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STABILITY OF γ-FACTORS FOR QUASI-SPLIT GROUPS

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Abstract One of the main obstacles in applying converse theorems to prove new cases of functoriality is that of stability of γ-factors for a certain class of L-functions obtained from the 'Langlands–Shahidi' method, where the γ-factors are defined inductively by means of 'local coefficients'. The problem then becomes that of stability of local coefficients upon twisting the representation by a highly ramified character. In this paper we first establish that the inverses of certain local coefficients are, up to an abelian γ-factor, genuine Mellin transforms of partial Bessel functions of the type we analysed in our previous paper. The second main result is then the resulting stability of the local coefficients in this situation, which include all the cases of interest for functoriality. Hopefully, the analysis given here will open the door to a proof of the general stability and the equality of γ-factors obtained from different methods through integration over certain quotient spaces whose generic fibres are closed. They do not seem to have been studied before in any generality.

Keywords: γ-factors; local coefficients; stability; Mellin transform; Bessel functions

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1. Introduction

One of the main obstacles in applying converse theorems [**6, 8**] to prove new cases of functoriality is that of stability of γ-factors for a certain class of L-functions obtained from the method developed in [**22–24**], where the γ-factors are defined inductively by means of 'local coefficients' (cf. [**24**]). Thus the problem becomes that of stability of local coefficients upon twisting the representation by a highly ramified character. This paper is the sequel to the authors' first paper [**11**] which together prove a general theorem on stability. More precisely, in this paper we establish that the inverses of certain local coefficients are, up to an abelian γ-factor, genuine Mellin transforms of appropriate partial Bessel functions whose asymptotic expansions were established in our first paper [**11**] in the generality of every quasi-split group. This then proves stability in all the cases of interest in functoriality (Theorem 6.1 and Corollary 6.2). Moreover, the analysis given here opens the door to a proof of the equality of γ-factors obtained from different methods through integration over certain quotient spaces whose generic fibres are closed. They do not seem to have been studied before in any generality.

To explain our result, let G be a quasi-split connected reductive algebraic group over a p-adic field k and let K be the splitting field of G. Let $B = TU$ be a Borel subgroup of G over k and denote by P a self-associate maximal parabolic of G over k. Assume $P \supset B$. Let $P = MN$ be a Levi decomposition uniquely determined by letting $T \subset M$. Let $U_M = U \cap M$. There exists a dense open subset $N' \subset N$ such that the space of orbits $U_M \backslash N'$ under the conjugation action is a variety, i.e. has a quotient structure (cf. § 4). Let ξ be the K-rational character of M defined by $\xi(m) = \det(\mathrm{Ad}_{\mathfrak{n}}(m))$ for $\mathfrak{n}$ the Lie algebra of N. Let Δ be the set of simple roots of the maximal split subtorus A_0 of T determined by B. Assume α is the unique simple root of A_0 appearing in N and let $\Omega = \Delta \setminus \{\alpha\}$. If $\tilde{\alpha}$ is a root of T restricting to α, let $K_{\tilde{\alpha}}$ be its splitting field. Finally, let $w_0 = w_\ell w_\ell^\Omega$, where w_ℓ and w_ℓ^Ω are the long elements of the Weyl groups of A_0 in G and M, respectively. Note that G being quasi-split, the Weyl group $W(A_0, G)$ of A_0 in G can be considered as the subgroup of elements $w \in W(T, G)$ for which $\mathrm{Int}(w)$ sends A_0 to itself. Given $w \in W(A_0, G)$, then $\ell(w)$, the length of w, is the number of indivisible (restricted) positive roots which are sent to negative ones under w. On the other hand, if we denote the length of w as an element in $W(T, G)$ by $\tilde{\ell}(w)$, then $\tilde{\ell}(w)$ is the number of non-restricted positive roots which go to negative roots under w.

There are two main results in this paper. The first is Theorem 4.22, which expresses the local coefficient as a genuine Mellin transform of a Bessel function of the type we analysed in [**11**]. This is given under several simplifying hypotheses, including the dimension and rank conditions $\dim(U_M \backslash N) = \mathrm{rank}(Z_G \backslash T_w) = 2$. As the statement is somewhat technical, we refer the reader to the end of § 4 for the precise formulation. The second main result is the resulting stability of the local coefficients in this situation. The best way to formulate our result is in terms of Bruhat double cosets in $G(k)$ as we do in Theorem 6.1, which we paraphrase here.

Theorem. *Let π be an irreducible admissible generic representation of $M(k)$ and let $C_\psi(s, \pi)$ be the corresponding local coefficient (cf. equation (2.4) here). There exists a unique Bruhat double coset $\bar{B}(k)\bar{w}\bar{N}(k)U_M(k)$ of $G(k)$ with respect to $\bar{B} = B^-$ and $B' = T\bar{N}U_M$ which intersects $N(k)$ in an open set; then $\bar{w}\alpha < 0$. Assume there exists a simple root β such that $\bar{w}\beta < 0$ but $\bar{w}(\theta) > 0$, where $\theta = \Delta \setminus \{\alpha, \beta\}$. Moreover, assume* $\dim(U_M \backslash N') = \dim(N) - \tilde{\ell}(w_0\bar{w}) = 2$ *where $\tilde{\ell}$ denotes the length function on the Weyl group of T in G as above. Then $C_\psi(s, \pi)$ is stable, i.e. if π_1 and π_2 are two such representations sharing the same central character, then*

$$C_\psi(s, \pi_1 \otimes \tilde{\nu}) = C_\psi(s, \pi_2 \otimes \tilde{\nu}),$$

for all sufficiently highly ramified characters ν of $K^\times$, identified as a character of $M(k)$ by $\tilde{\nu}(m) = \nu(\xi(m))$.

Let us explain the steps of the proofs of Theorem 4.22 and Theorem 6.1 and thereby give a brief outline of the paper. In § 3 we reformulate the integral representation of the local coefficient, as in Theorem 6.2 of [**26**], so that it can be used to prove stability in a number of cases of quasi-split groups. This section is written under the Assumption 3.6 necessary for the results of [**26**]. In § 4 we begin to restrict to the cases which are needed for the proof of functoriality for all the groups whose connected L-groups have classical groups as their derived groups. The main assumption which is necessary in order to use the result of our first paper [**11**] is that the semisimple rank of a certain parabolic subgroup of G defined by our data is equal to 2, i.e. $\mathrm{rank}(Z_G\backslash T_w) = 2$ (notation as in § 3). We must also assume that $\dim(U_M\backslash N) = 2$, where $U_M\backslash N$ is the set of orbits of N under conjugation by $U_M = U \cap M$. Under these assumptions, Theorem 6.2 of [**26**] (Proposition 3.10 here) reduces to a genuine Mellin transform of a Bessel function that is attached to a maximal parabolic subgroup of M, leading to only two asymptotic directions, infinity and zero, as analysed in [**11**]. This is our Theorem 4.22. We begin § 4 with a number of technical assumptions, notably equation (4.2), which have a great simplifying effect on the calculations. Then § 5 is devoted to removing these assumptions via a case-by-case analysis, showing that they follow from our main assumptions on the rank of $Z_G\backslash T_w$ and the dimension of $U_M\backslash N$. Finally, § 6 reformulates Theorem 4.22 in terms of the geometry of Bruhat cells with respect to $\bar{B}$ and $B' = T\bar{N}U_M$. In fact, the result can be formulated completely in terms of the unique double coset $\bar{B}\bar{w}B'$ of highest dimensional intersection with N, yielding Theorem 6.1. Theorem 6.1 and Corollary 6.2 cover all the cases of interest in functoriality. In § 7 we illustrate our results in the case of quasi-split unitary groups. (See also our comments on the quasi-split SO_{2n} and GSpin_{2n}.)

On the other hand, there are many other cases where the Bessel function is not attached to a maximal parabolic subgroup of M. This is in particular the case for the γ-factors for Sym^2 and Λ^2 L-functions for $\mathrm{GL}_n(k)$. In these cases the parabolic support for Bessel functions in M is no longer maximal and here is where at present there is little interpretation of the general expression obtained in Theorem 6.2 of [**26**], although the results in § 3 are valid in this more general context. Understanding the geometry of $U_M\backslash N$, where the integration takes place, is the first step. One must note that it is only for a (Zariski) dense open set $N' \subset N$ that $U_M\backslash N'$ has the structure of a quotient variety (cf. [**15**]) and thus $U_M(k)\backslash N'(k)$ is a manifold.

Sundaravaradhan [**29**] has now proved Assumption 4.1 of [**26**] (Assumption 3.6 here) at least for all the split groups and their self-associate maximal parabolic subgroups, as well as some other results related to this paper (cf. Remark 6.4 below). Beside giving us a better understanding of this problem and general stability, this clearly opens the road to proving the equality of γ-factors and ε-factors obtained from the Langlands–Shahidi method and those of the Rankin–Selberg method (cf. equation (6.38) of [**26**]).

It should finally be commented that the proofs given here are independent from any particular model or specific Bruhat decomposition for the group and may be considered as a coordinate free statement and proof of stability in the maximal parabolic Bessel support cases. Moreover, the conditions of our main Theorem 4.22 depend only on the isogeny class of the derived group of G and are thus easy to verify (cf. Corollary 6.2 and § 7).

Since we began this project, particularly through its application in proofs of functoriality [**9**, **10**], there has been an increased interest in establishing the stability of γ-factors through other methods. The most successful has been for the γ-factors for the local 'standard' L-functions of the classical groups that arise through the doubling method by Rallis, Soudry and Brenner [**5**, **19**]. From the analysis of Lapid and Rallis [**17**] it is known that these γ-factors agree with the ones we consider here where they overlap, namely generic representations of quasi-split classical groups (our Corollary 6.2). For the classical groups, the doubling method is broader, requiring neither for the groups to be quasi-split nor the representations to be generic. In this aspect, their results are more general than ours. On the other hand, our Theorem 4.22, which expresses the local coefficients as the Mellin transform of certain Bessel functions, is within the broader context of all γ-factors arising from the Langlands–Shahidi method associated to a maximal parabolic subgroup, and covers other groups whose derived groups have the same isogeny classes as the classical ones. Moreover, the basis of our Theorem 4.22, namely Theorem 6.2 of [**26**] and our results in § 3 here, are valid in the context of all γ-factors arising from the Langlands–Shahidi method, such as the symmetric and exterior square mentioned above. While we are not in a position to establish stability for these more general γ-factors yet, we believe our analysis will be the beginning of this endeavour.

2. Preliminaries

Throughout this paper G will denote a quasi-split connected reductive algebraic group over a non-archimedean local field k of characteristic zero. We use $\mathcal{O}$ to denote its ring of integers and let $\mathcal{P}$ be its maximal ideal. As in [**11**], which this paper is a sequel to, we let $\Gamma = \mathrm{Gal}(\bar{k}/k)$ and if K is the splitting Galois extension of G we set $\Gamma_K = \mathrm{Gal}(K/k)$. Any unexplained notation will be referred to either [**11**] or [**26**].

It is by now standard (cf. [**1**, **24**, **26**]) that local coefficients, whose stability is the goal of this paper to establish, depend only on the derived group G_{D} of G and, as explained in [**26**], one may replace G by a possibly larger group with the same derived group for which $H^1(k, Z_G) = 1$, where Z_G is the centre of G. Consequently, throughout this paper we shall assume $H^1(k, Z_G) = 1$.

Let $B = TU$ be a Borel subgroup of G defined over k with T a maximal torus and U its unipotent radical. Let P be a maximal parabolic subgroup and let $P = MN$ be a Levi decomposition with $N \subset U$. We shall make the decomposition unique by demanding $T \subset M$. Let $U_M = U \cap M$.

Next, let A_0 be the maximal split subtorus of T. We then have $\tilde{\Phi} = \Phi(T, G)$ and $\Phi = \Phi(A_0, G)$, the sets of non-restricted and restricted roots of T in G. Using U we then have the sets of positive and simple roots $\tilde{\Phi}^+$ and $\tilde{\Delta}$ of $\tilde{\Phi}$, respectively as in [**11**]. The Galois group Γ then acts on $\tilde{\Phi}$ and $\tilde{\Delta}$, decomposing them to a finite number of Γ-orbits. They will be the same as Γ_K-orbits. We finally have Φ^+ and Δ for similar objects in Φ.

Let $(x_{\tilde{\alpha}})_{\tilde{\alpha} \in \tilde{\Delta}}$ be a k-splitting of G as defined in [**11**], i.e. a collection of $K_{\tilde{\alpha}}$-isomorphisms from $\mathbb{G}_a$ to corresponding root subgroups $U_{\tilde{\alpha}}$ satisfying conditions (i) and (ii) in § 1.1 of [**11**]. Here $K_{\tilde{\alpha}}$ is the splitting field of $\tilde{\alpha}$. Given a non-trivial character ψ of k, we then define a generic character of $U(k)$ by

$$\psi(u) = \psi\biggl(\sum_{\tilde{\alpha}\in\tilde{\Delta}} u_{\tilde{\alpha}}\biggr), \tag{2.1}$$

where the simple root subgroups generating $U(K)$ are now

$$\{x_{\tilde{\alpha}}(u_{\tilde{\alpha}}) \mid u_{\tilde{\alpha}} \in K_{\tilde{\alpha}}\}_{\tilde{\alpha}\in\tilde{\Delta}}.$$

We recall that to get $U(k)$ we require $\sigma(u_{\tilde{\alpha}}) = u_{\sigma(\tilde{\alpha})}$ for all $\sigma \in \Gamma_K$. This is equivalent to $x_{\tilde{\alpha}}$ being a k-splitting. One can, of course, use the notation and the definition of splitting given in [**26**], but this paper being a sequel to [**11**], we have chosen to follow [**11**]. The notions are equivalent. We finally recall that our fixed splitting also determines a natural choice of Weyl group representatives as explained in [**23**,**24**]. For any Weyl group W that arises, we will let $\tilde{w}$ denote elements of the abstract group W and let $w \in G(k)$ denote this choice of representative of $\tilde{w}$. They all lie in the derived group of G.

The parabolic subgroup P being maximal, we use α to denote the unique simple root whose root subgroup sits in N. As in [**26**], throughout this paper we shall assume P is self-associate. This means that $\bar{N} = w_\ell N w_\ell^{-1} = N^-$, where N^- is the opposite subgroup to N. Equivalently $\tilde{w}_\ell(\alpha) = -\alpha$ and $\tilde{w}_\ell(\Omega) = -\Omega$ if $\Omega = \Delta\backslash\{\alpha\}$. Here w_ℓ is the corresponding representative for the longest Weyl group element $\tilde{w}_\ell$ in $W(A_0, G)$.

We use $\tilde{w}_0$ to denote the element $\tilde{w}_0 = \tilde{w}_\ell \tilde{w}_\ell^{\Omega}$, where $\tilde{w}_\ell^{\Omega}$ is the longest Weyl group element in $W(A_0, M)$. Then $\tilde{w}_0(\Omega) = \Omega$, P being self-associate, while $\tilde{w}_0(\alpha) < 0$.

Let $X(M)_k$ be the group of k-rational characters of M. Referring to [**24**] we have

$$\mathfrak{a} = \mathrm{Hom}(X(M)_k, \mathbb{R})$$

and $\mathfrak{a}^* = X(M)_k \otimes_{\mathbb{Z}} \mathbb{R}$ as well as $\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C}$. We also recall $H_{P(k)} = H_P : M(k) \to \mathfrak{a}$ defined by

$$q^{\langle \chi, H_P(m)\rangle} = |\chi(m)|_k$$

for all $\chi \in X(M)_k$.

Throughout this paper π denotes an irreducible admissible ψ-generic representation of $M(k)$. Given $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, let

$$I(\nu, \pi) = \mathrm{Ind}_{M(k)N(k)}^{G(k)}(\pi \otimes q^{\langle \nu, H_P(\cdot)\rangle} \otimes 1).$$

We denote the space for this representation by $V(\nu, \pi)$.

Since P is maximal, we shall use $s\hat{\alpha}$ to denote an arbitrary element of $\mathfrak{a}^*_{\mathbb{C}}$ modulo that of the complex dual of the real Lie algebra of the split centre of G, where $s \in \mathbb{C}$ and $\hat{\alpha} = \langle \rho_P, \alpha\rangle^{-1}\rho_P$ as in [**26**] (denoted $\tilde{\alpha}$ there). Here ρ_P is half the sum of the roots in N. We then set $I(s, \pi)$ for $I(s\hat{\alpha}, \pi)$.

We now recall the intertwining operator $A(s, \pi) : I(s, \pi) \to I(-s, w_0(\pi))$, as defined in equation (2.6) of [**26**], for example, given by

$$A(s, \pi) f(g) = \int_{N(k)} f(w_0^{-1} n g)\, \mathrm{d}n \tag{2.2}$$

for $g \in G(k)$, where $f \in V(s,\pi)$, since P is self-associate. If $\lambda_\psi(s,\pi)$ is the canonical Whittaker functional for $I(s,\pi)$ defined by equation (2.7) of [**26**], then the local coefficient $C_\psi(s,\pi)$ is defined by (equation (2.8) of [**26**])

$$\lambda_\psi(s,\pi) = C_\psi(s,\pi)\lambda_\psi(-s,w_0(\pi))A(s,\pi). \tag{2.3}$$

Then, rewriting equation (2.9) of [**26**],

$$C_\psi(s,\pi) = \lambda_G(\psi,w_0)^{-1}\prod_{i=1}^{m}\gamma(is,\pi,\tilde{r}_i,\bar{\psi}), \tag{2.4}$$

where $\gamma(s,\pi,r_i,\psi)$, $1 \leqslant i \leqslant m$, is the γ-function attached to π, r_i and ψ. Here r_i is the ith irreducible component of the adjoint action of ${}^LM$, the L-group of M, on the Lie algebra ${}^L\mathfrak{n}$ of ${}^LN$, the L-group of N. We refer to [**24**] and particularly Theorem 3.5 of [**24**] for the ordering of r_i, inductive definition of $\gamma(s,\pi,r_i,\psi)$, the constant $\lambda_G(\psi,w_0)$ and equation (2.9) of [**26**].

The aim of this paper is to prove the stability of $\gamma(s,\pi,r_i,\psi)$ under highly ramified twists in a number of important cases. It follows from the inductive definition of $\gamma(s,\pi,r_i,\psi)$ that one only needs to prove this for the corresponding $C_\psi(s,\pi)$.

3. An integral representation for local coefficients

In this section we shall reformulate Theorem 6.2 of [**26**] so that it can be used to prove stability in a number of cases of quasi-split groups. It will cover all the cases which are needed for the proof of functoriality for all the groups whose connected L-groups have classical groups as their derived groups. This includes all the quasi-split classical groups [**9, 10, 16**] and general spin groups [**1**].

As in [**26**], we start with the important Bruhat decomposition

$$w_0^{-1}n = mn'\bar{n} \tag{3.1}$$

valid for almost all $n \in N(k)$, where $m \in M(k)$, $n' \in N(k)$ and $\bar{n} \in \bar{N}(k)$. Decomposition (3.1) is crucial as we will need to integrate over orbits of elements of N satisfying (3.1) under conjugation by U_M. We will repeatedly refer to this.

