = \mathcal{T}^S(n-2)$ was needed in the following applications:

(iv) Automorphy of monomial representations of GL_3 [**33**] and non-normal cubic base change for GL_2 obtained by Jacquet, Piatetski-Shapiro and Shalika [**36**], used by Langlands and Tunnel in their work on the automorphy of tetrahedral and octahedral Galois representations [**46, 69**]
(v) The symmetric square lift from GL_2 to GL_3 by Gelbart and Jacquet [**24**]
(vi) The tensor lifting from $GL_2 \times GL_2$ to GL_4 by Ramakrishnan [**64**]
(v) The non-generic lifting from GSp_4 to GL_4 by Pitale, Saha, and Schmidt [**60**].

4. Limiting the ramification of the twists

In this family of converse theorems, we place severe restrictions on the ramification of the twists used. We now fix S a finite set of places and require $S \supset S_\infty$, i.e., that S contains all archimedean places. We then let

$$\mathcal{T}_S(m) = \{\pi' \in \mathcal{T}(m) \mid \pi'_v \text{ is unramified for all } v \notin S\}$$

so now we consider twists that are unramified *outside* a finite set of places of k.

The most general theorem we have of this type is the following [**18**].

THEOREM 4.1. *Let $n \geq 3$. Suppose S is a finite set of places as above such that the ring $\mathfrak{o}_S$ of S-integers of k has class number one. Suppose $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_S(n-1)$. Then π is quasi-automorphic in the sense that there is an automorphic representation Π of $GL_n(\mathbb{A})$ such that $\pi_v \simeq \Pi_v$ for all $v \notin S$ for which π_v is unramified.*

Note that for fields with class number 1, such as the field of rational numbers $\mathbb{Q}$, this gives a converse theorem for GL_n with *unramified* twists. There are ideas on how to remove the class number restriction in general, but these have not been implemented as of yet.

The proof of this theorem begins as all converse theorems do with a use of Mellin and spectral inversion. The restriction on the ramification of the twists is compensated for by the use of

(i) The theory of the conductor of representations of $GL_n(k_v)$ by Jacquet, Piatetski-Shapiro and Shalika [**35**]
(ii) Facts about the generation of congruence subgroups in $G_S = \prod_{v \in S} GL_n(k_v)$ [**3, 4**]

The latter can be viewed as additional arithmetic input to our spectral approach.

We begin with the use of the theory of the conductor. For $v \notin S$, let $\xi_v^\circ \in V_{\pi_v}$ be the new vector (or the essential vector) fixed by $K_1(\mathfrak{p}_v^{n_v})$ where n_v is the local conductor of π_v as in [**35**]. We view $\mathfrak{n} = \prod_{n \notin S} \mathfrak{p}_v^{n_v} \subset \mathfrak{o}_S$ as the S-conductor of π. Let $\xi^\circ = \prod_{v \notin S} \xi_v^\circ \in V_{\pi^S}$ be fixed.

For each $\xi_S \in \prod_{v \in S} V_{\pi_v} = V_{\pi_S}$ we can form $U_{\xi_S \otimes \xi^\circ}$ and $V_{\xi_S \otimes \xi^\circ}$ as in the proof of the basic converse theorem. Since the vector ξ° is unramified as far as $GL_{n-1}(\mathbb{A}^S)$ is concerned, using just the unramified twists in the hypothesis of the converse theorem, we can apply Mellin and spectral inversion to obtain

$$U_{\xi_S \otimes \xi^\circ} \begin{pmatrix} h & \\ & 1 \end{pmatrix} = V_{\xi_S \otimes \xi^\circ} \begin{pmatrix} h & \\ & 1 \end{pmatrix}$$

but now only for $h \in SL_{n-1}(\mathbb{A}_S) K_{n-1}^S(\mathfrak{o})$, and as above this can be extended to $U_{\xi_S \otimes \xi^\circ}(g) = V_{\xi_S \otimes \xi^\circ}(g)$ for $g \in G_S K_0^S(\mathfrak{n})$, where

$$K_0^S(\mathfrak{n}) = \prod_{v \notin S} K_0(\mathfrak{p}_v^{n_v}) \subset G^S.$$

We now use the adelic–classical correspondence to view these functions as classical functions on G_S, which we denote U_{ξ_S} and V_{ξ_S}. In terms of the mirabolic subgroups with respect to which U and V were formed, we now have

- U_{ξ_S} is left invariant under $P_n(\mathfrak{o}_S)$
- V_{ξ_S} is left invariant under $Q_n(\mathfrak{n})$

where we now view $\mathfrak{n} \subset \mathfrak{o}_S \subset k_S = \prod_{v \in S} k_v$ and then $P_n(\mathfrak{o}_S)$ and $Q_n(\mathfrak{n})$ are the projections of $GL_n(k) \cap G_S K_0^S(\mathfrak{n})$ and $GL_n(k) \cap G_S K_0^S(\mathfrak{n})$ to G_S.