Given n satisfying (3.1) we write

$$m = u_1twu_2 \tag{3.2}$$

with $t \in T(k)$, $u_i \in U_M(k)$, $i = 1,2$, and w the representative for an appropriate Weyl group element in $W(A_0,M)$ giving the Bruhat decomposition of m in $M(k)$. Moreover, assume that $wu_2w^{-1} \in U_M^-$; this determines u_1 and u_2 uniquely.

(3.3) *If $m = u_1twu_2$ corresponds to $n \in N(k)$ satisfying (3.1), then for any $m_0 \in M(k)$*

$$m_1 = w_0(m_0)u_1twu_2m_0^{-1}$$

corresponds to $n_1 = m_0nm_0^{-1} \in N(k)$, where $w_0(m_0) = w_0^{-1}m_0w_0$.

For our calculations we may suppose that w is such that its Bruhat double coset

$$C_M(w) = U_M(k)T(k)wU_M(k)$$

intersects the set of all possible m whose n satisfy (3.1) in a subset of highest possible dimension. In fact, then by (3.3), w will have highest length among those Bruhat cells which have a non-empty intersection with the subset of all m whose n satisfy (3.1). Observe that *a priori*, there may be several w satisfying this property, but we will rule this out now.

We start with some general consequences of the stratification of G by the Bruhat decomposition. Let G be a connected reductive group. Let H be a closed connected subgroup of G. Fix a Borel subgroup B of G. Write $B = TU$. Let $B' = TU'$ be another Borel subgroup sharing T as a maximal torus. Given $w \in W = W(T, G)$, let $C(w) = BwB'$ be the corresponding Bruhat double coset with respect to B and B'. Then $G = \coprod_{w\in W} C(w)$ will be the corresponding Bruhat decomposition. This then defines a stratification $H = \coprod_{w\in W} H(w)$ of H by locally closed subspaces, where $H(w) = H \cap C(w)$.

Let $\bar{w} \in W$ be an element such that $H(\bar{w})$ is open in H. Since H is irreducible and closed $\overline{H(\bar{w})} = H$, where $\overline{H(\bar{w})}$ is the Zariski closure of $H(\bar{w})$. In particular, if $H(\bar{w}_1)$ is another open strata of H, then $\bar{w}_1 = \bar{w}$.

Moreover, since

$$\overline{C(\bar{w})} = C(\bar{w}) \coprod \Big(\coprod_{\substack{w\in S(\bar{w})\\ w\neq\bar{w}}} C(w) \Big)$$

one gets

$$H(\bar{w}) \coprod \Big(\coprod_{\substack{w\in S(\bar{w})\\ w\neq\bar{w}}} H(w) \Big) = H \cap \overline{C(\bar{w})} \supset \overline{H \cap C(\bar{w})}.$$

Here $S(\bar{w})$ ($A_{\bar{w}}$ in Borel's notation [**3**] and $X_{\bar{w}}$ in Springer's [**28**]) is a subset of W containing $\bar{w}$ for which $\ell(w) < \ell(\bar{w})$ for $w = \bar{w}$ as specified in § 1.4 of [**11**] or § 8.5.4 of [**28**]. On the other hand,

$$\overline{H \cap C(\bar{w})} = \overline{H(\bar{w})} = \coprod_{w} H(w) = H.$$

Thus

$$H = H(\bar{w}) \coprod \Big(\coprod_{\substack{w\in S(\bar{w})\\ w\neq\bar{w}}} H(w) \Big). \tag{3.4}$$

We collect this as the following result.

Proposition 3.1. *There exists a unique Bruhat double coset $C(\bar{w})$ such that $H(\bar{w}) = H \cap C(\bar{w})$ is open in H. Every other Bruhat double coset intersecting H is of dimension less than that of $C(\bar{w})$. In other words, if $H(w) \neq \phi$ and $w \neq \bar{w}$, then $\tilde{\ell}(w) < \tilde{\ell}(\bar{w})$, where $\tilde{\ell}$ denotes the length function on $W = W(T, G)$.*

We will apply this in the cases that $H = N$ and $H = U$. More precisely, assume n and m satisfy (3.1) and (3.2). Then

$$w_0 u_1^{-1} w_0^{-1} n = w_0 t w_0^{-1}(w_0 w u_2 w^{-1} w_0^{-1})(w_0 w n' w^{-1} w_0^{-1}) w_0 w \bar{n}.$$

Set $u' = w_0 u_1^{-1} w_0^{-1}$ and $n_1 = u' n (u')^{-1} \in N$. Then $n_1 \in \bar{B} w_0 w \bar{N} U_M$ as well. Note that n determines n_1 uniquely. Conversely, given $n_1 \in \bar{B} w_0 w \bar{N} U_M$ one can write $n_1 = \bar{b} w_0 w \bar{n} u$ uniquely by imposing that $wuw^{-1} \in U_M$. Then $n = u n_1 u^{-1}$ will satisfy (3.1) and therefore the map $n \mapsto n_1$ is a bijection on the set of all n satisfying (3.1).

Let $\bar{C}(\bar{w}) = \bar{B}\bar{w}\bar{N}U_M$ be the Bruhat double coset intersecting N in an open set. Then by the disjointness of the Bruhat decomposition, we have $\bar{w} = w_0 w$. This then gives the following result.

Proposition 3.2. *Let $\bar{C}(\bar{w})$ be the unique Bruhat double coset intersecting N in an open set. Then*

$$\bar{w} = w_0 w, \tag{3.2.1}$$

where w is as in (3.2). In particular, $\tilde{\bar{w}} \in W(A_0, G)$ and $\bar{B}(k)\bar{w}\bar{N}(k)U_M(k) \subset G(k)$.

To use Theorem 6.2 of [**26**] to establish stability, we must analyse the integral appearing there, which is reproduced in (3.10.1) here. For this integral to be non-vanishing, which is expected for generic values of s, the m that appear in the integration must support a Bessel function in the sense of [**11**], at least for a set whose complement has measure zero. On the other hand, with m and n related by (3.1) and (3.2), Proposition 3.2 says that there is a unique w with this property. In the split case, as noted before, Rajan Sundaravaradhan has shown that the w in Proposition 3.2 supports a Bessel function as needed [**29**]. In the cases of interest to us, this will follow from the rank assumption we impose in § 4. For the remainder of this section we will simply assume that the w of Proposition 3.2 supports a Bessel function.

Suppose now that w supports a Bessel function. For this we appeal to discussions in § 2.2 of [**11**]. We will redefine the appropriate subsets as follows. Let

$$T_w = \{t \in T \mid w\tilde{\mu}(t) = 1 \text{ for all } \tilde{\mu} \in \tilde{\Delta} \text{ for which } w\mu > 0, \text{ where } \mu = \tilde{\mu} \mid A_0\}, \tag{3.5}$$

where as usual we embed the (relative) Weyl group W of A_0 in the Weyl group $\tilde{W}$ of T (relative to either G or M) by realizing W as the subgroup of all those $w \in \tilde{W}$ which fix the annihilator of A_0 in the character module $X(T)$ of T (in G or M respectively), i.e. those w such that $\mathrm{Int}(w)$ sends A_0 to itself.

Let $\theta = \theta_w = \{\mu \in \Delta \mid w\mu > 0\} \subset \Delta$. Then

$$T_w = \{t \in T \mid w\tilde{\mu}(t) = 1 \text{ for all } \mu \in \theta \text{ with } \mu = \tilde{\mu} \mid A_0\}. \tag{3.6}$$

For w to support a Bessel function on $M(k)$, we must have $w = w_\ell^\Omega w_\ell^\theta$ (cf. § 2.2 of [**11**]), where w_ℓ^Ω and w_ℓ^θ are the long Weyl group elements of M and M_θ, the Levi subgroup of M containing T and spanned by θ, respectively, and $\Omega = \Delta\backslash\{\alpha\}$. Observe that then T_w is precisely the centre of the Levi component $M_{w(\theta)}$ of the parabolic subgroup $P_{w(\theta)}^M$ associated to P_θ^M by w, containing T. When $w(\theta) = \theta$, i.e. P_θ^M is self-associate, then T_w

is precisely the centre of M_θ. (Here the superscript M in P_θ^M signifies that $P_\theta^M \subset M$.) We use $P_\theta = P_\theta^M N$ for the corresponding parabolic in G.

For $m = u_1 t w u_2$ to be in the support of a Bessel function, for all $\tilde{\mu} \in \tilde{\Delta}$ we must have

$$\psi\biggl(\prod_{\sigma\in\mathcal{A}_{K_{\tilde{\mu}}/k}} x_{\sigma(\tilde{\mu})}(\sigma(y))\biggr) = \psi(\mathrm{Tr}_{K_{\tilde{\mu}}/k}(y)) = \psi(\mathrm{Tr}_{K_{\tilde{\mu}}/k}(w\tilde{\mu}(t)y)) \tag{3.7}$$

for $y \in K_{\tilde{\mu}}$. The set $\mathcal{A}_{K_{\tilde{\mu}}/k}$ is the set of all the k-injections of $K_{\tilde{\mu}}$ into $\bar{k}$. This then implies that $t \in T_w(k)$ since the image of $\mathrm{Tr}_{K_{\tilde{\mu}}/k}$ is not compact.

We collect this information as follows.

Proposition 3.3. *For $m = u_1 t w u_2 \in M(k)$ to support a Bessel function, one must have $w = w_\ell^\Omega w_\ell^\theta$, where θ is the subset of all $\mu \in \Delta$ for which $w\mu > 0$, and t must belong to $T_w(k)$.*

Our intention is to use Theorem 6.2 of [**26**] to prove the stability of local coefficients $C_\psi(s,\pi)$ defined by (2.3) and thus local γ-factors (cf. [**22**–**24**]) under highly ramified twists in the generality of every quasi-split group. To use Theorem 6.2 of [**26**] we need to assume P is self-associate which we will assume from this point on. All the cases of interest fit into this situation.

As explained earlier we may and will also assume that $H^1(k, Z_G) = 1$. We can then use Lemma 5.2 of [**26**], but not in the form there.

Lemma 3.4. *Let $\tilde{\alpha}$ be a root restricting to α. Then there exists a bijection $\alpha^\vee$ from $\bar{k}^\times$ onto $Z_G(\bar{k})\backslash Z_M(\bar{k})$ such that*

$$(\sigma\tilde{\alpha})(\alpha^\vee(t)) = \sigma(t) \tag{3.4.1}$$

for all $t \in \bar{k}^\times$ and $\sigma \in \Gamma$. The map $\alpha^\vee$ depends on the choice of $\tilde{\alpha}$. Suppose $H^1(k, Z_G) = 1$ so that $(Z_G\backslash Z_M)(k) = Z_G(k)\backslash Z_M(k)$. Then $\alpha^\vee$ descends to a bijection from $K_{\tilde{\alpha}}^\times$ onto $Z_G(k)\backslash Z_M(k)$ satisfying (3.4.1) for all $t \in K_{\tilde{\alpha}}^\times$ and $\sigma \in \Gamma$.

Proof. The proof is as in Lemma 5.2 of [**26**], except that we require

$$(\sigma\tilde{\alpha})(\alpha^\vee(t)) = \sigma(t),$$

for $t \in \bar{k}^\times$ and all $\sigma \in \Gamma$ for a fixed $\tilde{\alpha} \in \tilde{\Delta}$ restricting to α, while $\tilde{\gamma}(\alpha^\vee(t)) = 1$ for all other non-restricted roots $\tilde{\gamma}$ which do not restrict to α. This guarantees that $\alpha^\vee(t) \in Z_G(\bar{k})\backslash Z_M(\bar{k})$. The embedding from $Z_G(\bar{k})\backslash Z_M(\bar{k})$ into $T_{\mathrm{ad}}(\bar{k})$ is obtained by applying $\tilde{\alpha}$, giving an injection of $Z_G(\bar{k})\backslash Z_M(\bar{k})$ into $\bar{k}^\times$.

If $H^1(k, Z_G) = 1$, we then conclude as in Lemma 5.2 of [**26**], that $\alpha^\vee(t) \in Z_G(k)\backslash Z_M(k)$ for all t in $K_{\tilde{\alpha}}^\times$. We thus get an injection from $K_{\tilde{\alpha}}^\times$ into $Z_G(k)\backslash Z_M(k) \subset T_{\mathrm{ad}}(k)$. The embedding from $Z_G(k)\backslash Z_M(k)$ into $T_{\mathrm{ad}}(k)$ is again obtained by applying $\tilde{\alpha}$, giving an injection of $Z_G(k)\backslash Z_M(k)$ into $K_{\tilde{\alpha}}^\times$. Thus $\alpha^\vee$ defines a bijection onto $Z_G(k)\backslash Z_M(k)$ satisfying (3.4.1). □

Remark. Changing σ to $\sigma\tau$, $\tau \in \Gamma$, in (3.4.1), which will change $\tilde{\alpha}$ to $\tau\tilde{\alpha}$ in its Γ-orbit, now replaces $K_{\tilde{\alpha}}^\times$ with $K_{\tau\tilde{\alpha}}^\times$ and $\alpha^\vee$ with $\alpha^\vee \cdot \tau^{-1}$ from $\tau(K_{\tilde{\alpha}}^\times) = K_{\tau\tilde{\alpha}}^\times$ again into $Z_G(k)\backslash Z_M(k)$. Thus changing $\tilde{\alpha}$ to $\tau\tilde{\alpha}$ will change $\alpha^\vee$ to $\alpha^\vee \cdot \tau^{-1}$, but their images remain the same.

We also need the following lemma. Its proof came out of discussions with Harder and Labesse and, although quite natural, does not seem to have appeared anywhere before. Kottwitz seems to be aware of it. His proof, which is similar to ours, has not yet appeared in any form.

Lemma 3.5. *Let G be a connected reductive group over k. Let M be a Levi subgroup in G. Let Z_M and Z_G be centres of M and G, respectively. If $H^1(k, Z_G) = 1$, then $H^1(k, Z_M) = 1$.*

Proof. Consider the exact sequence

$$1 \to Z_G \to Z_M \to Z_G \backslash Z_M \to 1.$$

It is enough to show that $H^1(k, Z_G\backslash Z_M) = 1$. We therefore need to show that if G is adjoint, then $H^1(k, Z_M) = 1$ for the centre of every Levi subgroup. By induction we may assume P is maximal and let α be the unique simple root whose root group lies in N. Let $X(Z_M)$ and $X(T)$ be the character modules of Z_M and T, $M \supset T$, respectively. Then $X(Z_M) \subset X(T)$ and $X(Z_M)$ is the $\mathbb{Z}$-span of all those roots $\tilde{\alpha} \in X(T)$ which restrict to α. Let $K_{\tilde{\alpha}}$ be the splitting field of $\tilde{\alpha}$; then it is one of Z_M. Moreover, $\Gamma = \mathrm{Gal}(\bar{k}/k)$ permutes the roots $\tilde{\alpha}$ among themselves in one orbit and therefore is an induced Γ-module from a $\Gamma_{\tilde{\alpha}}$-module $\tilde{X}$, where $\Gamma_{\tilde{\alpha}} = \mathrm{Gal}(\bar{k}/K_{\tilde{\alpha}})$. Choose a split torus $\tilde{Z}_M$, defined and split over $K_{\tilde{\alpha}}$, such that $\tilde{X} = X(\tilde{Z}_M)$. Then by [**2**, § 5.1], $Z_M = \mathrm{Res}_{K_{\tilde{\alpha}}/k}\, \tilde{Z}_M$. By Shapiro's lemma,

$$H^1(k, Z_M) = H^1(K_{\tilde{\alpha}}, \tilde{Z}_M) = 1,$$

completing the lemma. □

Remark. The converse is not true as $G = \mathrm{SL}(2)$ gives a counterexample.

To proceed we recall a number of important subgroups inside $U_M(k) = U(k) \cap M(k)$. Given n in $N(k)$ satisfying (3.1), i.e. $w_0^{-1}n = mn'\bar{n}$, we let $U_{M,n}(k)$ be the centralizer of n in $U_M(k)$, i.e.

$$U_{M,n}(k) = \{u \in U_M(k) \mid unu^{-1} = n\}.$$

Observe that $U_{M,n}(k) = U_{M,\bar{n}}(k)$, the later being defined the same way. Next, let

$$U^t_{M,m}(k) = \{u \in U_M(k) \mid w_0(u)mu^{-1} = m\}$$

be the twisted centralizer of m in $U_M(k)$, where $w_0(u) = w_0^{-1}uw_0$. Note that if $U_{M,n}$ and $U^t_{M,m}$ are the corresponding algebraic groups, then the notation is justified since k-points of these groups are precisely $U_{M,n}(k)$ and $U^t_{M,m}(k)$, respectively. Clearly, $U_{M,n}(k) \subset U^t_{M,m}(k)$. Finally, let

$$U'_{M,m}(k) = \{u \in U_M(k) \mid mum^{-1} \in U_M(k) \text{ and } \psi(mum^{-1}) = \psi(u)\}.$$

Moreover, suppose $u \in U^t_{M,m}(k)$; then $w_0(u) = mum^{-1}$ and therefore

$$\psi(mum^{-1}) = \psi(w_0(u)) = \psi(u)$$

by compatibility of ψ and w_0 which is valid by our choices (cf. Proposition 9.3.5 of [**28**]). Thus $u \in U'_{M,m}(k)$. We therefore have

$$U_{M,n}(k) \subseteq U^t_{M,m}(k) \subseteq U'_{M,m}(k).$$

We finally recall the following (Assumption 4.1 of [**26**]).

Assumption 3.6. *Assume $n \in N(k)$ satisfies (3.1). Then except for n in a set of measure zero with respect to $\mathrm{d}n$, $U_{M,n}(k) = U'_{M,m}(k)$.*

Remark. Since we completed the results in this paper, a full proof of Assumption 3.6 has been announced by Sundaravaradhan [**29**]. Granting this, the results that follow, as well as those in [**26**], are now unconditional.

Now assume $m = u_1twu_2$ as in (3.2) with w supporting a Bessel function and $t \in T_w$ (cf. Proposition 3.3). In this case $U'_{M,n}(k)$ has a simple description. (We do not need Assumption 3.6 for this.)

As in [**11**], we let $U^+_{M,w}$ be the subgroup of U_M generated by simple roots which are sent to positive ones by w. Then $U_M = U^+_{M,w}U^-_{M,w}$ and $U^+_{M,w}$ normalizes $U^-_{M,w}$.

Lemma 3.7. *Let $m = u_1twu_2$, $t \in T_w(k)$, $u_2 \in U^-_{M,w}(k)$. Then*

$$U^+_{M,w}(k) = u_2U'_{M,m}(k)u_2^{-1}.$$

This in particular justifies the notation of k-points $U'_{M,m}(k)$ for $U'_{M,m}$.

Proof. If $u \in U'_{M,m}(k)$, then $u_2uu_2^{-1}$ must belong to $U^+_{M,w}(k)$. In fact, $mum^{-1} \in U_M(k)$ implies $wu_2uu_2^{-1}w^{-1} \in U_M(k)$ or $u_2uu_2^{-1} \in U^+_{M,w}(k)$. Thus $u_2U'_{M,m}(k)u_2^{-1} \subset U^+_{M,w}(k)$.

Now suppose $u \in U^+_{M,w}(k)$. Then we need to show $u_2^{-1}uu_2 \in U'_{M,m}(k)$. Consider

$$\begin{aligned} mu_2^{-1}uu_2m^{-1} &= u_1twu_2u_2^{-1}uu_2u_2^{-1}(tw)^{-1}u_1^{-1} \\ &= u_1twu(tw)^{-1}u_1^{-1} \in U_M(k). \end{aligned}$$

Thus

$$\psi(mu_2^{-1}uu_2m^{-1}) = \psi(w(w^{-1}twuw^{-1}t^{-1}w)w^{-1}) = \psi(u)$$

using compatibility and the fact that $w^{-1}tw$ centralizes $U^+_{M,w}(k)$ since w supports the corresponding Bessel function. This completes the lemma. □

Corollary 3.8. *Under Assumption 3.6*

$$U_{M,n}(k) = u_2^{-1}U^+_{M,w}(k)u_2$$

for $m = u_1twu_2$ corresponding to an n satisfying (3.1).

Remark 3.9. It is not clear whether one can even prove $U'_{M,m}(k) = U^+_{M,m}(k)$ for $m = tw$ in general, that is, not satisfying (3.1). In fact, suppose $u \in U^+_{M,m}(k)$ and $w_0(u)twu^{-1} = tw$. Then $w_0(u)w^{-1}(u^{-1})tw = tw$ which requires $w_0wu^{-1}w^{-1}w_0^{-1} = u$. If $\tilde{w}_\ell$ is the longest element of the Weyl group of A_0 in G, then we must have $\tilde{w}_\ell\tilde{w}^\theta_\ell(\alpha') = \alpha'$ for every $\alpha' > 0$ for which $\tilde{w}_\ell \tilde{w}_\ell^\theta(\alpha') > 0$. Although this is true for rank two parabolic subgroups in classical groups, it fails for the M_θ in G, where $G = E_8$ and the derived group of M_θ is of type E_6. Thus, if we consider the pair (E_8, M) with $M_{\mathrm{D}} = E_7$ and assume that an n appears for which $m = u_1 t w u_2$ with $(M_\theta)_{\mathrm{D}} = E_6$, then $U^t_{M,m}(k)$ is strictly smaller than $U'_{M,m}(k)$ and thus the assumption fails. One hopes that such elements in the context of this pair constitute a set of measure zero with respect to $\mathrm{d}n$. This is clearly the case for split groups by [**29**].

We finally need to state a version of Theorem 6.2 of [**26**] that can be used to prove stability for arbitrary quasi-split groups.

We again let Z^0_M denote the image of $\alpha^\vee$, but this time, as we agreed in Lemma 3.4,

$$K^\times_{\tilde{\alpha}} \overset{\alpha^\vee}{\simeq} Z^0_M = Z_G(k)\backslash Z_M(k).$$

We will also use $\psi_{K_{\tilde{\alpha}}}$ to denote $\psi_k \circ \mathrm{Tr}_{K_{\tilde{\alpha}}/k}$ and as in (6.16) of [**26**], we write

$$u_\alpha = \sum_{\tilde{\alpha}} u_{\tilde{\alpha}} \in k$$

(denoted by x_α in [**26**]). Then

$$\psi_{K_{\tilde{\alpha}}}(u_{\tilde{\alpha}}) = \psi_k \cdot \mathrm{Tr}_{K_{\tilde{\alpha}}/k}(u_{\tilde{\alpha}}) = \psi_k(u_\alpha).$$

Next, as in equations (6.33)–(6.36) of [**26**], we have that for $z = \alpha^\vee(t)$

$$q^{\langle s\hat{\alpha}, H_M(z)\rangle} = |\hat{\alpha}(\alpha^\vee(t))|^s_k = |\hat{\alpha}(\alpha^\vee(t))|^{s/[K_{\tilde{\alpha}}:k]}_{K_{\tilde{\alpha}}},$$

where to avoid confusion we use $\hat{\alpha}$ to denote $\langle \rho_P, \alpha\rangle^{-1}\rho_P$ (previously denoted by $\tilde{\alpha}$ in [**24, 26**]). Again, since

$$t \mapsto |\hat{\alpha}(\alpha^\vee(t))|_{K_{\tilde{\alpha}}}$$

is an unramified character of $K^\times_{\tilde{\alpha}}$, we can define $\langle \hat{\alpha}, \alpha^\vee\rangle \in \mathbb{C}$ such that

$$|\hat{\alpha}(\alpha^\vee(t))|_{K_{\tilde{\alpha}}} = |t|^{\langle \hat{\alpha}, \alpha^\vee\rangle}_{K_{\tilde{\alpha}}}$$

and therefore

$$q^{\langle s\hat{\alpha}, H_M(z)\rangle} = |t|^{\langle \hat{\alpha}, \alpha^\vee\rangle s/[K_{\tilde{\alpha}}:k]}_{K_{\tilde{\alpha}}}.$$

Assume now that π is an irreducible admissible ψ-generic representation of M (cf. [**24, 26**]). Let ω_π be its central character. Then $\omega_\pi(w_0\omega_\pi^{-1})$ is a character of $Z_G(k)\backslash Z_M(k)$ which we will consider as one of $K^\times_{\tilde{\alpha}}$ via the isomorphism $\alpha^\vee : K^\times_{\tilde{\alpha}} \simeq Z^0_M$ (cf. Remark 4.11 later). We then use

$$\gamma_{K_{\tilde{\alpha}}}(2\langle \hat{\alpha}, \alpha^\vee\rangle s/[K_{\tilde{\alpha}} : k], \omega_\pi(w_0\omega_\pi^{-1}), \psi_{K_{\tilde{\alpha}}})$$

to denote the corresponding γ-function.

The second part of Theorem 6.2 of [**26**] which is of interest to us can be now reformulated as follows.

Proposition 3.10. *Assume $H^1(k, Z_G) = 1$ and that Assumption 3.6 (Assumption 4.1 of [**26**]) is valid, i.e. $U_{M,n}(k) = U'_{M,m}(k)$, for almost all n satisfying (3.1). Moreover, suppose $\omega_\pi(w_0\omega_\pi^{-1})$ is ramified as a character of $K_{\tilde\alpha}^\times$. Then*

$$C_\psi(s,\pi)^{-1} = \gamma_{K_{\tilde\alpha}}(2\langle\hat\alpha,\alpha^\vee\rangle s/[K_{\tilde\alpha}:k], \omega_\pi(w_0\omega_\pi^{-1}), \psi_{K_{\tilde\alpha}}) \times \int_{Z_M^0 U_M(k)\backslash N(k)} j_{\tilde v,\bar N_0}(m)\omega_{\pi_s}^{-1}(u_{\tilde\alpha})(w_0\omega_{\pi_s})(u_{\tilde\alpha})q^{\langle s\hat\alpha+\rho, H_M(m)\rangle}\,\mathrm{d}n, \quad (3.10.1)$$

*where $j_{\tilde v,\bar N_0}(m) = j_{\tilde v,\bar N_0}(m, y_0)$ with $\mathrm{ord}_{K_{\tilde\alpha}}(y_0) = -d - f$, where d and f are conductors of $\psi_{K_{\tilde\alpha}}$ and $\omega_\pi^{-1}(w_0\omega_\pi)$, respectively. We recall that $j_{\tilde v,\bar N_0}(m, y_0)$ is defined by equations (6.24) and (6.21) of [**26**], i.e.*

$$j_{\tilde v,\bar N_0}(m) = j_{\tilde v,\bar N_0}(m, y_0) = \int_{U_{M,n}(k)\backslash U_M(k)} W_{\tilde v}(mu^{-1})\varphi_{\bar N_0}(zu\bar n u^{-1}z^{-1})\psi(u)\,\mathrm{d}u, \quad (3.10.2)$$

with $z = \alpha^\vee(y_0^{-1}u_{\tilde\alpha})$, $W_{\tilde v}(e) = 1$. Moreover, if $\tilde\alpha$ is the non-restricted simple root used to identify Z_M^0 with $K_{\tilde\alpha}^\times$, then $u_{\tilde\alpha}$ is the coordinate of $w_0^{-1}\bar n w_0$ at the root $\tilde\alpha$ by means of our fixed splitting $\{x_{\tilde\alpha}\}$. The subgroup $\bar N_0$ is a sufficiently large open compact subgroup of $\bar N$ and $\varphi_{\bar N_0}$ is its characteristic function.

Proof. Exactly as in Theorem 6.2 of [**26**]. □

Remark. This reformulation is necessary if we are to use this result to prove stability for non-split quasi-split groups. It is this form of the Mellin transform to which the asymptotic expansion proved in [**11**] can be applied.

To be able to use asymptotics of partial Bessel functions proved in [**11**], we need to show that the domain of integration in the definition of $j_{\tilde v,\bar N_0}(m)$ is independent of m. This was proved in [**10**,**16**] by looking at special matrix presentations or root coordinates. Here we will prove the following general lemma.

Lemma 3.11. *The domain of integration of*

$$j_{\tilde v,\bar N_0}(m) = \int_{U_{M,n}(k)\backslash U_M(k)} W_{\tilde v}(mu^{-1})\varphi_{\bar N_0}(uz\bar n z^{-1}u^{-1})\psi(u)\,\mathrm{d}u \quad (3.11.1)$$

is independent of m, and in fact depends only on $\bar N_0$ and y_0.

Proof. Recall that $z = \alpha^\vee(y_0^{-1}u_{\tilde\alpha})$. Since $\mathrm{ord}_{K_{\tilde\alpha}}(y_0)$ is fixed, we need to consider those u for which $u\alpha^\vee(u_{\tilde\alpha})\bar n\alpha^\vee(u_{\tilde\alpha})^{-1}u^{-1}$ belongs to a fixed open compact subgroup of $\bar N_0$, defined only by y_0 and $\bar N_0$. Write

$$\bar n = \prod_{\tilde\mu} w_0 x_{\tilde\mu}(u_{\tilde\mu})w_0^{-1}\bar n_1, \quad (3.11.2)$$

where the product runs over all non-restricted roots $\tilde\mu$ restricting to α and $\bar n_1$ belongs to the derived group of $\bar N$. Then

$$\alpha^\vee(u_{\tilde\alpha})w_0 x_{\tilde\mu}(u_{\tilde\mu})w_0^{-1}\alpha^\vee(u_{\tilde\alpha})^{-1} = w_0 x_{\tilde\mu}(\tilde\mu(w_0(\alpha^\vee(u_{\tilde\alpha})))u_{\tilde\mu})w_0^{-1}. \quad (3.11.3)$$

Now, by equation (6.28) of [**26**] and equation (3.4.1) here

$$\begin{aligned}\tilde{\mu}(w_0(\alpha^\vee(u_{\tilde{\alpha}}))) &= \tilde{\mu}(\alpha^\vee(u_{\tilde{\alpha}})^{-1}) = \tilde{\mu}(\alpha^\vee(u_{\tilde{\alpha}}^{-1})) \\ &= (\sigma_j\tilde{\alpha})(\alpha^\vee(u_{\tilde{\alpha}}^{-1})) = \sigma_j(u_{\tilde{\alpha}}^{-1}) \\ &= u_{\sigma_j\tilde{\alpha}}^{-1} = u_{\tilde{\mu}}^{-1} \end{aligned} \tag{3.11.4}$$

for some $\sigma_j \in \mathcal{A}_{K_{\tilde{\alpha}}/k}$. By 8.1.12 (2) of [**28**], there exists a non-zero constant $d_{\tilde{\mu}}$, depending on $\tilde{\mu}$ and our splitting giving the Weyl group representatives, such that (3.11.3) equals

$$w_0 x_{\tilde{\mu}}(1) w_0^{-1} = x_{\tilde{\mu}}(d_{\tilde{\mu}}). \tag{3.11.5}$$

We thus conclude that the coordinate of

$$\alpha^\vee(u_{\tilde{\alpha}})\bar{n}\alpha^\vee(u_{\tilde{\alpha}})^{-1}$$

at each root subgroup attached to $w_0(\tilde{\mu})$ is equal to $d_{\tilde{\mu}}$ and is therefore independent of n (and m).

Next, we need to consider

$$u\alpha^\vee(u_{\tilde{\alpha}})\bar{n}\alpha^\vee(u_{\tilde{\alpha}})^{-1}u^{-1} = u x_{w_0(\tilde{\mu})}(d_{\tilde{\mu}})u^{-1}$$

for $u \in U_{M,n}(k)\backslash U_M(k)$. This means to consider

$$x_{\tilde{\gamma}'}(u_{\tilde{\gamma}'})x_{w_0(\tilde{\mu})}(d_{\tilde{\mu}})x_{\tilde{\gamma}'}(u_{\tilde{\gamma}'})^{-1} \tag{3.11.6}$$

as $\tilde{\gamma}'$ runs over non-restricted roots whose root subgroups are in $U_{M,n}\backslash U_M$. Consequently, by Proposition 8.2.3 of [**28**], (3.11.6) equals

$$x_{w_0(\tilde{\mu})}(d_{\tilde{\mu}})\prod x_{iw_0(\tilde{\mu})+j\tilde{\gamma}'}(C_{w_0(\tilde{\mu}),\tilde{\gamma}',i,j}d_{\tilde{\mu}}^i u_{\tilde{\gamma}'}^j), \tag{3.11.7}$$

where the product runs over all positive integers i and j for which $iw_0(\tilde{\mu}) + j\tilde{\gamma}'$ is a root. Here $C_{w_0(\tilde{\mu}),\tilde{\gamma}',i,j}$ is the corresponding structure constant. Since $U_{M,n} = U_{M,\bar{n}}$, which implies that $\tilde{\gamma}'$ and $w_0(\tilde{\mu})$ do not commute, one concludes that at least one term in the product over i and j in (3.11.7) must appear. Consequently, the compactness of $\bar{N}_0$ implies a bound on $|u_{\tilde{\gamma}'}|$, giving a domain of integration for (3.11.1) depending only on $\bar{N}_0$ and y_0. We finally point out that the choice $\tilde{\gamma}'$ among the non-restricted roots restricting to the same root is irrelevant since $x_{\tilde{\gamma}'}$ lies in a compact set if and only if $\sigma(x_{\tilde{\gamma}'})$ does for all $\sigma \in \mathit{\Gamma}$. The lemma is now complete. □

4. The integral representation as a Mellin transform

In this section we shall show that, under certain restrictive assumptions, the integral representation given in equation (3.10.1) can be written as a Mellin transform of a Bessel function [**11**]. The stability of γ-factors then follows as in [**1**, **7**, **10**, **16**]. This will cover all the cases treated so far, as well as that of any quasi-split group whose L-group has a classical derived group [**1**, **9**, **10**, **16**, **27**].

The cases for which stability is proved here for their corresponding γ-factors all satisfy a dimension condition for $U_M\backslash N$ and in fact are among those where $U_M\backslash N$ has the smallest non-trivial dimension. (While the geometry of $U_M\backslash N$ is quite fascinating, in this paper we restrict ourselves to this special case.) Let us now be more precise.

By Rosenlicht's generic quotient theorem (cf. [**15**, **20**, **21**]), N contains a Zariski open dense subset N' such that the quotient $U_M\backslash N'$, the equivalence classes of elements of N' under conjugation by U_M, exists. Then the orbits of U_M in N' are all equidimensional (cf. § 6 of [**3**]) and of maximal possible dimension (Theorem 19.5 and Corollary 19.6 of [**15**]). Moreover, since U_M is connected so is the closed subgroup $U_{M,n}$ for each $n \in N$. On the other hand, since k is perfect, $H^1(k, U_{M,n}) = 1$ and therefore U_M-conjugation and $U_M(k)$-conjugation are the same. (That H^1 is trivial when k is perfect follows from the fact that the consecutive quotients in a composition series for a unipotent group are isomorphic over k to $\bar{k}$.) Therefore,

$$(U_M\backslash N)(k) = U_M(k)\backslash N(k)$$

as well as

$$(U_M\backslash N')(k) = U_M(k)\backslash N'(k).$$

Here $(U_M\backslash N)(k)$ represents the conjugacy classes which have a k-rational representative, i.e. which intersect $N(k)$, while $(U_M\backslash N')(k)$ are the k-points of the quotient variety $U_M\backslash N'$. The k-points of the quotient variety $U_M\backslash N'$ will differ from $(U_M\backslash N)(k)$ only on a set of measure zero.

Proposition 4.1. *Under Assumption 3.6,* $\dim(U_M\backslash N') = \tilde{\ell}(w_0) - \tilde{\ell}(w)$, *where* $w = w_0^{-1}\bar{w}$ *with* $\bar{w}$ *the unique Weyl group element for which* $\bar{C}(\bar{w})$ *intersects* N *in an open subset. Conversely, if* $\dim(U_M\backslash N') = \tilde{\ell}(w_0) - \tilde{\ell}(w)$, *then Assumption 3.6 is valid.*

Proof. The generic stabilizer of elements $n \in N(k)$ is by definition the centralizer $U_{M,n}(k)$. Under Assumption 3.6, Corollary 3.8 gives that $U_{M,n}(k) \simeq U^+_{M,w}(k)$. Note that these are all affine algebraic groups. Hence the dimension of the U_M orbit through a generic n is $\dim(U_M) - \dim(U^+_{M,w}) = \dim(U^-_{M,w})$. But $\dim(U^-_{M,w}) = \tilde{\ell}(w)$. Hence by Corollary 19.6 of [**15**] we have that $\dim(U_M\backslash N') = \dim(N) - \tilde{\ell}(w)$. Since $\dim(N) = \tilde{\ell}(w_0)$ we obtain $\dim(U_M\backslash N') = \tilde{\ell}(w_0) - \tilde{\ell}(w)$ as desired. The converse is a consequence of the inclusion $U_{M,n} \subset U'_{M,n}$ and the connectedness of these unipotent groups. □

We shall now assume that $\dim(U_M\backslash N') = 2$, that is, $\tilde{\ell}(w_0) - \tilde{\ell}(w) = 2$. Equivalently, the quotient manifold $U_M(k)\backslash N'(k)$ will be of dimension 2. For simplicity, we shall write $\dim(U_M\backslash N) = 2$ with the understanding that only $U_M\backslash N'$ has a quotient structure.

To choose a representative for almost all orbits in $U_M\backslash N$ we proceed as follows. Let $M = M(\bar{k})$ act on $\mathfrak{n}(\bar{k})$, the Lie algebra of $N = N(\bar{k})$ by adjoint action and let $V_1 \subset \mathfrak{n}(\bar{k})$ be the subspace obtained by roots restricting to $\tilde{\alpha}$, when considered as roots of the split centre of $M = M(\bar{k})$. Here $\tilde{\alpha}$ is the non-restricted root restricting to α we fixed earlier. Then the action on V_1 is irreducible. Let $\tilde{h}$ be the highest weight of this representation. It will be a non-restricted highest root, restricting to a positive root h. We shall call h

(respectively $\tilde{h}$) the highest root (respectively a highest non-restricted root) restricting to α (respectively $\tilde{\alpha}$). (Note the two different meanings of restrictions.) Since α is simple, α and h will be distinct roots, $\tilde{\alpha}$ and $\tilde{h}$ being the lowest and highest weights respectively in V_1, as long as the rank is larger than one, which we can assume.

We fix a lexicographic order on the positive roots $\tilde{\Phi}^+$ and therefore one-parameter subgroups of U. More precisely, we will choose an order with initial point $\tilde{\alpha}$ and we order the roots in V_1 from $\tilde{\alpha}$ to $\tilde{h}$, which then induces one from α to h.