We now need our arithmetic input, which is a consequence of stable algebra for GL_n due to Bass [**3**], namely that for $n \geq 3$ the subgroups $P_n(\mathfrak{o}_S)$ and $Q_n(\mathfrak{n})$ generate the Hecke congruence subgroup $\Gamma_1(\mathfrak{n})$ of G_S. This fact was used by Bass, Milnor, and Serre in their solution of the congruence subgroup problem for SL_n [**4**] and is the reason for our restriction to the case of $n \geq 3$. Once we know this, then from the equality $U_{\xi_S}(g_S) = V_{\xi_S}(g_S)$ we have that U_{ξ_S} is a classical automorphic form on G_S with respect to the congruence subgroup $\Gamma_1(\mathfrak{n})$. Thus the map $\xi_S \mapsto U_{\xi_S}$ embeds π_S into the space of classical automorphic forms $\mathcal{A}(\Gamma_1(\mathfrak{n})\backslash G_S)$.

Finally, one again uses the classical–adelic correspondence, the strong approximation theorem and the class number 1 assumption to lift back to automorphic forms on $GL_n(\mathbb{A})$ obtain the statement given in the converse theorem.

What happens in the seemingly simpler case when $n = 2$, which was omitted from the above due to our use of stable algebra? For GL_2 the generation of congruence subgroups is a much more subtle issue. However, this was investigated by Vasserstein early on [**70**] and was used by Piatetski-Shapiro in his first converse theorem, that in his 1971 paper [**55**]. It is again a converse theorem with restricted ramification. The statement is a bit complicated.

Let $\pi = \otimes' \pi_v$ be an irreducible admissible representation of $GL_2(\mathbb{A})$ as in the general framework of converse theorems. For each place v we let n_v be the local conductor of π_v in the sense of Casselman [**10**].

We begin with a finite set of places S such that the ring of S-integers $\mathfrak{o}_S$ has class number one as above. We then take two other finite sets of finite places T and P such that

(i) for each $v \in T$, the local representation π_v has conductor $n_v > 0$
(ii) for each $v \in P$, the local representation π_v has conductor $n_v = 0$, that is, is unramified
(iii) for $\mathfrak{n} = \prod_{v \notin S \cup T} \mathfrak{p}_v^{n_v}$ and a suitable $\nu = 0, 1$ the group $\Gamma_\nu(\mathfrak{n})$ contains a set of generators of type (P, r) as in Vasserstein, that is, each generator $\gamma = (\gamma_{i,j})$ has the property that the lower right entry $\gamma_{2,2}$, if non-zero, satisfies $\mathrm{ord}_v(\gamma_{2,2}) \leq r e_v$ where $e_v = 1$ for $v \in P$ and $e_v = 0$ for $v \notin P$.

For the twisting set, we take $\mathcal{T}$ to be the set of idele class characters ω (automorphic forms on $GL_1(\mathbb{A})$) satisfying

(i) ω_v is unramified for $v \notin S \cup T \cup P \cup \{2\}$
(ii) the degree of ramification d_v of ω_v satisfies

$$d_v \leq \begin{cases} 2r & v \in P \\ n_v & v \in T \end{cases}$$

where r is the integer in the definition of generators of type (P, r) above.

Finally, we assume that for there is no pair of idele class characters μ and ν such that $\pi_v \simeq \pi_v(\mu_v, \nu_v)$ for all $v \in S$ or $v \notin S$ with π_v unramified. This condition says that π is not nearly equivalent to a global principal series representation in a strong sense.