For $\gamma \in \Phi^+$, let $\tilde{\Phi}_\gamma$ be the set of non-restricted roots restricting to γ. If we fix one $\tilde{\gamma} \in \tilde{\Phi}_\gamma$ then for $u_\gamma \in K_{\tilde{\gamma}}$ we define

$$x_\gamma(u_\gamma) = \prod_{\tilde{\mu} \in \tilde{\Phi}_\gamma} x_{\tilde{\mu}}(u_{\tilde{\mu}}),$$

where $u_{\sigma(\tilde{\gamma})} = \sigma(u_{\tilde{\gamma}})$ for all $\sigma \in \mathrm{Gal}(K_{\tilde{\gamma}}/k)$ and where the product is taken with respect to our fixed lexicographic order on $\tilde{\Phi}^+$. When γ is a simple root whose associated rank one subgroup G^γ is isomorphic to $\mathrm{Res}_{K_{\tilde{\gamma}}/k}\,\mathrm{SL}_2$ this notation is consistent with that of [**11**, § 1.1].

We start with the following lemma.

Lemma 4.2. *Every $u \in U_M(k)$ commutes with $x_h(u_h)$.*

Proof. We shall use the fact that $h + \gamma$ is not a root for any γ whose root group lies in U_M. Clearly, no $h + j\gamma$, $j > 1$, can be a root since $h + \gamma$ is not one (cf. Proposition 9, § 3, Chapter VI of [**4**] for non-reduced root systems). Similarly, no $ih + \gamma$, $i > 1$, can be a root. We now consider the possibility of roots of type i and $j \geqslant 2$. If G is not split, then $2(h + \gamma)$ will not be a root since $h + \gamma$ is not. Other possibilities cannot be non-split. Now, assume G is split. The only possible roots where every simple root appears in their expressions with multiplicity 2 or higher are the highest roots in G_2, F_4 and E_8. It then follows by inspection that there are no positive roots h and γ such that $ih + j\gamma$, $i, j \geqslant 2$, is one of the these highest roots. The lemma now follows from Proposition 8.2.3 of [**28**]. □

Let $n \in N(k)$ be such that $w_0^{-1} n = m n' \bar{n}$ and $m = u_1 t w u_2$. If w supports a Bessel function, then by Proposition 3.3 w must be of the form $w = w_\ell^\Omega w_\ell^\theta$, where $\Omega = \Delta \backslash \{\alpha\}$ and $t \in T_w$.

In this paper we will be only interested in the case where the rank of $Z_G \backslash T_w$ is equal to 2 for the w in Proposition 4.1. In fact, even under the assumption that $\dim(U_M \backslash N) = 2$, this semisimple rank could be bigger than 2. One only needs to consider the Levi subgroup $M = \mathrm{GL}_2 \times \mathrm{GL}_2$ inside $G = \mathrm{GL}_4$. Then $\mathrm{rank}(Z_G \backslash T_w) = 3$, but $\dim(U_M \backslash N) = 2$ (Bruhat decomposition). Note that still $\tilde{\ell}(w_0) - \tilde{\ell}(w) = 2$ and thus $\dim(U_M \backslash N') = \tilde{\ell}(w_0) - \tilde{\ell}(w) = 2$ does not necessarily imply $\mathrm{rank}(Z_G \backslash T_w) = 2$. It should be pointed out that in order to apply the asymptotic formulae proved in [**7**,**11**] for Bessel functions, as to conclude stability for corresponding γ-factors, we need to assume that $\mathrm{rank}(Z_G \backslash T_w) = 2$. It is a fascinating problem to prove a more explicit integral representation for our γ-factors, as those we obtain here in the rank 2 case, for ranks higher than 2.

From this point on we shall assume that the w of Proposition 4.1 satisfies the rank condition $\mathrm{rank}(Z_G \backslash T_w) = 2$. Then $\theta = \Delta \backslash \{\alpha, \beta\}$ for some $\beta \in \Delta$, $\beta \neq \alpha$. Consequently, the parabolic subgroup of $M = M_\Omega$ whose standard Levi subgroup is generated by θ is maximal, thus putting us in the situation considered in [**7**, **11**]. In particular, *this assumption implies that w supports a Bessel function.*

We shall now make the following assumptions.

(4.1) *The parabolic subgroup whose standard Levi subgroup is generated by θ is self-associate in $M = M_\Omega$, or equivalently $w_\ell^\Omega(\theta) = -\theta$ and $w_\ell^\Omega(\beta) = -\beta$.*

(4.2) $w_\ell^\theta(\alpha) = \alpha$.

We expect that if $\dim(U_M\backslash N) = 2$, then (4.2) is automatically satisfied. We, in fact, verify this in a case by case analysis later in § 5. It should be pointed out that in general (4.2) is false. One only needs to consider

$$\Omega = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\} \supset \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\} = \theta$$

for E_6 (Bourbaki's notation [**4**]).

Remark 4.3. Under the assumption that $\operatorname{rank}(Z_G\backslash T_w) = 2$, so that $\Delta = \{\alpha\} \cup \{\beta\} \cup \theta$, then one has

$$\tilde{\ell}(w_0) = |\{\tilde{\eta} > 0 \mid \tilde{\eta}|_{A_M} = m\tilde{\alpha}|_{A_M} \text{ for some } m \in \mathbb{Z}^+\}|$$

and

$$\tilde{\ell}(w) = |\{\tilde{\gamma} > 0 \mid \tilde{\gamma}|_{A_{M_\theta}} = m\tilde{\beta}|_{A_{M_\theta}} \text{ for some } m \in \mathbb{Z}^+\}|.$$

So in the case of non-exceptional groups G, by which we mean that the derived group G_{D} is a quasi-split form of a split group of type A, B, C or D, with $\tilde{\alpha} = \alpha_1$ and $\tilde{\beta} = \alpha_2$ in Bourbaki's numbering, one can compute from this that $\tilde{\ell}(w_0) - \tilde{\ell}(w) = 2$. Hence in these cases, $\operatorname{rank}(Z_G\backslash T_w) = 2$ actually implies that $\dim(U_M\backslash N') = 2$. For exceptional groups this in no longer the case.

Lemma 4.4.

(a) *Assume (4.1) is valid. Then $\tilde{w}_0^2 = \tilde{w}^2 = 1$.*

(b) *Assume both (4.1) and (4.2) are valid. Then $-\tilde{w}_0\tilde{\alpha} = \tilde{h}$, where $\tilde{h}$ is the highest weight in V_1 as in the proof of Lemma 4.2. Moreover, $\tilde{w}\tilde{\alpha} = \tilde{h}$.*

Proof. (a) follows from the fact that $\tilde{w}_\ell|\langle\Omega\rangle = \tilde{w}_\ell^\Omega$ and $\tilde{w}_\ell^\Omega|\langle\theta\rangle = \tilde{w}_\ell^\theta$ since Ω and θ are self-associate in Δ and Ω, respectively. Here $\langle\cdot\rangle$ means the $\mathbb{Z}$-span.

(b) We first show that $\tilde{w}_0\tilde{\alpha} = -\tilde{h}$. In fact, write $\tilde{w}_0\tilde{\alpha} = -\tilde{\alpha} - \tilde{\nu}$, where $\tilde{\nu}$ restricts to an element in $\langle\Omega\rangle$, the $\mathbb{Z}$-span of roots in Ω. Now, given $\tilde{\gamma}$ restricting to an element in Ω, the adjoint action of $x_{\tilde{w}_0\tilde{\gamma}}(1)$ will send the weight space $\tilde{w}_0\tilde{\alpha}$ to $\tilde{w}_0\tilde{\alpha} + \tilde{w}_0\tilde{\gamma} = -\tilde{\alpha} - \tilde{\nu} + \tilde{w}_0\tilde{\gamma}$. This requires $\tilde{\nu} - \tilde{w}_0\tilde{\gamma}$ to be a non-negative $\mathbb{Z}$-linear combination of roots restricting to roots in Ω for all $\tilde{\gamma}$ as above. Thus $\tilde{w}_0\tilde{\alpha} = -\tilde{h}$.

Now consider $\tilde{w}_0\tilde{w}\tilde{\alpha} = \tilde{w}_\ell\tilde{w}_\ell^\theta\tilde{\alpha} = -\tilde{\alpha}$, using (4.2). This implies $\tilde{w}\tilde{\alpha} = -\tilde{w}_0^{-1}\tilde{\alpha} = -\tilde{w}_0\tilde{\alpha} = \tilde{h}$, completing the lemma. □

Lemma 4.5. *Suppose $w_\ell^\theta(\alpha) = \alpha$. Then $\alpha + \gamma$ is not a root for any root γ whose root subgroup is contained in $U_{M,w}^+$, i.e. for any root γ whose root group lies in U_M and for which $w(\gamma) > 0$.*

Proof. Let $W = W(A_0, G)$. Fix a W-invariant inner product $\langle\cdot,\cdot\rangle$ on $X(A_0)_{\mathbb{Q}}$ (cf. [**4**]). Then

$$\langle\alpha,\gamma\rangle = \langle\tilde{w}_0\tilde{w}(\alpha), \tilde{w}_0\tilde{w}(\gamma)\rangle, \tag{4.5.1}$$

where we insist to use Weyl group elements and not their representatives in the group fixed earlier. Then, using $\tilde{w}_0 = \tilde{w}_\ell\tilde{w}_\ell^\Omega$, $\tilde{w} = \tilde{w}_\ell^\Omega\tilde{w}_\ell^\theta$ and $(\tilde{w}_\ell^\Omega)^2 = 1$, we conclude that $\tilde{w}_0\tilde{w} = \tilde{w}_\ell\tilde{w}_\ell^\theta$. Observe that now

$$\tilde{w}_0\tilde{w}(\alpha) = \tilde{w}_\ell(\tilde{w}_\ell^\theta(\alpha)) = \tilde{w}_\ell(\alpha).$$

Since Ω is self-associate $\tilde{w}_\ell(\alpha) = -\alpha$. On the other hand, $\tilde{w}_0\tilde{w}(\gamma) = \eta > 0$. Thus (4.5.1) implies

$$\langle\alpha,\gamma\rangle = -\langle\alpha,\eta\rangle.$$

If $\langle\alpha,\gamma\rangle > 0$, then $\alpha - \gamma$ will be a root which is not possible since α and γ are both positive. (This is valid whether the root system is reduced or not, cf. [**4**].) On the other hand $\langle\alpha,\gamma\rangle < 0$ implies $\alpha - \eta$ is a root which is again absurd. Thus $\langle\alpha,\gamma\rangle = 0$. Now, suppose $\alpha + \gamma$ is a root. Then

$$s_\alpha(\alpha + \gamma) = \gamma - \alpha$$

must be a root, a contradiction. This completes the lemma. □

Corollary 4.6. *$w_\ell^\theta(\alpha) = \alpha$ if and only if $\langle\alpha,\gamma\rangle = 0$ for all roots γ whose root groups are inside U_M and $w(\gamma) > 0$, where $w = w_\ell^\Omega w_\ell^\theta$. Here $\langle\cdot,\cdot\rangle$ is a W-invariant inner product on $X(A_0)_{\mathbb{Q}}$.*

Proof. For the converse write $\tilde{w}_\ell^\theta = \tilde{w}_{\eta_1}\cdots\tilde{w}_{\eta_\ell}$ in the shortest way, where the η_i are (simple) roots in θ. Then

$$\tilde{w}_\ell^\theta(\alpha) = \tilde{w}_{\eta_1}\cdots\tilde{w}_{\eta_{\ell-1}}(\alpha - 2\langle\alpha,\eta_\ell\rangle/\langle\eta_\ell,\eta_\ell\rangle\eta_\ell) = \alpha$$

since $\langle\alpha,\eta_i\rangle = 0$ for $\eta_i \in \theta$. □

Here we need not assume $\dim(U_M\backslash N) = 2$ or $\operatorname{rank}(Z_G\backslash T_w) = 2$ and the result is valid for any $\theta \subset \Omega \subset \Delta$ as long as Ω is self-associate in Δ.

Lemma 4.7. *Assume $\operatorname{rank}(Z_G\backslash T_w) = 2$ and (4.1) and (4.2) are valid. Then for n of the form $n = x_\alpha(u_\alpha)x_h(u_h)$, with $u_\alpha \in K_{\tilde{\alpha}}^\times$ and $u_h \in K_{\tilde{h}}^\times$, satisfying (3.1) we have that Assumption 3.6 holds, i.e. $U_{M,n} = U'_{M,m}$. More precisely $U_{M,n} = U_{M,w}^+$. In particular, u_2 normalizes both $U'_{M,m}$ and $U_{M,w}^+$.*

Proof. By Lemma 4.2 we know that $U_M(k)$ centralizes $x_h(u_h)$. Hence we need only consider the centralizer of $x_\alpha(u_\alpha)$ with $u_\alpha \in K_{\tilde{\alpha}}^\times$. Write $w_0^{-1}n = mn'\bar{n}$ with $m = u_1twu_2$.

We first show that every $u \in U_{M,w}^+$ commutes with $x_\alpha(u_\alpha)$ under the assumptions in the statement of the lemma. It is enough to consider $u = x_\gamma(u_\gamma)$ with $x_\gamma(u_\gamma) \in U_{M,w}^+$.

We already showed that $\alpha+\gamma$ is not a root. To apply Proposition 8.2.3 of [**28**], we need to show no $i\alpha+j\gamma$, $i,j\geqslant 1$, are roots. But this can be argued exactly as in Lemma 4.2. Hence $U^+_{M,w}\subset U_{M,n}$.

On the other hand, we always have $U_{M,n}\subset U'_{M,m}$. By Lemma 3.7 we know that $U'_{M,m}=u_2^{-1}U^+_{M,w}u_2$, so that

$$\dim(U^+_{M,w})\leqslant\dim(U_{M,n})\leqslant\dim(U'_{M,m})=\dim(U^+_{M,w}).$$

Since these are all connected affine algebraic groups, this implies that $U_{M,n}=U'_{M,m}$, i.e. Assumption 3.6 holds.

The fact that u_2 normalizes both $U'_{M,m}$ and $U^+_{M,w}$ then follows from Lemma 3.7 and Corollary 3.8. □

Proposition 4.8. *Under Assumption 3.6, assume in addition that* $\dim(U_M\backslash N)=2$. *Then the set*

$$R=\{x_\alpha(u_\alpha)x_h(u_h)\mid u_\alpha\in K^\times_{\tilde\alpha},\ u_h\in K^\times_{\tilde h}\},$$

is a set of representatives for $U_M(k)\backslash N(k)$, outside a set of measure zero. (Since $i\tilde\alpha+j\tilde h$, $i,j\in\mathbb{Z}^+$, may still be a root, the order of the product needs to be fixed as it is.) If we also allow $u_h=0$, then the set

$$R'=\{x_\alpha(u_\alpha)x_h(u_h)\mid u_\alpha\in K^\times_{\tilde\alpha},\ u_h\in K_{\tilde h}\}$$

will be a set of representatives for $U_M(k)\backslash N'(k)$ when N' is the largest open subset of N giving a quotient structure, again outside a set of measure zero.

We note that one can replace $K^\times_{\tilde\alpha}$ and $K^\times_{\tilde h}$ with their conjugates under $\mathcal{A}_{K_{\tilde\alpha}/k}$ and $\mathcal{A}_{K_{\tilde h}/k}$, respectively, so that the representatives are independent of the choice of basis.

Proof. We have fixed a lexicographic order on the positive roots, and therefore one-parameter subgroups of U, such that the roots in V_1 are ordered from $\tilde\alpha$ to $\tilde h$. This then induces an order from α to h. We must first show that no two different $n=x_\alpha(u_\alpha)x_h(u_h)$ are conjugate by elements of $U_M(k)$. Since $U_M(k)$ centralizes $x_h(u_h)$, we only need to consider conjugation for $x_\alpha(u_\alpha)$.

Write $u\in U_M(k)$ as a product of $x_\eta(u_\eta)$ according to our fixed order for roots η generating U_M using Proposition 8.2.1 of [**28**]. We now appeal to Proposition 8.2.3 of [**28**], applied to non-restricted roots, to conjugate $x_\alpha(u_\alpha)$ by each of $x_\eta(u_\eta)$ to express $ux_\alpha(u_\alpha)u^{-1}$ as a product of $x_\nu(u_\nu)$ as ν runs over roots in N between α and h, or rather roots between $\tilde\alpha$ and $\tilde h$. Since the decomposition of u is unique with no repetition (Proposition 8.2.1 of [**28**]), there will be no two terms belonging to the same root ν between α and h coming from consecutive conjugation by factors in u, as dictated by Proposition 8.2.3 of [**28**]. Consequently, no cancellation will happen and the number of factors in $n=x_\alpha(u_\alpha)x_h(u_h)$ between α and h will increase upon conjugation by a $u\notin U_{M,n}(k)$. It thus cannot equal another such representative. Hence the elements in the set R represent distinct $U_M(k)$ orbits in $N(k)$. Let us identify R with the set of orbits it represents.

Consider now the subgroup $\tilde U_{\alpha,h}$ of N generated by $x_\alpha(u_\alpha)$ and $x_h(u_h)$ with $u_\alpha,u_h\in\bar k$. This is a closed connected affine subgroup of N. Unless $x_\alpha(u_\alpha)$ and $x_h(u_h)$ commute,

$\tilde{U}_{\alpha,h}$ will be of dimension greater than two. (This already happens when $G = \mathrm{Sp}_{2n}$ and $M = \mathrm{GL}_1 \times \mathrm{Sp}_{2n-2}$.) Moreover, an arbitrary element $u \in \tilde{U}_{\alpha,h}(\bar{k})$ can be written as

$$u = x_\alpha(u_\alpha)x_h(u_h)\prod x_\gamma(u_\gamma) \tag{4.8.1}$$

with $u_\alpha, u_h, u_\gamma \in \bar{k}$, where γ runs over all restricted roots which are restrictions of non-restricted roots $\tilde{\gamma}$ of the form $\tilde{\gamma} = i\tilde{\alpha} + j\tilde{h}$ with $i, j \in \mathbb{Z}$, $i, j > 0$. The product is taken with respect to our fixed lexicographic order. Setting $u_\gamma = 0$ for all γ occurring in (4.8.1), we see that the set

$$U_{\alpha,h} = \{x_\alpha(u_\alpha)x_h(u_h) \mid u_\alpha, u_h \in \bar{k}\}$$

is Zariski closed in $\tilde{U}_{\alpha,h}$. Consequently, the subset $U'_{\alpha,h} \subset U_{\alpha,h}$ defined by the conditions $u_\alpha \neq 0$ and $u_h \neq 0$ is locally closed in the affine variety $\tilde{U}_{\alpha,h}$ and is therefore a constructible set. Note that the previous argument is still valid over $\bar{k}$ and therefore the elements in $U'_{\alpha,h}$ represent distinct orbits under U_M conjugation.