THEOREM 4.2. *Let π and the sets of places S, P, and T be as above. Suppose $L(s, \pi \otimes \omega)$ is nice for all $\omega \in \mathcal{T}$. Then π is quasi-autormophic in the sense that there is an automorphic representation Π of $GL_2(\mathbb{A})$ such that $\pi_v \simeq \pi_v$ for all $v \in S$ and all $v \notin S$ such that π_v is unramified.*

The proof of this is essentially the same as that outlined above, although one must be quite careful with the generation of the congruence subgroup. The theory of the conductor for GL_2 was already known thanks to work of Casselman [**10**] (and independently P-S's student Novodvorsky [**53**]) and as noted the needed results on generation of congruence subgroups were due to Vasserstein (another student of

P-S). So, in some sense, one could say that P-S also brought this family of converse theorems with him from the Soviet Union as well.

As far as I know, there have been few applications of these theorems limiting the ramification of the twists. The converse theorem for GL_n with $n \geq 3$ discussed above was designed to attack Langlands functoriality conjecture for the classical groups, but where the analytic properties of the L-functions were to be controlled via integral representations. Once we combined forces with Kim and Shahidi and realized that there was sufficient control of the L-functions coming out of the Langlands-Shahidi method [**65**], then we were able to establish functoriality for generic representations of the classical groups to GL_n using the more basic converse theorem discussed in the previous section. There is a recent paper of Pitale, Saha, and Schmidt that uses Theorem 3.3 to lift from GSp_4 to GL_4 using integral representations to control the L-function [**60**].

There have been recent developments in the converse theorem for GL_2. In 1995 Conrey and Farmer proved some classical converse theorems for $\Gamma_0(N)$ with small N which require no twists, but an Euler product [**23**]. They replace the twists by a use of the Euler product and knowledge of specific generators for $\Gamma_0(N)$; this is in keeping with the ideas of Piatetski-Shapiro in that they do use facts about generation of congruence subgroups. More recently, Booker and then Booker and Krishnamurthy have developed converse theorems with a relaxing of the analytic properties of the twists depending on the ramification [**6, 7**]. They do this by passing from multiplicative twists, as in the statements of the converse theorem, to the consideration of additive twists, which they then control by analytic number theoretic methods. In fact, what they show is that they can control the analytic properties of many twists in terms of only a few. Their relaxed hypothesis is that the twisted L-functions are meromorphic (ratios of entire functions), but the twists by unramified characters must be entire. They show that under these seemingly weaker hypotheses, in fact all twists are entire and hence satisfy the hypotheses of either Weil's converse theorem classically or that of Jacquet and Langlands adelically. I refer the reader to their papers for more details.

5. Speculation on GL_1 twists

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We begin by stating the following conjecture of Piatetski-Shapiro, which one can find in [**18, 19, 58**].

CONJECTURE 5.1. Let π be an irreducible admissible representation of $GL_n(\mathbb{A})$ as in Section 2. Suppose that $L(s, \pi \otimes \omega)$ is nice for all $\omega \in \mathcal{T}(1)$, that is, for all idele class characters ω. Then π is quasi-automorphic in the sense that there is an automorphic representation Π of $GL_n(\mathbb{A})$ such that $\Pi_v \simeq \pi_v$ for all finite places v where both π and Π are unramified and such that $L(s, \pi \otimes \omega) = L(s, \Pi \otimes \omega)$ and $\varepsilon(s, \pi \otimes \omega) = \varepsilon(s, \Pi \otimes \omega)$ for all ω.

Note that if $n = 2$ or $n = 3$, then we can conclude that π itself is cuspidal and automorphic by the results in Section 3. For $n \geq 4$ this is no longer possible. In fact P-S has given examples of continuous families of representations π_λ of $GL_n(\mathbb{A})$ for $n \geq 4$ for which $L(s, \pi_\lambda \otimes \omega)$ and $\varepsilon(s, \pi_\lambda \otimes \omega)$ are independent of λ [**56, 19**].

The idea is that the representation π encodes an Euler product $L(s, \pi) = \prod L(s, \pi_v)$ and twisted Euler products $L(s, \pi \otimes \omega) = \prod L(s, \pi_v \otimes \omega_v)$. The hypothesis of this conjecture is that the Euler products are nice for all ω. The conclusion is that the Euler product is automorphic, that is, there is an automorphic parameter Π such that $L(s,\pi\otimes\omega)=L(s,\Pi\otimes\omega)$ and $\varepsilon(s,\pi\otimes\omega)=\varepsilon(s,\Pi\otimes\omega)$ for all ω. This is very much in the spirit of the classical results of Hecke and Weil that had as their input a Dirichlet series. Here the central object is the Euler product more than the analytic parameter π.

Such a simple test of automorphy would have many applications to the problem of functoriality and other number theoretic questions. On the functoriality side, if we combine this with the Langlands-Shahidi method we would immediately get all tensor functorialities from $GL_n \times GL_m$ to GL_{nm}, leading to a proof of the Ramanujan conjecture in general. Some of the arithmetic applications can be found in Richard Taylor's 2002 ICM talk and its "long version" [**67, 68**].

As we indicated, one can already find the examples on GL_4 related to this conjecture in P-S's 1975 University of Maryland preprint [**56**]. So perhaps this too he brought with him when he left the Soviet Union.

6. Converse theorems with poles

Piatetski-Shapiro took a small foray in the question of converse theorems when one relaxes the condition (1) of being *nice* to allow a finite number of poles. The first converse theorem of this type was due to Hamburger who showed that the Riemann zeta function was characterized by its analytic properties. Hamburger allowed a finite number of poles in his hypotheses. We take the following statement of Hamburger's Theorem from [**59**].

THEOREM 6.1 (Hamburger). *Let* $D(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$ *be absolutely convergent for* $Re(s)=\sigma>1$ *and suppose* $D(s)=G(s)/P(s)$ *where* $G(s)$ *is an entire function of finite growth and* $P(s)$ *is a polynomial. Suppose that*

$$\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}D(s)=\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}D'(1-s)$$

where $D'(1-s)=\sum_{n=1}^{\infty}\frac{b_n}{n^{1-s}}$*, the series being absolutely convergent for* $Re(s)<-\alpha<0$*. Then* $D(s)=c\zeta(s)$ *for some constant c,*

The polynomial $P(s)$ accounts for the possible poles of $D(s)$. A similar characterization of Dirichlet L-series was given by Gurevič [**26**], although it was weaker than the conclusion of Hamburger, in the sense that there was a finite dimensional space of solutions to the functional equation.

In a 1995 paper with his student Ravi Raghunathan, P-S revisited the theorems of Hamburger and Gurevič and give a simplified proof that was "in keeping with the spirit of Tate's thesis and the modern theory of automorphic forms" [**59**].

As a thesis problem, P-S gave Raghunathan the problem of extending this result to the GL_2 situation of Hecke. Raghunathan worked in the classical context and combined the ideas of Hecke and those of Bochner [**5**] to establish the following result [**61, 62**].

THEOREM 6.2 (Raghunathan). *Let* $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *be absolutely convergent for* $Re(s) > c > 0$ *and let* $a_0 \in \mathbb{C}$. *Set*

$$L(s) = (2\pi)^{-s}\Gamma(s)D(s) - \frac{a_0}{s}.$$

Assume $L(s)$ *is relatively nice in the sense*

(1) $D(s)$ *can be continued to a meromorphic function of the form* $G(s)/P(s)$ *where* $G(s)$ *is an entire function and* $P(s)$ *is a polynomial*
(2) $L(s)$ *has finite order on lacunary vertical strips*
(3) $L(s)$ *satisfies the functional equation* $L(s) = L(k - s)$.

Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi n z}$ *for* $z \in \mathfrak{H}$. *Then if* $k > 2$, k *even,* $f(z)$ *is a modular form of weight* k *for* $SL_2(\mathbb{Z})$; *if* $k = 2$ *then* $f(z)$ *is the holomorphic part of the non-holomorphic Eisenstein series of weight* 2 *for* $SL_2(\mathbb{Z})$.

Later we realized that Weissauer had established a similar result for modular forms with level following Weil [**72**]. Raghunathan has also established the analogue of Theorem 6.2 for the Maaß functional equation [**63**]. Recently, interesting work on converse theorems for GL_2 allowing essentially arbitrary sets of poles has been done by Booker and Krishnamurthy using the relation between additive and multiplicative twists [**8, 9**].

The interest in this theorem, and theorems like it, was they would give a vehicle for functoriality if one was only able to show that certain L-functions had a finite number of poles. At the time, this was what was expected to be available from the Langlands-Shahidi method, hence the desirability of such results. We thought briefly about how to generalize this result to GL_n for $n \geq 3$ in an adelic context, but were not successful. Eventually, we did not need such a theorem since we "twisted away" any poles by twisting with a suitably ramified character, a small variant of the converse theorem outlined in [**20**] and implemented in [**16, 17, 22**].