By Lemma 4.7 the generic stabilizer of an element $n \in U'_{\alpha,h}$ is $U^+_{M,w}$. Hence, as in Proposition 4.1, the dimension of an U_M orbit through a generic element of $U'_{\alpha,h}$ is $\dim(U_M) - \dim(U^+_{M,w}) = \tilde{\ell}(w)$. Since $U_M = U^+_{M,w}U^-_{M,w}$ we can identify $U^+_{M,w}\backslash U_M \simeq U^-_{M,w}$. Consider the map

$$U'_{\alpha,h} \times U^-_{M,w} \hookrightarrow N$$

sending $(x_\alpha(u_\alpha)x_h(u_h), u^-) \mapsto (u^-)^{-1}x_\alpha(u_\alpha)x_h(u_h)u^-$. Since $U'_{\alpha,h} \times U^-_{M,w}$ is a constructible set, by Chevalley's theorem (cf. [**18**, Chapter 2, § 6]) we know that the closure of its image is a closed subvariety of N and its image contains a Zariski open subset of its closure. Since $\dim(U_{\alpha,h} \times U^-_{M,w}) = 2 + \tilde{\ell}(w) = \dim(N)$ and the map is injective, we see that the image contains a Zariski open subset of N. Hence $\mathrm{Im}(U'_{\alpha,h} \times U^-_{M,w}) \cap N'$ is Zariski open in N'. Composing with the projection map to $U_M\backslash N'$, and using that $R = U'_{\alpha,h}(k)$, we obtain the first statement of the proposition.

For the second statement, note that if we let $U''_{\alpha,h}$ be the larger open subset of $U_{\alpha,h}$ defined only by $u_\alpha \neq 0$, then still the stabilizers in U_M are the same for all elements of $U''_{\alpha,h}$ and the above discussion applies to this set as well. Since $R' = U''_{\alpha,h}(k)$ the second statement of the proposition follows. □

We finally have the following lemma.

Lemma 4.9. *Assume* $\dim(U_M\backslash N) = 2$, $\mathrm{rank}(Z_G\backslash T_w) = 2$ *and that (4.1) and (4.2) are valid. Then the simple root* β *appears in the expression of* h *in terms of simple roots.*

Proof. Since the subgroup generated by

$$\{x_\beta(u_\beta) \mid u_\beta \in K^\times_{\tilde{\beta}}\}$$

is not contained in $U_{M,n} = U^+_{M,w}$, conjugation by $x_{\tilde{\beta}}(1)$ must send a weight space $\tilde{\alpha} + \tilde{\eta}$ in V_1 under the adjoint action to another weight space $\tilde{\alpha} + \tilde{\eta} + \tilde{\beta}$. This must then be of the form $\tilde{h} - \sum h_i\tilde{\gamma}_i$, for some non-restricted simple roots $\tilde{\gamma}_i$ in U_M and integers $k_i \geqslant 0$, $\tilde{h}$ being the highest weight of this action which completes the lemma. □

We shall now proceed to compute the integral in (3.10.1) as a genuine Mellin transform as in [**1, 10, 16**] to which techniques of [**7, 11**] can be applied to prove the stability of γ-factors. We start with the following lemma.

Lemma 4.10. *Fix $\tilde{\beta}$ restricting to β such that it appears in the expression of the highest root $\tilde{h}$ in V_1. Then there exist embeddings $\alpha^\vee$ and $\beta^\vee$ from $\bar{k}^\times$ into $Z_G(\bar{k})\backslash Z_M(\bar{k})$ and $Z_M(\bar{k})\backslash T_w(\bar{k})$, respectively, such that*

$$\sigma\tilde{\alpha}(\alpha^\vee(q)) = \sigma(q), \tag{4.10.1}$$
$$\sigma\tilde{\alpha}(\beta^\vee(r)) = \sigma(r^{-1}) \tag{4.10.2}$$

and

$$\sigma\tilde{\beta}(\beta^\vee(r)) = \sigma(r), \tag{4.10.3}$$

for all $q, r \in \bar{k}^\times$ and $\sigma \in \Gamma$. Assume $H^1(k, Z_G) = 1$. Moreover, assume that the splitting fields $K_{\tilde{\alpha}}$ and $K_{\tilde{\beta}}$ of $\tilde{\alpha}$ and $\tilde{\beta}$ are equal. Then there exist embeddings $\alpha^\vee$ and $\beta^\vee$ from $K_{\tilde{\alpha}}^\times = K_{\tilde{\beta}}^\times$ into $Z_G(k)\backslash Z_M(k)$ and $Z_M(k)\backslash T_w(k)$, respectively, such that (4.10.1), (4.10.2), and (4.10.3) hold for all $q, r \in K_{\tilde{\alpha}}^\times$ and $\sigma \in \Gamma$. Moreover,

$$\alpha^\vee : K_{\tilde{\alpha}}^\times \cong Z_M^0 = Z_G(k)\backslash Z_M(k), \tag{4.10.4}$$
$$Z_M(k) \cong K_{\tilde{\alpha}}^\times \times Z_G(k), \tag{4.10.5}$$

with $\alpha^\vee(K_{\tilde{\alpha}}^\times) \cap Z_G(k) = \{1\}$ while $\alpha^\vee(K_{\tilde{\alpha}}^\times) \subset (Z_G \cap G_{\mathrm{D}})(\bar{k})\backslash(Z_M \cap G_{\mathrm{D}})(\bar{k})$;

$$T_w(k) \cong K_{\tilde{\alpha}}^\times \times K_{\tilde{\beta}}^\times \times Z_G(k) \cong K_{\tilde{\beta}}^\times \times Z_M(k), \tag{4.10.6}$$

with $\beta^\vee(K_{\tilde{\beta}}^\times) \cap Z_M(k) = \{1\}$ while $\beta^\vee(K_{\tilde{\beta}}^\times) \subset (Z_M \cap M_{\mathrm{D}})(\bar{k})\backslash(T_w \cap M_{\mathrm{D}})(\bar{k})$, and

$$(\alpha^\vee, \beta^\vee) : K_{\tilde{\alpha}}^\times \times K_{\tilde{\beta}}^\times \cong Z_G(k)\backslash T_w(k) \tag{4.10.7}$$

as well as

$$K_{\tilde{\alpha}}^\times \times K_{\tilde{\beta}}^\times \cong (Z_G(k)\backslash Z_M(k)) \times (Z_M(k)\backslash T_w(k)), \tag{4.10.8}$$

where

$$Z_M(k)\backslash T_w(k) \cong (Z_G(k)\backslash Z_M(k))\backslash(Z_G(k)\backslash T_w(k)). \tag{4.10.9}$$

Finally, we note that the Weyl group $W(A_0, G)$ acts only on $K_{\tilde{\alpha}}^\times$ and $K_{\tilde{\alpha}}^\times \times K_{\tilde{\beta}}^\times$ in (4.10.5) and (4.10.6), respectively, and leaves $Z_G(k)$ fixed pointwise.

Proof. The map $\alpha^\vee$ is obtained exactly as in Lemma 3.4. For $\beta^\vee$, besides (4.10.2) and (4.10.3), we require again that $\tilde{\gamma}(\beta^\vee(r)) = 1$ for all non-restricted roots $\tilde{\gamma}$ restricting to roots in θ; otherwise the proof proceeds as for Lemma 3.4.

For assertions (4.10.4)–(4.10.9), we appeal to Lemma 3.5 to conclude that $H^1(k, Z_M) = H^1(k, T_w) = 1$. Consequently, we get, just as before, a split exact sequence of k-points

$$1 \to Z_G(k) \to T_w(k) \to Z_G(k)\backslash T_w(k) \to 1,$$

with analogous exact sequences for $Z_G(k)$, $Z_M(k)$ and $Z_G(k)\backslash Z_M(k)$ as well as for $Z_M(k)$, $T_w(k)$ and $Z_M(k)\backslash T_w(k)$.

The map $(\alpha^\vee, \beta^\vee)(q,r) = \alpha^\vee(q)\beta^\vee(r)$ defines a bijection from $K_{\tilde\alpha}^\times \times K_{\tilde\beta}^\times$ onto $Z_G(k)\backslash T_w(k)$, this quotient being two dimensional. Using the previous splittings, we get $T_w(k) \cong K_{\tilde\alpha}^\times \times K_{\tilde\beta}^\times \times Z_G(k)$. The facts that $\alpha^\vee(K_{\tilde\alpha}^\times) \cap Z_G(k) = \beta^\vee(K_{\tilde\beta}^\times) \cap Z_M(k) = \{1\}$ are consequences vanishing of roots in Δ and Ω on $Z_G(k)$ and $Z_M(k)$, respectively.

Finally, to show $\alpha^\vee(K_{\tilde\alpha}^\times) \subset (Z_G \cap G_{\mathrm{D}})(\bar k)\backslash(Z_M \cap G_{\mathrm{D}})(\bar k)$ we only need to observe that

$$\begin{aligned} Z_M^0 = Z_G(k)\backslash Z_M(k) &= (Z_G\backslash Z_M)(k) \\ &= (Z_G \cap G_{\mathrm{D}}\backslash Z_M \cap G_{\mathrm{D}})(k) \\ &\subset (Z_G \cap G_{\mathrm{D}})(\bar k)\backslash(Z_M \cap G_{\mathrm{D}})(\bar k), \end{aligned}$$

similarly for $\beta^\vee(K_{\tilde\beta}^\times) \subset (Z_M \cap M_{\mathrm{D}})(\bar k)\backslash(T_w \cap M_{\mathrm{D}})(\bar k)$. □

Remark 4.11. Writing $Z_M(k) \cong K_{\tilde\alpha}^\times \times Z_G(k)$, we can write the central character ω_π of π as $\omega_\pi = \eta \otimes \omega_{\pi,0}$, via

$$\omega_\pi((\alpha^\vee(q), z)) = \eta(q)\omega_{\pi,0}(z), \tag{4.3}$$

realizing $K_{\tilde\alpha}^\times$ in $Z_M(k)$ through $\alpha^\vee$.

We shall now add the equality

$$K_{\tilde\alpha} = K_{\tilde\beta} \tag{4.4}$$

as in Lemma 4.10 to the list of our assumptions.

To write out integral (3.10.1) as a genuine Mellin transform, we need to replace the domain of integration $Z_M^0 U_M(k)\backslash N(k)$ by integration over a torus (see [**6**,**10**]). By Proposition 4.8, for purposes of integration we can replace $U_M(k)\backslash N(k)$ by the set R since this differs from $U_M(k)\backslash N(k)$ by a set of measure zero. To proceed we will choose a base point $n_0 \in R$ and consider the space of conjugates of n_0 under $T_w(\bar k)$, or more precisely, under elements of the form $b = \alpha^\vee(q)\beta^\vee(r) \in T_w(\bar k)$. As we will see, this space will be two dimensional and since $\dim(U_M\backslash N) = 2$ this will then cover $U_M(k)\backslash N(k)$ up to a set of measure zero, Hence we will replace the domain of integration $Z_M^0 U_M(k)\backslash N(k)$ by the domain $(Z_M^0\backslash T_w(\bar k) \cdot n_0) \cap R$.

Without loss of generality, we may assume our base point $n_0 \in N$ satisfies (3.1), since this is an open condition, which we emphasize here as

$$w_0^{-1} n_0 = m_0 n_0' \bar n_0. \tag{4.5}$$

We may assume that we have m_0 of the form $m_0 = w t_0 u_2$. As we shall see, there is no loss of generality in assuming $t_0 \in Z_G(k) \cap G_{\mathrm{D}}(k)$ which is a finite set.

For each $n \in N$ satisfying (3.1), i.e. $w_0^{-1} n = m n' \bar n$, let $u_{\tilde\alpha} \in K_{\tilde\alpha}$ be the coordinate such that $x_{\tilde\alpha}(u_{\tilde\alpha})$ appears in the decomposition of $w_0^{-1}\bar n w_0$ as in Proposition 8.2.1 of [**28**]. We may assume $u_{\tilde\alpha} \in K_{\tilde\alpha}^\times$. (This is one of the ingredients going into equation (3.10.1) of Proposition 3.10.) We then use $u_{\tilde\alpha,0}$ to denote the corresponding coordinate appearing in the decomposition of $w_0^{-1}\bar n_0 w_0$.

Writing $n = bn_0b^{-1}$, (4.5) implies

$$w_0^{-1}n = w_0(b)m_0b^{-1} \cdot bn_0'b^{-1} \cdot b\bar{n}_0b^{-1}, \tag{4.6}$$

where $w_0(b) = w_0^{-1}bw_0$. Thus, under such conjugation, m_0 will change to $m = w_0(b)m_0b^{-1}$. Moreover,

$$w_0^{-1}b\bar{n}_0b^{-1}w_0 = w_0(b)[w_0^{-1}\bar{n}_0w_0]w_0(b)^{-1} \tag{4.7}$$

implies that $u_{\tilde{\alpha},0}$ will change to $\tilde{\alpha}(w_0(b))u_{\tilde{\alpha},0}$ under this conjugation.

Lemma 4.12. *Suppose $b = \alpha^\vee(q)\beta^\vee(r)$. Let $n_{\tilde{\beta}} = n_\beta$ be the multiplicity of $\tilde{\beta}$ in $-w_0\tilde{\alpha} = \tilde{h}$, which is independent of the choice of $\tilde{\alpha}$. Then*

$$\tilde{\alpha}(w_0(b)) = q^{-1}r^{1-n_\beta} \tag{4.12.1}$$

and therefore, under conjugation by b, $u_{\tilde{\alpha},0}$ will transform to $q^{-1}r^{1-n_\beta}u_{\tilde{\alpha},0}$.

Proof. We need to calculate $\tilde{\alpha}(w_0(\alpha^\vee(q))w_0(\beta^\vee(r)))$. First,

$$\tilde{\alpha}(w_0(\alpha^\vee(q))) = \tilde{w}_0\tilde{\alpha}(\alpha^\vee(q)) = \tilde{w}_\ell\tilde{w}_\ell^\Omega\tilde{\alpha}(\alpha^\vee(q)). \tag{4.12.2}$$

Write $\tilde{w}_\ell^\Omega\tilde{\alpha} = \tilde{\alpha} + \tilde{\nu}$, with $\tilde{\nu}$ restricting to an element in the $\mathbb{Z}$-span $\langle\Omega\rangle$ of roots in Ω. Then $\tilde{w}_\ell(\tilde{\alpha} + \tilde{\nu}) = -\tilde{\alpha} - \tilde{\mu}$ with again $\tilde{\mu}$ restricting to an element in $\langle\Omega\rangle$. Now

$$(-\tilde{\alpha} - \tilde{\mu})(\alpha^\vee(q)) = (-\tilde{\alpha})(\alpha^\vee(q)) = q^{-1}$$

since all the roots in Ω act trivially on Z_M^0.

For $\tilde{\alpha}(w_0(\beta^\vee(r)))$ we note that $\tilde{w}_0\tilde{\alpha} = -\tilde{\alpha} - \tilde{\mu}$ and thus

$$\begin{aligned}(\tilde{w}_0\tilde{\alpha})(\beta^\vee(r)) &= (-\tilde{\alpha} - \tilde{\mu})(\beta^\vee(r)) = r\tilde{\mu}(\beta^\vee(r))^{-1} \\ &= r \cdot r^{-n_\beta} = r^{1-n_\beta},\end{aligned}$$

completing the lemma, except that we need to observe that $n_{\tilde{\beta}} = n_\beta$ is independent of the choice of $\tilde{\alpha}$ (and thus $\tilde{\beta}$). For that, note that $\sigma\tilde{w}_0 = \tilde{w}_0$ for all $\sigma \in \Gamma$. Thus $\tilde{w}_0(\sigma\tilde{\alpha}) = \sigma\tilde{w}_0(\sigma\tilde{\alpha}) = \tilde{w}_0\tilde{\alpha}$. Consequently, $\tilde{\beta}$ and $\sigma\tilde{\beta}$ appear in $-\tilde{w}_0\tilde{\alpha}$ and $-\tilde{w}_0(\sigma\tilde{\alpha})$, respectively, with the same multiplicities. □

The next ingredient we need to compute is $m = w_0(b)m_0b^{-1}$. We may assume $m_0 = wt_0u_{2,0}$. Then

$$\begin{aligned}m = w_0(b)m_0b^{-1} &= w \cdot (w_0w)(b)t_0b^{-1} \cdot bu_{2,0}b^{-1} \\ &= w \cdot (w_0w)(b)b^{-1}t_0 \cdot bu_{2,0}b^{-1} \\ &= wtu_2\end{aligned} \tag{4.8}$$

with $t = (w_0w)(b)b^{-1}t_0$ and $u_2 = bu_{2,0}b^{-1}$. We shall prove the following lemma.

Lemma 4.13. *Under assumptions (4.1), (4.2) and (4.4), for* $b = \alpha^\vee(q)\beta^\vee(r)$ *we have*

$$w_0 w(b) = b^{-1} \tag{4.13.1}$$

and hence $t = b^{-2}t_0$.

Proof. Clearly, (4.1) implies that $\tilde{w}_\ell(\beta) = -\beta$. In fact, since M is self-associate in G, $\tilde{w}_\ell \mid \langle \Omega \rangle = \tilde{w}_\ell^\Omega$ which implies $\tilde{w}_\ell(\beta) = -\beta$ by self-associativity of θ in Ω. Here $\langle \Omega \rangle$ is the $\mathbb{Z}$-span of Ω. We shall now compute the effect of roots on $w_0 w(b)$. We get

$$\begin{aligned}\tilde{\alpha}(\tilde{w}_0\tilde{w}(b)) &= \tilde{\alpha}(\tilde{w}_\ell \tilde{w}_\ell^\theta(b)) = \tilde{w}_\ell \tilde{w}_\ell^\theta \tilde{\alpha}(b)\\ &= \tilde{w}_\ell \tilde{\alpha}(b) = (-\tilde{\alpha})(\alpha^\vee(q)\beta^\vee(r))\\ &= q^{-1}r\end{aligned}$$

using $\tilde{w}_\ell^\theta \tilde{\alpha} = \tilde{\alpha}$ implied by (4.2).

Next we have

$$\tilde{\beta}(w_0 w(b)) = \tilde{w}_\ell \tilde{w}_\theta^\ell \tilde{\beta}(b) = \tilde{w}_\ell(\tilde{w}_\theta^\ell \tilde{\beta})(b).$$

Write $\tilde{w}_\theta^\ell \tilde{\beta} = \tilde{\gamma}$. Then $\tilde{\gamma}$ restricts to a root in the $\mathbb{Z}$-span $\langle \Omega \rangle$ of Ω. Write $\tilde{\gamma} = \tilde{\beta} + \tilde{\mu}$, where $\tilde{\mu}$ is in the $\mathbb{Z}$-span $\langle \theta \rangle$ of θ. Then

$$\tilde{w}_\ell(\tilde{\gamma}) = \tilde{w}_\ell(\tilde{\beta}) + \tilde{w}_\ell(\tilde{\mu}) = -\tilde{\beta} + \tilde{w}_\ell(\tilde{\mu}).$$

Note that $\tilde{w}_\ell(\tilde{\mu})$ is now a $\mathbb{Z}$-linear combination of simple roots restricting to roots in θ. Consequently,

$$\tilde{\beta}(w_0 w(b)) = (-\tilde{\beta})(\alpha^\vee(q)\beta^\vee(r)) = r^{-1}.$$

All other simple roots act trivially. We therefore can write $w_0 w(b) = \alpha^\vee(q_1)\beta^\vee(r_1)$. Computing $\tilde{\alpha}$ and $\tilde{\beta}$ on $w_0 w(b)$, one gets $r_1 = r^{-1}$ and $q_1 = q^{-1}$. This completes the lemma. □

Next, we need to compute $j_{\tilde{v},\bar{N}_0}(m)$ for $m = wtu_2 = wb^{-2}t_0u_2$. We may and will assume that $u_2 \in U_{M,w}^-$. Recall that $n_0 = x_\alpha(u_{\alpha,0})x_h(u_{h,0})$, $u_\alpha \in K_{\tilde{\alpha}}^\times$, $u_h \in K_{\tilde{h}}^\times$ satisfying (4.5). Then

$$j_{\tilde{v},\bar{N}_0}(m) = \int_{U_{M,n}(k)\backslash U_M(k)} W_{\tilde{v}}(wtu_2u^{-1})\varphi_{\bar{N}_0}(uz\bar{n}z^{-1}u^{-1})\psi(u)\,du. \tag{4.9}$$

We need the following lemma.

Lemma 4.14. *Under assumptions (4.1) and (4.2),* u_2 *belongs to the derived group of* U_M. *In particular* $\psi(uu_2^{-1}) = \psi(u)$. *Moreover,* u_2 *centralizes* $U_{M,w}^+(k) = U_{M,n}(k) = U'_{M,m}(k)$.

Proof. Since $w_\ell^\theta(\alpha) = \alpha$, Corollary 4.6 implies that $\langle \alpha, \gamma \rangle = 0$ for all $\gamma \in \theta$. If every root in θ is perpendicular to β, then Δ will be reducible with the relevant part being the rank two system generated by α and β since P is maximal. They can be handled case by case if need be. They will not be of interest to us. Thus we may assume there exists a simple root $\delta \in \theta$ such that $\langle \beta, \delta \rangle \neq 0$. Since β and δ are both simple this implies that $\beta + \delta$ is a root. Let $u = x_\delta(u_\delta) \in U^+_{M,w}(k) = U_{M,n}(k)$, $u_\delta \neq 0$ (cf. Lemma 3.7 and Lemma 4.7). Since by Lemma 4.7, u_2 normalizes $U^+_{M,w}(k)$, $u_2^{-1}uu_2 \in U^+_{M,w}(k)$ and consequently

$$u_2^{-1}uu_2u^{-1} \in U^+_{M,w}(k).$$

But $uu_2u^{-1} \in U^-_{M,w}(k)$ since $U^+_{M,w}(k)$ normalizes $U^-_{M,w}(k)$, from which we conclude that $u_2^{-1}uu_2u^{-1} = 1$ or $uu_2 = u_2u$. This is in fact true for any u in $U^+_{M,w}(k)$ and thus u_2 centralizes $U^+_{M,w}(k)$.

Since $u_2 \in U^-_{M,w}$, we can write $u_2 = u_1'u_2'$, where $u_1' = \prod_{\tilde\beta} x_{\tilde\beta}(u_{\tilde\beta})$ with the product over roots restricting to the simple root β, and u_2' in the derived group of U_M. Since $\beta+\delta$ is a root, Propositions 8.2.1 and 8.2.3 of [**28**] imply that if u and u_2 commute, then at least $u_{\tilde\beta} = 0$ for all $\tilde\beta$. Thus $u_2 = u_2'$, completing the lemma. □

Using Lemma 4.14, we can write (4.9) as

$$j_{\tilde v,\bar N_0}(wtu_2) = \int_{U_{M,n}(k)\backslash U_M(k)} W_{\tilde v}(wtu^{-1})\varphi_{\bar N_0}(uzu_2\bar n u_2^{-1}z^{-1}u^{-1})\psi(u)\,\mathrm{d}u \tag{4.10}$$

or

$$j_{\tilde v,\bar N_0}(wtu_2) = \int_{U_{M,n}(k)\backslash U_M(k)} W_{\tilde v}(wtu^{-1})\varphi_{\bar N_0}(uz\bar n' z^{-1}u^{-1})\psi(u)\,\mathrm{d}u, \tag{4.11}$$

where $\bar n' = u_2\bar n u_2^{-1}$. We set

$$j'_{\tilde v,\bar N_0}(wt) = \int_{U_{M,n}(k)\backslash U_M(k)} W_{\tilde v}(wtu^{-1})\varphi_{\bar N_0}(uz\bar n' z^{-1}u^{-1})\psi(u)\,\mathrm{d}u. \tag{4.12}$$

We observe that since the $x_{w_0(\tilde\mu)}$-coordinates of $\bar n$ and $\bar n'$, as $\tilde\mu$ runs over roots restricting to α, are the same, (3.11.3) will be the same for $\bar n$ and $\bar n'$ and thus Lemma 3.11 applies. Consequently, $j'_{\tilde v,\bar N_0}(wt)$ becomes another partial Bessel function, replacing $j_{\tilde v,\bar N_0}(m)$.

Remark 4.15. This is a generalization of the case of symplectic groups (as well as unitary ones [**16**]) discussed by means of equation (7.20)–(7.25) in pp. 2115, 2116 of [**26**]. In view of Lemma 3.11 one need not know the precise coordinates of $\bar n' = u_2\bar n u_2^{-1}$, $H\bar n H^{-1}$ in the notation of [**26**], given in the case studied there.

Now consider

$$j_{\tilde v,\bar N_0}(m) = j'_{\tilde v,\bar N_0}(w\alpha^\vee(q^{-2})\beta^\vee(r^{-2})t_0). \tag{4.13}$$

We in fact have the following lemma.

Lemma 4.16. *With assumptions the same as in Lemma 4.13 one has*

$$j_{\tilde v,\bar N_0}(m) = j'_{\tilde v,\bar N_0}(w\beta^\vee(r^{-2})t_0)\eta(q^{-2}),$$

where $j_{\tilde v,\bar N_0}$ and η are defined by (3.10.2) and (4.3), respectively.

Proof. Using (4.13) we need to determine $w\alpha^\vee(q^{-2})w^{-1}$. Since $\alpha^\vee(q^{-2})$ is in $Z_M(k)$ and since w is in the Weyl group of T in M, $w\alpha^\vee(q^{-2})w^{-1} = \alpha^\vee(q^{-2})$. The lemma now follows by Remark 4.11. □

Corollary 4.17. *One has*

$$j_{\tilde{v},\bar{N}_0}(m)\omega_\pi^{-1}(u_{\tilde{\alpha}})(w_0\omega_\pi)(u_{\tilde{\alpha}}) = j'_{\tilde{v},\bar{N}_0}(w\beta^\vee(r^{-2})t_0)\eta^{n_\beta-1}(r^2)\eta(u_{\tilde{\alpha},0}^{-2}),$$

where $u_{\tilde{\alpha},0}$ is the corresponding $u_{\tilde{\alpha}}$ for n_0.

Proof. We only need to calculate

$$\omega_\pi^{-1}(u_{\tilde{\alpha}})w_0\omega_\pi(u_{\tilde{\alpha}})$$

which is easily checked, using Lemma 4.12, to equal

$$\omega_\pi^{-2}(\alpha^\vee(q^{-1}r^{1-n_\beta}u_{\tilde{\alpha},0})) = \eta(q^2)\eta^{n_\beta-1}(r^2)\eta(u_{\tilde{\alpha},0}^{-2}),$$

proving the corollary. □

We now prove the following lemma.

Lemma 4.18. *We have*

$$j'_{\tilde{v},\bar{N}_0}(w\beta^\vee(r^{-2})t_0) = j'_{\tilde{v},\bar{N}_0}(\beta^\vee(r^2)wt_0)\eta^{2-n_\beta}(r^2). \tag{4.18.1}$$

Proof. We need to calculate

$$\tilde{\beta}(w\beta^\vee(r^{-2})w^{-1}) = w^{-1}\tilde{\beta}(\beta^\vee(r^{-2})).$$

Note that

$$\begin{aligned}\tilde{w}^{-1}\tilde{\beta} &= (\tilde{w}_\ell^\theta)^{-1}(\tilde{w}_\ell^\Omega)^{-1}\tilde{\beta}\\ &= (\tilde{w}_\ell^\theta)^{-1}(-\tilde{\beta}) = -(\tilde{w}_\ell^\theta)^{-1}(\tilde{\beta}) = -(\tilde{\beta}+\tilde{\nu}_1),\end{aligned}$$

where $\tilde{\nu}_1$ restricts to a non-negative $\mathbb{Z}$-linear combination of roots in θ. Thus

$$\tilde{\beta}(w\beta^\vee(r^{-2})w^{-1}) = (-\tilde{\beta})(\beta^\vee(r^{-2})) = r^2.$$

Moreover, by Lemma 4.4,

$$\begin{aligned}\tilde{\alpha}(w\beta^\vee(r^{-2})w^{-1}) &= \tilde{w}\tilde{\alpha}(\beta^\vee(r^{-2})) = \tilde{h}(\beta^\vee(r^{-2}))\\ &= \tilde{\alpha}(\beta^\vee(r^{-2}))\tilde{\beta}(\beta^\vee(r^{-2}))^{n_\beta} = r^2r^{-2n_\beta}.\end{aligned}$$

In view of the last statement in Lemma 4.10 this implies that

$$w\beta^\vee(r^{-2})w^{-1} = \alpha^\vee(r^{4-2n_\beta})\beta^\vee(r^2), \tag{4.18.2}$$

from which we conclude

$$j'_{\tilde{v},\bar{N}_0}(w\beta^\vee(r^{-2})t_0) = j'_{\tilde{v},\bar{N}_0}(\beta^\vee(r^2)wt_0)\eta^{2-n_\beta}(r^2),$$

completing the lemma. □

Corollary 4.19. *With assumptions as before one has*

$$j_{\tilde{v},\bar{N}_0}(m)\omega_\pi^{-1}(u_{\tilde{\alpha}})w_0\omega_\pi(u_{\tilde{\alpha}}) = j'_{\tilde{v},\bar{N}_0}(\beta^\vee(r^2)wt_0)\eta(r^2)\eta(u_{\tilde{\alpha},0}^{-2}).$$

Proof. Combine Corollary 4.17 and Lemma 4.18. □

We now need to calculate the modulus characters in (3.10.1) of Proposition 3.10.

Lemma 4.20. *With assumptions as before, let $m = w\alpha^\vee(q^{-2})\beta^\vee(r^{-2})t_0u_2$ with t_0 taken in $Z_G(k)\cap G_{\mathrm{D}}(k)$. Let n_α be the multiplicity of α in 2ρ, sum of the positive roots in N. Then*

(a) $$q^{-\langle s\hat{\alpha},H_M(\alpha^\vee(u_{\tilde{\alpha}}))\rangle+\langle s\hat{\alpha},H_M(w_0^{-1}\alpha^\vee(u_{\tilde{\alpha}})w_0)\rangle+\langle s\hat{\alpha},H_M(m)\rangle} = |r^2|^{n_\alpha s/2\langle\rho,\alpha\rangle}|u_{\tilde{\alpha},0}^2|^{-n_\alpha s/2\langle\rho,\alpha\rangle}, \quad (4.20.1)$$

where $m_0 = wt_0u_{2,0}, w_0^{-1}n_0 = m_0n_0'\bar{n}_0$ is a fixed representative with $u_{\tilde{\alpha},0}$ the corresponding $u_{\tilde{\alpha}}$ in $\bar{n}_0$;

(b) *suppose $\langle\tilde{\beta},\tilde{\alpha}\rangle = -1$; then (4.20.1) is equal to $|r^2|^s|u_{\tilde{\alpha},0}|^{-2s}$.*

Proof. (a) Recall that (cf. [**26**])

$$\omega_{\pi_s} = \omega_\pi\otimes q^{\langle s\hat{\alpha},H_M(\cdot)\rangle}.$$

By Lemma 4.12

$$u_{\tilde{\alpha}} = q^{-1}r^{1-n_\beta}u_{\tilde{\alpha},0}. \quad (4.20.2)$$

Using (4.20.2) and (4.18.2), and the fact that $H_M(KU_M(k)) = 1$, we calculate the left-hand side of (4.20.1) as q^ℓ, where

$$\begin{aligned}\ell &= \langle s\hat{\alpha},H_M(\alpha^\vee(r^{4-2n_\beta})\alpha^\vee(q^{-2}))\rangle - 2\langle s\hat{\alpha},H_M(\alpha^\vee(q^{-1})\alpha^\vee(r^{1-n_\beta})\alpha^\vee(u_{\tilde{\alpha},0}))\rangle\\ &= \langle s\hat{\alpha},H_M(\alpha^\vee(r^{4-2n_\beta})\alpha^\vee(q^{-2})\alpha^\vee(q^2)\alpha^\vee(r^{2n_\beta-2})\alpha^\vee(u_{\tilde{\alpha},0}^{-2}))\rangle\\ &= \langle s\hat{\alpha},H_M(\alpha^\vee(r^2u_{\tilde{\alpha},0}^{-2}))\rangle \end{aligned} \quad (4.20.3)$$

from which (4.20.1) is equal to

$$|2\rho(\alpha^\vee(r^2u_{\tilde{\alpha},0}^{-2}))|^{s/2\langle\rho,\alpha\rangle}. \quad (4.20.4)$$

Here, we have applied Lemma 4.10 to conclude that $\langle s\hat{\alpha},H_M(\beta^\vee(r^2))\rangle = 0$ since $\beta^\vee(r^2)\in(Z_M\cap M_{\mathrm{D}})(\bar{k})\backslash(T_w\cap M_{\mathrm{D}})(\bar{k})$. We note that $\langle s\hat{\alpha},H_M(t_0)\rangle = 0$ since t_0 is central.

Write

$$2\rho = \sum_{\tilde{\alpha}|\alpha} n_{\tilde{\alpha}}\tilde{\alpha} + \sum_{\tilde{\gamma}|\gamma\in\Omega} n_{\tilde{\gamma}}\tilde{\gamma},$$

where the first sum runs over all the non-restricted $\tilde{\alpha}$ restricting to α and the second over non-restricted $\tilde{\gamma}$ restricting to roots in Ω. It then follows from Lemma 4.10 that (4.20.4) is equal to

$$|r^2|^{n_\alpha s/2\langle\rho,\alpha\rangle}|u_{\tilde{\alpha},0}^2|^{-n_\alpha s/2\langle\rho,\alpha\rangle}, \quad (4.20.5)$$

as desired.

(b) Suppose $\langle\tilde\beta,\tilde\alpha\rangle = -1$. Let

$$\tilde t = \alpha^\vee(t)\beta^\vee(t)H_{\tilde\alpha}(t)^{-1},$$

where $H_{\tilde\alpha}$ is the standard coroot at $\tilde\alpha$. Then

$$\begin{aligned}\tilde\alpha(\tilde t) &= \tilde\alpha(\alpha^\vee(t))\tilde\alpha(\beta^\vee(t))\tilde\alpha(H_{\tilde\alpha}(t)^{-1})\\ &= t\cdot t\cdot t^{-2} = 1.\end{aligned}$$

Similarly,

$$\tilde\beta(\tilde t) = t^{-1}t^{-\langle\tilde\beta,\tilde\alpha\rangle} = 1.$$

Since $w_\theta(\alpha) = \alpha$, $\langle\tilde\gamma,\tilde\alpha\rangle = 0$ for all $\tilde\gamma$ restricting to $\gamma\in\theta$ and $\tilde\gamma(\alpha^\vee(t)\beta^\vee(t^{-1})) = 1$, and thus $\tilde\gamma(\tilde t) = 1$. Consequently, $\alpha^\vee(t)\beta^\vee(t^{-1})H_{\tilde\alpha}(t)^{-1}\in Z_G$. Thus

$$2\rho(\alpha^\vee(t)\beta^\vee(t)) = 2\rho(H_\alpha(t)).$$

□

Now by Lemma 4.10, $\beta^\vee(K_{\tilde\beta}^\times)\cap Z_M(k) = \{1\}$ and $\beta^\vee(K_{\tilde\beta}^\times)$ sits inside a quotient of the derived group of $M(\bar k)$ on which ρ, being a character of $M(\bar k)$, acts trivially. Thus $2\rho(\alpha^\vee(t)) = 2\rho(H_\alpha(t))$ and consequently (4.20.5) equals

$$\begin{aligned}|2\rho(\alpha^\vee(r^2u_{\tilde\alpha,0}^{-2}))|^{s/2\langle\rho,\alpha\rangle} &= |2\rho(H_\alpha(r^2u_{\tilde\alpha,0}^{-2}))|^{s/\langle2\rho,\alpha\rangle}\\ &= |r^2u_{\tilde\alpha,0}^{-2}|^s,\end{aligned}$$

as desired.

Remark 4.21. In all the cases studied in [**1**,**9**,**10**,**16**] we have $\langle\tilde\beta,\tilde\alpha\rangle = -1$ and therefore case (b) is always valid.

Finally, we need to compute the invariant measure

$$q^{\langle\rho,H_M(m)\rangle}\,\mathrm{d}n, \tag{4.14}$$

for $m = wtu_2 = \alpha^\vee(q^{-2})w\beta^\vee(r^{-2})t_0u_2$. Again as before by (4.18.2) and the discussion below (4.20.4)

$$q^{\langle\rho,H_M(m)\rangle}\,\mathrm{d}n = q^{\langle\rho,H_M(\alpha^\vee(r^{4-2n_\beta})\alpha^\vee(q^{-2}))\rangle}\,\mathrm{d}n. \tag{4.15}$$

As before, (4.15) is equal to

$$|2\rho(\alpha^\vee(r^{4-2n_\beta}q^{-2}))|^{1/2}\,\mathrm{d}n = |r^{4-2n_\beta}q^{-2}|^{n_\alpha/2}\,\mathrm{d}n. \tag{4.16}$$

Using conjugation by $b = \alpha^\vee(q)\beta^\vee(r)$ as before and the fact that $\tilde h = -w_0\tilde\alpha$, equations (4.10.1) and (4.10.2) imply that our representative can now be written as

$$x_\alpha(qr^{-1}u_{\tilde\alpha,0})x_h(qr^{n_\beta-1}u_{\tilde h,0}). \tag{4.17}$$

To integrate over $Z_M^0 U_M(k)\backslash N(k)$, we may assume $qr^{-1} = 1$ or $q = r$. Our representative (4.17) will then reduce to

$$x_\alpha(u_{\tilde\alpha,0})x_h(r^{n_\beta}u_{\tilde h,0}). \tag{4.18}$$

The measure $\mathrm{d}n$ is then the additive measure on the variable $r^{n_\beta}u_{\tilde h,0}$, i.e.

$$\mathrm{d}n = |u_{\tilde h,0}|\,|r^{n_\beta}|\,\mathrm{d}^\times(r^{n_\beta}).$$

We can now write (4.16) as

$$|u_{\tilde h,0}|\,|r^{2-2n_\beta}|^{n_\alpha/2}|r^{n_\beta}|\,\mathrm{d}^\times(r^{n_\beta}) = |u_{\tilde h,0}|\,|r^{n_\alpha+n_\beta-n_\alpha n_\beta}|\,\mathrm{d}^\times(r^{n_\beta}). \tag{4.19}$$

To continue, we make our final hypothesis

$$n_\beta = 2. \tag{4.20}$$

Then (4.19) equals

$$|r^2|^{1-(n_\alpha/2)}\,\mathrm{d}^\times(r^2).$$

We now gather everything together, using all the lemmas and corollaries proved in this section, as well as assumptions (4.1), (4.2), (4.4) and (4.20), to conclude that the integral in (3.10.1) can be written as

$$\eta(u_{\tilde\alpha,0})^{-2}|u_{\tilde\alpha,0}|^{-2s}|u_{\tilde h,0}|\int_{r^2\in K_{\tilde\alpha}^\times} j'_{\tilde v,\bar N_0}(\beta^\vee(r^2)wt_0)\eta(r^2)|r^2|^{n_\alpha s/\langle 2\rho,\alpha\rangle}|r^2|^{-(n_\alpha/2)+1}\,\mathrm{d}^\times(r^2), \tag{4.21}$$

since $K_{\tilde h} = K_{\tilde\alpha}$ by Lemma 4.4 (b). The function j' is defined in (4.12).