7. Local converse theorems

There is a version of the converse theorem that one can formulate over a non-archimedean local field or a finite field. So now let F be a non-archimedean local field. The local converse theorems are stated not in terms of the local L-function, but rather the local γ-factor. This factor is related to the local L– and ε–factors. If π is an irreducible admissible generic representation of $GL_n(F)$ and π' an irreducible admissible generic representation of $GL_m(F)$ then

$$\gamma(s, \pi \times \pi', \psi) = \frac{\varepsilon(s, \pi \times \pi', \psi)L(1 - s, \tilde{\pi} \times \tilde{\pi}')}{L(s, \pi \times \pi')}$$

where ψ is a non-trivial additive character of F. By analogy with the global converse theorem, we will have a twisting set $\mathcal{T}_\ell$. For $m < n$ let us let

$$\mathcal{T}_\ell(m) = \coprod_{1 \leq d \leq m} \{\pi' \mid \pi' \text{ is a (super)cuspidal representation of } GL_d(F)\}$$

and let $\mathcal{T}_\ell \subset \mathcal{T}_\ell(m)$ for some m. Then the local converse theorem addresses the following type of question.

Question: Let π_1 and π_2 be irreducible admissible representations of $GL_n(F)$ having the same central character. Suppose that $\gamma(s, \pi_1 \times \pi', \psi) = \gamma(s, \pi_2 \times \pi', \psi)$ for all $\pi' \in \mathcal{T}_\ell$. What is the relation between π_1 and π_2? Is $\pi_1 \simeq \pi_2$?

In an early Comptes Rendus note of Jacquet, Piatetski-Shapiro, and Shalika [**34**] we (essentially) find the following statement: "Le quasi-charactère central de π et les facteurs $\gamma(s, \pi \times \sigma; \psi)$ pour $1 \leq r \leq n-2$ ($r = 1$ si $n = 2$) et σ cuspidale déterminent la class de π". This then implies the following version of the local converse theorem.

Theorem 7.1. *Let π_1 and π_2 be irreducible admissible representations of $GL_n(F)$ with the same central character. Suppose that $\gamma(s, \pi_1 \times \pi', \psi) = \gamma(s, \pi_2 \times \pi', \psi)$ for all $\pi' \in \mathcal{T}_\ell(n-1)$. Then $\pi_1 \simeq \pi_2$.*

This is the local analogue of Theorem 3.1. While Jacquet, Piatetski-Shapiro and Shalika never published the proof of this statement, Piatetski-Shapiro explained the result and its purely local proof to me during the course of our collaboration, quite possibly when I was still a student. This result was rediscovered by Henniart in 1993 [**30**], with a very similar proof to the one explained to me by P-S, and appears in an appendix to the paper by Laumon, Rapoport, and Stuhler on the local Langlands conjecture for $GL_n(F)$ for F a function field over a finite field [**48**]. It is used to show that the local Langlands conjecture, phrased in terms of L-function, is unique. It was subsequently used by Harris and Taylor for the same purpose [**28**].

Even though covered by the remark in the Comptes Rendus note, Piatetski-Shapiro gave the local version of Theorem 3.3 to his student Jeff Chen as his thesis problem. Chen then gave a purely local proof of the following result in his 1996 thesis [**11, 12**].

Theorem 7.2 (Chen). *Let π_1 and π_2 be irreducible admissible representations of $GL_n(F)$ having the same central character. Suppose that $\gamma(s, \pi_1 \times \pi', \psi) = \gamma(s, \pi_2 \times \pi', \psi)$ for all $\pi' \in \mathcal{T}_\ell(n-2)$. Then $\pi_1 \simeq \pi_2$.*

One can also derive this from Theorem 3.3 by local-global arguments, as noted in [**19**], and perhaps it was this proof that P-S had in mind in [**34**].

Recently Chufeng Nien has given a proof of the local analogue of Jacquet's conjecture for F a finite field [**52**]. The result is the following.

Theorem 7.3 (Nien). *Let F be a finite field. Let π_1 and π_2 be irreducible admissible representations of $GL_n(F)$ with the same central character. Suppose that $\gamma(s, \pi_1 \times \pi', \psi) = \gamma(s, \pi_2 \times \pi', \psi)$ for all $\pi' \in \mathcal{T}_\ell\left(\left[\frac{n}{2}\right]\right)$. Then $\pi_1 \simeq \pi_2$.*

Her proof uses the theory of Bessel functions of representations, a technique that works well over the finite field and one that P-S was quite fond of. Jiang and Nien are working on extending this result to p-adic local fields. It is not at all clear how to generalize this to the global situation.