The integration is over $r^2 \in K_{\tilde\alpha}^\times$. In fact, as we recall from the statement of Lemma 4.12, $qr^{n_\beta-1}$ must lie in $K_{\tilde\alpha}^\times$ as $u_{\tilde\alpha,0}$ already is. Setting $n_\beta = 2$ by (4.20) and $q = r$, we conclude that we can let $r \in \bar k$ freely change so long as $r^2 \in K_{\tilde\alpha}^\times$. Now, changing r^2 to r, we can state the main result of this paper as follows.

Theorem 4.22. *Assume $H^1(k, Z_G) = 1$, $\dim(U_M\backslash N) = \mathrm{rank}(Z_G\backslash T_w) = 2$, and that Assumption 3.6 as well as assumptions (4.1), (4.2), (4.4) and (4.20) are all valid. Moreover, suppose $\omega_\pi(w_0\omega_\pi^{-1})$ is ramified as a character of $K_{\tilde\alpha}^\times$.*

(a) One has

$$C_\psi(s,\pi)^{-1} = \eta(u_{\tilde\alpha,0})^{-2}|u_{\tilde\alpha,0}|^{-2s}|u_{\tilde h,0}|\gamma_{K_{\tilde\alpha}}(2\langle\hat\alpha,\alpha^\vee\rangle s/[K_{\tilde\alpha}:k], \omega_\pi(w_0\omega_\pi^{-1}), \psi_{K_{\tilde\alpha}}) \times \int_{r\in K_{\tilde\alpha}^\times} j'_{\tilde v,\bar N_0}(\beta^\vee(r)wt_0)\eta(r)|r|^{(n_\alpha s/\langle 2\rho,\alpha\rangle)-(n_\alpha/2)+1}\,\mathrm{d}^\times r. \tag{4.22.1}$$

The element n_0 satisfying (4.5) is so normalized that $t_0 \in Z_G(k)\cap G_{\mathrm{D}}(k)$, where $m_0 = wt_0u_{2,0}$. The character η of $K_{\tilde\alpha}^\times$ is defined by (4.3). Here n_α is the multiplicity of α in 2ρ and the partial Bessel function $j'_{\tilde v,\bar N_0}$ is defined by (4.12). We observe that $u_{\tilde\alpha,0}$ and $u_{\tilde h,0}$ are structural and independent of π.

(b) Suppose $\langle\tilde{\beta},\tilde{\alpha}\rangle = -1$, then (4.22.1) equals

$$C_\psi(s,\pi)^{-1} = \eta(u_{\tilde{\alpha},0})^{-2}|u_{\tilde{\alpha},0}|^{-2s}|u_{\tilde{h},0}|\gamma_{K_{\tilde{\alpha}}}(2\langle\hat{\alpha},\alpha^\vee\rangle s/[K_{\tilde{\alpha}}:k],\omega_\pi(w_0\omega_\pi^{-1}),\psi_{K_{\tilde{\alpha}}}) \times \int_{K_{\tilde{\alpha}}^\times} j'_{\tilde{v},\bar{N}_0}(\beta^\vee(r)wt_0)\eta(r)|r|^{s-\langle\rho,\alpha\rangle+1}\,\mathrm{d}^\times r. \quad (4.22.2)$$

Fix a character ν of $K^\times$ and realize it as a character $\tilde{\nu}$ of $M(k)$ by

$$\tilde{\nu}(m) = \nu(\det(\mathrm{Ad}_{\mathfrak{n}}(m))),$$

where $\mathfrak{n}$ is the Lie algebra of $N(k)$. If π is an irreducible admissible representation of $M(k)$ with central character $\omega_\pi = \eta\otimes\omega_{\pi,0}$ as in Remark 4.11, then the central character of $\pi\otimes\tilde{\nu}$ is equal to

$$\begin{aligned}\omega_\pi\nu(\det(\mathrm{Ad}_{\mathfrak{n}}(\alpha^\vee(q),z))) &= \eta(q)\nu(\det(\mathrm{Ad}_{\mathfrak{n}}(\alpha^\vee(q))))\omega_{\pi,0}(z)\\ &= \eta\nu^a(q)\omega_{\pi,0}(z),\end{aligned}$$

where a is some fixed positive integer depending only on G and M. We now have the following corollary.

Corollary 4.23. *With assumptions as in Theorem 4.22, $C_\psi(s,\pi)$ is stable, i.e. if π_1 and π_2 are two irreducible admissible ψ-generic representations of $M(k)$ sharing the same central character, then*

$$C_\psi(s,\pi_1\otimes\tilde{\nu}) = C_\psi(s,\pi_2\otimes\tilde{\nu})$$

for all the sufficiently highly ramified characters ν of K^* extended to a $\tilde{\nu}$ on $M(k)$ as above.

Proof. Lemma 3.11 implies, exactly as in § 4.2.2 of [**10**], that the Bessel function $j'_{\tilde{v},\bar{N}_0}$ is a $j_{v,w,Y}$ as in [**11**]. The proof of Proposition 4.4 of [**10**] then goes through line by line when applied to our equation (4.22.1) (or (4.22.2)) by means of Theorem 7.1 of [**11**] and implies the corollary. □

Remark 4.24. The corollary is valid even without the assumption that $H^1(k,Z_G)=1$ since one can in fact extend G to a group $\tilde{G}$ for which $G_{\mathrm{D}} = \tilde{G}_{\mathrm{D}}$ and $H^1(k,Z_{\tilde{G}})=1$. The assertion then easily follows by extending the character $\tilde{\nu}$ of $M(k)$ to one of $\tilde{M}(k)$ which is still highly ramified since $M=\tilde{M}\cap G$ and the corresponding local coefficients are equal.

Remark 4.25. We expect that the assumption that $\dim(U_M\backslash N)=2$ and the semisimple rank of T_w is equal to 2 must automatically imply the validity of (4.2), (4.4) and (4.20), as well as $\langle\tilde{\beta},\tilde{\alpha}\rangle=-1$. We will discuss this in the next section.

5. Removing some conditions

In this section we will show that (4.1), (4.2), (4.4) and (4.20) are all automatically valid under Assumption 3.6 and the assumptions that $\dim(U_M\backslash N) = 2$ and $\operatorname{rank}(Z_G\backslash T_w) = 2$.

The conditions on the semisimple rank and the dimension are quite restrictive. In fact, if $d = \dim(U_M\backslash N)$, then considering the fibre bundle

$$N' \to U_M\backslash N'$$

whose fibres, $U_{M,n}\backslash U_M \simeq U_M \cdot n$, $n \in N'$, all have the same dimension by Rosenlicht's generic quotient theorem (cf. Theorem 19.5 and Corollary 19.6 of [**15**]), one concludes that

$$\dim N = \dim U_M - \dim U_{M,n} + d \tag{5.1}$$

for all $n \in N'$, or

$$\dim N \leqslant d + \dim U_M,$$

or

$$\dim U = \dim N + \dim U_M \leqslant d + 2\dim U_M.$$

Thus, we have the following proposition.

Proposition 5.1. $\dim U \leqslant d + 2\dim U_M$.

We now add the restriction that $\dim(U_M\backslash N) = 2$ as well.

If we assume G is split over F and $d = 2$, then Proposition 5.1 implies

$$\operatorname{card}(\tilde{\Phi}^+) \leqslant 2(1 + \dim U_M). \tag{5.2}$$

One can easily check that maximal parabolic subgroups of G with $G_{\mathrm{D}} = G_2$ or $G_{\mathrm{D}} = F_4$ cannot entertain inequality (5.2).

Let $G_{\mathrm{D}} = E_6$. Inequality (5.2) then reduces to $\dim U_M \geqslant 17$ and the only possibilities are for $\alpha \in \{\alpha_1, \alpha_2, \alpha_6\}$, i.e. one of the external nodes (as always, in Bourbaki's numbering). In the cases that $\alpha = \alpha_1$ or $\alpha = \alpha_6$, M will not be self-associate.

Suppose $G_{\mathrm{D}} = E_7$. Then (5.2) reduces to $\dim U_M \geqslant 31$ and the only possibility here is $\alpha = \alpha_7$.

Finally, assume $G_{\mathrm{D}} = E_8$. Then (5.2) implies that $\dim U_M \geqslant 59$ which can only happen if $\alpha = \alpha_8$.

We point out that by Corollary 4.6, (4.2) is valid if and only if β is adjacent to α.

Note that in all these cases $\langle\tilde{\beta}, \tilde{\alpha}\rangle = -1$. Also, we can explicitly compute that in all these cases that $\tilde{\ell}(w_0) - \tilde{\ell}(w) > 2$, in violation of Proposition 4.1. Hence the conditions that $\dim(U_M\backslash N) = 2$ and $\operatorname{rank}(Z_G\backslash T_w) = 2$ are incompatible for these E-type exceptional groups. The last assertion is easily checked to hold even if β is not adjacent to α.

We observe that in these last two cases the corresponding main L-functions are those of standard L-functions of E_6 and E_7, respectively.

Finally, assume that G is the quasi-split form of E_6. Then by Proposition 5.1, we may assume $\tilde{\alpha} \in \{\alpha_1, \alpha_2\}$. If $\tilde{\alpha} = \alpha_1$ then for $\tilde{\beta} = \alpha_3$ it is easily checked that $\tilde{\ell}(w_0) - \tilde{\ell}(w) = 24 - 9 = 15 > 2$. For $\tilde{\alpha} = \alpha_2$ and $\tilde{\beta} = \alpha_4$ again $\tilde{\ell}(w_0) - \tilde{\ell}(w) = 21 - 9 > 2$.

We summarize the relevant parts of this discussion as the following.

Proposition 5.2. *Suppose $G_{\mathrm{D}} = E_7$ or E_8 and let $P = MN$ be a maximal parabolic of G. Assume $\dim(U_M\backslash N) = 2$. Then P is self-associate and $w_\ell^\theta(\alpha) = \alpha$, where $\alpha = \alpha_7$ or α_8 and $\beta = \alpha_6$ or α_7, respectively. If $G_{\mathrm{D}} = E_6$ and $\dim(U_M\backslash N) = 2$ then $w_\ell^\theta(\alpha) = \alpha$ for an adjacent β, but P may not be self-associate. However, in these cases, the conditions that $\dim(U_M\backslash N) = 2$ and $\operatorname{rank}(Z_G\backslash T_w) = 2$ are incompatible, and similarly for G_{D} the quasi-split form of E_6. Thus no exceptional group can satisfy our dimension and rank conditions simultaneously.*

Now assume G_{D} is not exceptional. To study the conditions in hand, we may restrict ourselves to an appropriate member of the isogeny class of G_{D}. In particular we will assume G_{D} is classical. We can then use the results from [**12**–**14**, **25**] to remove some of the conditions.

Let $G = G_{\mathrm{D}}$ be a quasi-split classical group of rank $r + \ell$ and assume $M = \mathrm{GL}_r \times G_\ell$, where G_ℓ is a quasi-split classical group of the same type as G of (semisimple) rank ℓ. If G is of either type B or C, then (5.2) implies

$$(r+\ell)^2/2 \leqslant 1 + \ell^2 + r(r-1)/2. \tag{5.3}$$

One can then see that this is possible for all ℓ if $r = 1$. On the other hand, if $r = 2$, then $\ell \geqslant 4$, and for $r > 2$, $\ell > 2r$.

When G is of type D, then (5.2) is equivalent to

$$[(r+\ell)^2 + (r+\ell)]/2 \leqslant 1 + (\ell^2 - \ell) + r(r-1)/2. \tag{5.4}$$

This then implies that if $r = 1$, then $\ell \geqslant 2$, while for $r > 1$, $\ell \geqslant 2r + 1$.

We point out that the case $r = 1$ corresponds to the cases of stability needed for functoriality (cf. [**1**, **9**, **10**, **16**]).

It is clear that $w_\ell^\theta(\alpha) = \alpha$ if and only if $\tilde{\alpha} = \alpha_1$ and $\tilde{\beta} = \alpha_2$ (Corollary 4.6). Using calculations in [**12**–**14**, **25**], we will show that $\operatorname{rank}(Z_G\backslash T_w) = 2$ will imply $\alpha = \alpha_1$ and thus $w_\ell^\theta(\alpha) = \alpha$.

To use the results from [**12**–**14**, **25**], we will assume r is even. If $n \in N$ corresponds to (X, Y), i.e. $n = n(X, Y)$ as in Lemma 2.1 of [**12**] or equation (3.2) of [**25**], then n satisfies equation (3.1) here if and only if $Y \in \mathrm{GL}_r(k)$ (Lemma 2.2 of [**12**] and Lemma 3.1 of [**25**]). Moreover, $n(gXh, gY\varepsilon(g)^{-1})$ also satisfies (3.1) for all $g \in \mathrm{GL}_r(k)$ and all $h \in G_\ell(k)$. Then

$$m(gXh, gY\varepsilon(g)^{-1}) = \operatorname{diag}(\varepsilon(g)\varepsilon(Y)g^{-1}, h^{-1}(I_{2\ell} - X'Y^{-1}X)h, gY\varepsilon(g)^{-1}) \tag{5.5}$$

will correspond to $n = n(gXh, gY\varepsilon(g)^{-1})$ by means of (3.1).

To proceed, write (3.1) as

$$w_0u_1^{-1}w_0^{-1}n = w_0tw_0^{-1}(w_0wu_2w^{-1}w_0^{-1})(w_0wn'w^{-1}w_0^{-1})w_0w\bar{n}, \tag{5.6}$$

where

$$m = u_1twu_2, \tag{5.7}$$

with $u_2 \in U^-_{M,w}(k)$. This gives a Bruhat decomposition for n as $n \in \bar{B}(k)w_0w\bar{N}(k)U_M(k)$.

Assume $\operatorname{rank}(Z_G\backslash T_w) = 2$ with α and β as before. We note that $\tilde{w}_0\tilde{w} = \tilde{w}_\ell\tilde{w}_\ell^\theta$ sends α and β to negative roots, but keeps other simple roots positive. We thus see that β will be the unique simple root which is sent to a negative one by w. To show that for $r \geqslant 2$ this will not be the case, we need to show that for such r, w in the Bruhat decomposition of $m = (m_r, m_\ell)$, $m_r \in \mathrm{GL}_r(k)$, $m_\ell \in G_\ell(k)$ will send more than one simple root to negative ones. This would be accomplished if one shows that the Weyl group elements appearing in the Bruhat decomposition of m_r and m_ℓ are both non-trivial for almost all n.

Suppose that $r \geqslant 2$, then inequalities (5.3) and (5.4) imply that $r < 2\ell$ and we will be in the setting in p. 288 of [**12**]. We now use the fact that the norm map of [**12**, **25**] is a surjection with finite fibres onto $\mathcal{C}^\vee$, where $\mathcal{C}^\vee$ is the set of conjugacy classes in $G_\ell(k)$ whose semisimple parts intersect $G_{r/2}(k)$, with $G_{r/2}$ identified as a subgroup of G_ℓ as in [**12**].

The discussion leading to equality (5.5) now allows us to conclude that the m_ℓ will generate an open set in $G_{r/2}(k)$ and that consequently the Weyl group element in the Bruhat decomposition of m_r will be non-trivial for almost all n satisfying (3.1). In fact, one of the consequences of the norm calculations in [**12**–**14**, **25**] is that the Weyl group elements in the m_ℓ are non-trivial if and only if the elements for the m_r are.

This basically sketches an argument towards proving $\operatorname{rank}(Z_G\backslash T_w) > 2$ if $r \geqslant 2$, at least when r is even. The odd case can be treated similarly. We collect these discussions in the following result. Note that in view of the last statement in Proposition 5.2, it also covers all the exceptional cases.

Proposition 5.3. *Under Assumption 3.6, assume also that*

$$\dim(U_M\backslash N) = \operatorname{rank}(Z_G\backslash T_w) = 2.$$

Then (4.1), (4.2), (4.4) and (4.20) are all valid. Moreover, $\langle\tilde{\beta}, \tilde{\alpha}\rangle = -1$.

6. Stability of local coefficients

The most basic statements of our stability results for local coefficients were given in Theorem 4.22 and Corollary 4.23 under a list of conditions. However the best way to formulate our results is in terms of Bruhat double cosets in G. This provides a formulation of our main result in a format which can be applied to the isogeny class of the derived group of G. Thus if the conditions of the theorem are verified for any member of the isogeny class, then it is valid for all, proving stability for every G whose derived group belongs to the given isogeny class.

We now state the main result of this paper as follows.

Theorem 6.1. *Let G be a quasi-split connected reductive algebraic group over k with $B = TU$, $P = MN$, $U \supset N$, $M \supset T$ and the representation π as before. Let $\bar{C}(\bar{w}) = \bar{B}\bar{w}B'$ be the unique Bruhat double coset with respect to $\bar{B} = B^-$ and $B' = T\bar{N}U_M$, intersecting N in an open set. Then $\bar{C}(\bar{w})$ is the Bruhat double coset of largest dimension intersecting N (Proposition 3.1), $\bar{w}\alpha < 0$ and $\bar{B}(k)\bar{w}\bar{N}(k)U_M(k)$ is* *the unique Bruhat double coset of $G(k)$ intersecting $N(k)$ in an open set. Assume there exists a simple root β such that $\bar{w}\beta < 0$ but $\bar{w}(\theta) > 0$, where $\theta = \Delta \setminus \{\alpha, \beta\}$. Moreover, assume* $\dim(U_M \backslash N) = \tilde{\ell}(w_0) - \tilde{\ell}(w_0\bar{w}) = 2$. *Then $C_\psi(s, \pi)$ is stable, i.e. if π_1 and π_2 are two such representations sharing the same central character, then*

$$C_\psi(s, \pi_1 \otimes \tilde{\nu}) = C_\psi(s, \pi_2 \otimes \tilde{\nu}),$$

for all sufficiently highly ramified characters ν of $K^\times$. Moreover, $\bar{w}\alpha = -\alpha$.

Proof. Using (3.4) one can write

$$N(k) = N(\bar{w})(k) \coprod \Big(\coprod_{\substack{w \in S(\bar{w}) \\ w \neq \bar{w}}} N(w)(k) \Big). \tag{6.1.1}$$

Since the k-topology is finer than the Zariski topology, $N(\bar{w})(k)$ will be open in the (relative) k-topology of $N(k)$. Thus they both have the same dimension upon realizing $N(\bar{w})(k)$ as a submanifold of $N(k)$. Moreover, since N is a product of affine spaces, $\dim_{\bar{k}} N = \dim_k N(k)$. On the other hand, $\dim_{\bar{k}} N(w) < \dim_{\bar{k}} N = \dim_{\bar{k}} N(\bar{w})$ for all $w \neq \bar{w}$ in $S(\bar{w})$, and since $\dim_k N(w)(k) \leqslant \dim_{\bar{k}} N(w)$, one concludes that $N(\bar{w})(k)$ is the only open strata in (6.1.1).