8. Final remarks

Beginning with his paper in the proceedings of the 1971 Budapest conference [**55**] the converse theorems and their applications to functoriality were a dominant theme in Piatetski-Shapiro's mathematical life. This is brought out not just by his work on these questions and their applications but also through his students.

Converse theorems remain a general approach to the question of functoriality when the target group is GL_n. While the approach to functoriality via the trace formula by Arthur has seen astounding success recently [**1**], the approach through converse theorems remains quite nimble. As Piatetski-Shapiro said "Arthur's approach is more general, but this approach is easier" (see Shahidi's contribution to [**15**]). Indeed, to establish functoriality from G to GL_n associated to an L-homomorphism R, one needs only control the analytic properties of an appropriate family of twisted L-functions associated to R. This was the strategy for the applications mentioned in Section 3. While we have probably exhausted the cases of functoriality we can prove by appealing to the Langlands-Shahidi method of controlling twisted L-functions, the control via integral representations is still a wide open, although difficult and not well understood. But whenever we have analytic control of a sufficient family of twisted L-functions, we can use the converse theorem to obtain functoriality.

The more one can limit the family $\mathcal{T}$ of twists needed, the more powerful a converse theorem becomes. In light of the above survey, the following questions come to mind.

(1) Can we further reduce the ranks of the twists in Theorems 3.2 and 3.3, beginning with rank $n-3$?

As mentioned in Section 3, the difficulty in the standard approach is that one has to resolve a question in non-abelian harmonic analysis combined with P-S's local trick. It seems to me that if one can resolve this, and reduce the rank of the twists to rank up to $n-3$, then whatever the new technique is, it should let you inductively arrive at the proof of Jacquet's Conjecture. The first case to try would be a converse theorem for GL_5 with twists by GL_1 and GL_2. I would not expect these techniques to go beyond Jacquet's Conjecture.

(2) Can we combine the techniques of Sections 3 and 4 to obtain hybrid theorems in which one combines say Theorems 3.3 and 4.1 to have a converse theorem where one twists by (essentially) unramified cusp forms of rank up to $n-2$?

We have made some attempts at this, but at the moment we have been unable to implement the generation of congruence groups from Theorem 4.1 in the context of restricting the rank and P-S's local trick. The techniques seem somewhat incompatible.

(3) What are the new conceptual ideas required for Conjecture 5.1?

As I mentioned above, I believe that variants of our standard techniques, without new fundamental insight, will at most get us as far as Jacquet's Conjecture. But, as observed by Taylor [**67, 68**], the pay off from Conjecture 5.1 would be tremendous.

(4) Do there exist converse theorems for other groups and what would their applications be?

I only know of a few of converse theorems for groups other than GL_n. The classical examples have been proven in a Dirichlet series context and not in terms of automorphic L-functions. These include a converse theorem for $SO_{n,1}$ by Maaß [**50**] and one for Sp_4 by Imai [**31**]. P-S wrote one paper related to this topic. A converse theorem for the three fold cover of GL_3 is embedded in the paper [**54**] with Patterson, which could be considered as a variant of Theorem 3.3 for GL_3. He also believed that there was a converse theorem for $U_{2,1}$ based on constructions he gave in a course at Yale in 1977; I currently have a student working on this. As for applications, they would include approaches for functoriality where the target group need not be GL_n, but this is far off at the moment.

The new approach to limiting the analytic properties of the twists that one finds in the work of Booker and Krishnamurthy are quite intriguing [**6, 7, 8, 9**]. These involve passing back and forth between multiplicative and additive twists. At present, this technique exists only for GL_2 where the twists are by characters. My understanding is that Booker and Krishnamurthy have ideas on how to implement this in higher rank settings. It would then be quite interesting to see how these techniques could be combined with the theorems of Piatetski-Shapiro presented here.

In his later years, as his Parkinson's progressed, P-S lost the ability to write and could only speak with difficulty and only during certain periods. But throughout this he continued to do mathematics. The one topic that dominated his thoughts in these days were new potential converse theorems for GL_n. Every few months he would come up with a new idea or new approach. Most of these did not play out in the end, but there remain a few that have not been ruled out.

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