Next, write $m = u_1 t w u_2$ as in (5.7) for n satisfying (3.1). Equation (5.6) then implies that

$$n \in \bar{B}(k) w_0 w \bar{N}(k) U_M(k).$$

Thus

$$\tilde{w}_0 \tilde{w} = \tilde{\bar{w}} = \tilde{w}_\ell \tilde{w}_\ell^\theta. \tag{6.1.2}$$

Clearly, $\bar{w}\alpha < 0$. Since $\bar{w}\beta < 0$ while $\bar{w}\theta > 0$, this implies that $\tilde{\bar{w}} = \tilde{w}_\ell \tilde{w}_\ell^\theta$. Writing $\tilde{w}_0 = \tilde{w}_\ell \tilde{w}_\ell^\Omega$, then implies $\tilde{w} = \tilde{w}_\ell^\Omega \tilde{w}_\ell^\theta$, which implies, in particular, that P_θ^M supports a Bessel function on $M(k)$ (see § 2.2 of [**11**]). (Also see the discussion just before Proposition 3.2.)

Since $\dim(U_M \backslash N) = \tilde{\ell}(w_0) - \tilde{\ell}(w) = 2$ and $\operatorname{rank}(Z_G \backslash T_w) = 2$, Propositions 4.1 and 5.3 and Corollary 4.23 now imply the stability assertion. □

Corollary 6.2. *Let G be a quasi-split connected reductive algebraic group over k such that the Γ-diagram of G_D is of either type B_n, C_n, D_n, ${}^2A_n$ or ${}^2D_n$ ($n \geqslant 4$). Let $P = MN$ be a self-associate maximal parabolic subgroup of G over k such that the unique simple root in N is the restriction of the root $\tilde{\alpha}_1$, the first root in the Dynkin diagram of the Chevalley type of the derived group of G in Bourbaki's numbering [**4**]. Let π be an irreducible admissible generic representation of $M(k)$. Then $C_\psi(s, \pi)$ is stable, that is, if ν is a character of $K^\times$, realized as a character $\tilde{\nu}$ of $M(k)$ by*

$$\tilde{\nu}(m) = \nu(\det(\mathrm{Ad}_{\mathfrak{n}}(m))),$$

then

$$C_\psi(s, \pi_1 \otimes \tilde{\nu}) = C_\psi(s, \pi_2 \otimes \tilde{\nu})$$

for any two such representations π_1 and π_2 with same central characters and all sufficiently highly ramified ν. Here $\mathfrak{n}$ is the Lie algebra of $N(k)$.

Proof. One needs to verify that the conditions of Theorem 6.1 are valid.

Observe that the conditions depend only on the isogeny class of G_{D}. Moreover, if $\tilde{N}$ and $\tilde{U}_{\tilde{M}}$ are the full inverse images of N and U_M under the defining $\bar{k}$-isomorphism $f : \tilde{G}_{\mathrm{D}} \to G_{\mathrm{D}}$ from the Chevalley type $\tilde{G}_{\mathrm{D}}$ of G_{D} to G_{D}, both of which are defined over k, then $\dim(\tilde{U}_{\tilde{M},\tilde{n}}) = \dim(U_{M,n})$, where $\tilde{U}_{\tilde{M},\tilde{n}}$ is the centralizer of $\tilde{n} = f^{-1}(n)$ in $\tilde{U}_{\tilde{M}}$. In fact, one only needs to observe that $f(\tilde{U}_{\tilde{M},\tilde{n}}) = U_{M,n}$, using the definitions of $\tilde{U}_{\tilde{M},\tilde{n}}$ and $U_{M,n}$.

Together with the fact that one is only interested in the lengths of the Weyl group elements as elements in $W(T,G)$, this then reduces our task to checking the conditions for the split form of G_{D}, i.e. here we may assume that G_{D} is a split classical group.

We can then show as in [**26**] that if $\tilde{\beta} = \alpha_2$ and θ is obtained by restricting the set $\tilde{\theta}$ of all other simple roots in $\tilde{\Delta}$ which do not restrict to α or β, then $\tilde{\bar{B}}\tilde{\bar{w}}\tilde{B}'$ intersects $\tilde{N}$ in an open set (cf. § 7), where $\bar{w} = w_\ell w_\ell^\theta$, $\tilde{\bar{w}} = f^{-1}(\bar{w})$, $\tilde{\bar{B}} = f^{-1}(\bar{B})$, and $\tilde{B}' = f^{-1}(B')$. Note that since $w_\ell = f(w_\ell^{\tilde{\Delta}})$ and $w_\ell^\theta = f(w_\ell^{\tilde{\theta}})$, then $\tilde{\bar{w}} = w_\ell^{\tilde{\Delta}} w_\ell^{\tilde{\theta}}$. Observe that all these representatives are chosen to lie in G_{D}. We now apply f to $\tilde{\bar{B}}\tilde{\bar{w}}\tilde{B}'$ to conclude that $\bar{B}\bar{w}B'$ intersects N in an open set as desired. This completes the proof. □

Corollary 6.3. *Let $\bar{C}(\bar{w})$ be the unique Bruhat cell intersecting N openly. Then under Assumption 3.6 we have*

$$\dim(U_M \backslash N) = \tilde{\ell}(w_0) - \tilde{\ell}(w_0\bar{w}).$$

Proof. From Proposition 4.1, under Assumption 3.6 we have that $\dim(U_M\backslash N) = \tilde{\ell}(w_0) - \tilde{\ell}(w)$. On the other hand, from (6.1.2) we know $w = w_0^{-1}\bar{w}$. Since P is self-associate, $w_0^{-1} = w_0$. The corollary now follows. □

Remark 6.4. Sundaravaradhan [**29**] has determined $\bar{w}$ for a given N in Theorem 6.1 without doing any explicit Bruhat decomposition. The algorithm is quite clever, simple and general. He also has proved that if $\bar{w}(\theta) > 0$ then $\bar{w}(\theta) = \theta$ and thus (4.1) is automatically satisfied. In particular, in part (a) of Theorem 6.1 we may replace $\bar{w}(\theta) = \theta$ with the seemingly weaker assumption $\bar{w}(\theta) > 0$.

Remark 6.5. The rational character

$$\xi(m) = \det(\mathrm{Ad}_{\mathfrak{n}}(m))$$

of M is not necessarily defined over k. In general, it is only defined over the splitting field K of G. On the other hand $\xi|A_0$ is defined over k.

7. Examples

If G_{n+1} is a split classical groups of Chevalley type B_{n+1}, C_{n+1}, or D_{n+1} the parabolic P arising in Corollary 6.2 is the maximal parabolic with Levi of the form $\mathrm{GL}_1 \times G_n$ associated to the root α_1 in the Bourbaki numbering. It and the associated Weyl group element $w_0 = w_\ell w_\ell^\Omega$ are given explicitly in § 4.2.1 of [**10**]. The root β as in Theorem 6.1 then corresponds to α_2 in Bourbaki's numbering. The associated Weyl group element

$w = w_\ell^\Omega w_\ell^\theta$ is then explicitly given after Proposition 4.1 of [**10**]. From these, $\bar{w} = w_0 w = w_\ell w_\ell^\theta$ is easily computed and the rank and dimension conditions easily verified. As we pointed out in § 5, this may well be the only situation in the classical groups for which our rank and dimension conditions are satisfied.

We next assume that $G = G_{n+1}$ is the quasi-split special orthogonal group SO_{2n+2}^* defined by a quadratic extension K of k, that is, the case ${}^2D_{n+1}$ of Corollary 6.2. This is particularly interesting since it should lead to functorial transfer from SO_{2n}^* to GL_{2n}, accounting for self-dual representations with non-trivial central character which still need to be treated (cf. [**9**,**10**] for the split cases). The parabolic subgroup that we are concerned with in connection with functoriality has $M = \mathrm{GL}_1 \times \mathrm{SO}_{2n}^*$ as its Levi subgroup. As we observed in the proof of Corollary 6.2, this satisfies all the conditions of our main theorem since its Chevalley type above does (cf. [**10**,**26**]). The cases of GSpin_{2n+2}^* (and Spin_{2n+2}^*) are the same, either using [**1**] or Corollary 6.2.

Our next example concerns the quasi-split group of type ${}^2A_{n+1}$, that is, the case of unitary groups. In fact, to prove the transfer of automorphic forms from U_n, the quasi-split unitary group in n variables, i.e. the unitary group defined by a quadratic extension K of k and of signature $(\frac{1}{2}n, \frac{1}{2}n)$ if n is even or $(\frac{1}{2}(n+1), \frac{1}{2}(n-1))$ if n is odd. Then one needs to consider the Levi subgroup $M = \mathrm{Res}_{K/k}\,\mathrm{GL}_1 \times U_n$ of $G = U_{n+2}$. In this case, stability has been proven in [**16**] in the even case by proving Theorem 4.22 directly for the unitary group. Here we will use Corollary 6.2 to conclude it as a special case of our more general results.

To apply the corollary, we need to consider the Chevalley form $\tilde{G}$ of $G = U_{n+2}$. Strictly speaking we should consider $\tilde{G} = \mathrm{SL}_{n+2}$ as the Chevalley form of $G = \mathrm{SU}_{n+2}$, but for our purposes we may consider $\tilde{G}(K) = \mathrm{GL}_{n+2}(K)$ and let $f : \mathrm{GL}_{n+2}(K) \xrightarrow{\sim} U_{n+2}(K)$ be the defining K-isomorphism between K-points of K-groups. Since both groups are defined over k,

$$\{\sigma \mapsto a_\sigma = f^{-\sigma} f\} \in H^1(\Gamma_K, \mathrm{Aut}(\tilde{G}(K))).$$

If $\sigma \neq 1$ is the non-trivial element of $\Gamma_K = \mathrm{Gal}(K/k)$ then $a_\sigma(h) = w\,{}^{\mathrm{t}}h^{-1} w^{-1}$ for $h \in \mathrm{GL}_{n+2}(K)$, where w is an appropriate permutation matrix which fixes the standard splitting of the upper triangular unipotent matrices in $\mathrm{GL}_{n+2}(K)$, i.e. a second diagonal matrix with alternating ± 1 as non-zero entries. Then

$$f(a_\sigma(\sigma(g))) = \sigma(f(g))$$

which leads to the standard definition

$$g = w\,{}^{\mathrm{t}}\sigma(g)^{-1} w^{-1}$$

for the k-points of U_{n+2} for which $\sigma(f(g)) = f(g)$.

To use Corollary 6.2 we only need to show that $\bar{\tilde{B}}\bar{\tilde{w}}\tilde{B}'$ intersects $\tilde{N}$ in an open set, where $\bar{\tilde{w}} = w_\ell^{\tilde{\Delta}} w_\ell^{\tilde{\theta}}$. Note that $\bar{\tilde{B}} = \tilde{\bar{B}}$ and $\bar{\tilde{w}} = \tilde{\bar{w}}$ and thus our notation agrees with that of the corollary. Here the parabolic subgroup $\tilde{P} = \tilde{M}\tilde{N}$ of $\tilde{G}$ has $\mathrm{GL}_1 \times \mathrm{GL}_n \times \mathrm{GL}_1$ as Levi subgroup, which restricts to $\mathrm{Res}_{K/k}\,\mathrm{GL}_1 \times U_n$ upon restriction of roots, giving the case of unitary groups of the corollary (as ${}^2A_{n+1}$). This means that we need to show that if $\tilde{w}_0^{-1}\tilde{n} = \tilde{m}\tilde{n}'\bar{\tilde{n}}$ and $\tilde{m} = \tilde{u}_1\tilde{t}\tilde{w}\tilde{u}_2$ then $\tilde{w}_0\tilde{w} = \bar{\tilde{w}} = w_\ell^{\tilde{\Delta}} w_\ell^{\tilde{\theta}}$ on a dense open set.

For the sake of simplicity of calculations we will take

$$\tilde{w}_0 = \begin{pmatrix} & & 1 \\ & I_n & \\ 1 & & \end{pmatrix}$$

by multiplying $\tilde{m}$ by an appropriate member of $\tilde{T}$, the subgroup of diagonal elements of GL_{n+2}. If

$$\tilde{n} = \begin{pmatrix} 1 & X & a \\ & I_n & Y \\ & & 1 \end{pmatrix}$$

with $X, Y \in K^n$ then

$$\tilde{w}_0^{-1}\tilde{n} = \tilde{m}\tilde{n}'\bar{\tilde{n}} \tag{7.1}$$

if and only if $a \in K^\times$.

A quick calculation using (7.1) then implies that if $\tilde{m} = \mathrm{diag}(b, m, c)$, with $b, c \in K^\times$ and $m \in \mathrm{GL}_n(K)$, then

$$m = I_n - a^{-1}YX.$$

All we need to check is that the dominant Bruhat double coset for m is attached to

$$\tilde{w} = \begin{pmatrix} & & 1 \\ & I_{n-2} & \\ 1 & & \end{pmatrix}.$$

This is an exercise in Gaussian elimination. There exists a unipotent upper triangular element u in $\mathrm{GL}_n(K)$ such that $uY = {}^{\mathrm{t}}(0, \ldots, 0, x)$ with $x \in K^\times$ for almost all $\tilde{n}$ and thus on a dense open set

$$Z = umu^{-1} = \begin{pmatrix} & & & 0 \\ & I_{n-1} & & \vdots \\ & & & 0 \\ z_1 & \cdots & z_{n-1} & z_n \end{pmatrix}$$

with $z_1, \ldots, z_n \neq 0$. Then

$$\tilde{w}Z = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n-1} & z_n \\ 0 & & & & 0 \\ \vdots & & I_{n-2} & & \vdots \\ 0 & & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Let

$$\bar{u}_1 = \begin{pmatrix} & & & & 0 \\ & & I_{n-1} & & \vdots \\ & & & & 0 \\ -z_1^{-1} & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then

$$\bar{u}_1\tilde{w}Z = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n-1} & z_n \\ 0 & & & & 0 \\ \vdots & & I_{n-2} & & \vdots \\ 0 & & & & 0 \\ 0 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \end{pmatrix}.$$

Next, let

$$\bar{u}_2 = \begin{pmatrix} & & & & & 0 \\ & & I_{n-1} & & & \vdots \\ & & & & & 0 \\ 0 & -\alpha_2 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

which implies

$$\bar{u}_2\bar{u}_1\tilde{w}Z = \begin{pmatrix} z_1 & z_2 & \cdots & z_{n-1} & z_n \\ 0 & & & & 0 \\ \vdots & & I_{n-2} & & \vdots \\ 0 & & & & 0 \\ 0 & 0 & \alpha_3 & \cdots & \alpha_n \end{pmatrix}$$

and inductively

$$\bar{u}\tilde{w}Z \in \tilde{B}_n \quad \text{or} \quad \tilde{w}(\tilde{w}^{-1}\bar{u}\tilde{w})Z \in \tilde{B}_n,$$

where

$$\bar{u} = \begin{pmatrix} & & & 0 \\ & I_{n-1} & & \vdots \\ & & & 0 \\ * & \cdots & * & 1 \end{pmatrix}.$$

Note that $\tilde{w}^{-1}\bar{u}\tilde{w} \in \tilde{B}_n$ and thus $Z \in \tilde{B}_n\tilde{w}\tilde{B}_n$ or $m \in \tilde{B}_n\tilde{w}\tilde{B}_n$. This implies that $\tilde{w}_0\tilde{w} = w_\ell^{\tilde{\Delta}}w_\ell^{\tilde{\theta}}$ as claimed. Thus the conditions of Theorem 6.1 are satisfied for the parabolic $\tilde{P}$ of $\tilde{G}$, and hence for the parabolic P of G obtained by restriction of roots.

Finally, as we observed in § 5, there are no cases of exceptional groups that satisfy the rank and dimension requirements of Theorem 6.1.

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References

1. M. Asgari and F. Shahidi, Generic transfer for general spin groups, *Duke Math. J.* **132** (2006), 137–190.
2. A. Borel, *Automorphic L-functions*, Proceedings of Symposia in Pure Mathematics, Volume 33 (II), pp. 27–61 (American Mathematical Society, Providence, RI, 1979).
3. A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics, Volume 126 (Springer, 1991).
4. N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4–6, Elements of Mathematics (Springer, 2002).
5. E. Brenner, Stability of the local gamma factor in the unitary case, preprint (2006; available at xxx.lanl.gov/pdf/math.NT/0608101).
6. J. W. Cogdell and I. I. Piatetski-Shapiro, Converse theorems for GL_n, *Publ. Math. IHES* **79** (1994), 157–214.
7. J. W. Cogdell and I. I. Piatetski-Shapiro, Stability of gamma factors for $SO(2n+1)$, *Manuscr. Math.* **95** (1998), 437–461.
8. J. W. Cogdell and I. I. Piatetski-Shapiro, Converse theorems for GL_n, II, *J. Reine Angew. Math.* **507** (1999), 165–188.
9. J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro and F. Shahidi, On lifting from classical groups to GL_N, *Publ. Math. IHES* **79** (2001), 5–30.
10. J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro and F. Shahidi, Functoriality for the classical groups, *Publ. Math. IHES* **99** (2004), 163–233.
11. J. W. Cogdell, I. I. Piatetski-Shapiro and F. Shahidi, Partial Bessel functions for quasi-split groups, in *Automorphic representations, L-functions and applications: progress and prospects*, pp. 95–128 (Walter de Gruyter, Berlin, 2005).
12. D. Goldberg and F. Shahidi, On the tempered spectrum of quasi-split classical groups, *Duke Math. J.* **92** (1998), 225–294.
13. D. Goldberg and F. Shahidi, On the tempered spectrum of quasi-split classical groups, II, *Can. J. Math.* **53** (2001), 244–277.
14. D. Goldberg and F. Shahidi, On the tempered spectrum of quasi-split classical groups, III, The odd orthogonal groups, *Forum Math.*, in press.
15. F. D. Grosshans, *Algebraic homogeneous spaces and invariant theory*, Lecture Notes in Mathematics, Volume 1673 (Springer, 1997).
16. H. Kim and M. Krishnamurthy, Stable base change from unitary groups to GL_N, *Int. Math. Res. Papers* **1** (2005), 1–52.
17. E. Lapid and S. Rallis, On the local factors of representations of classical groups, in *Automorphic representations, L-functions and applications: progress and prospects*, pp. 309–359 (Walter de Gruyter, Berlin, 2005).
18. H. Matsumura, *Commutative algebra* (Benjamin, Reading, MA, 1980).
19. S. Rallis and D. Soudry, Stability of the local gamma factor arising from the doubling method, *Math. Ann.* **333** (2005), 291–313.
20. M. Rosenlicht, Some basic theorems on algebraic groups, *Am. J. Math.* **78** (1956), 401–443.
21. C. S. Seshadri, Some results on the quotient space by an algebraic group of automorphisms, *Math. Ann.* **149** (1963), 280–310.
22. F. Shahidi, On certain L-functions, *Am. J. Math.* **103** (1981), 297–355.
23. F. Shahidi, Local coefficients as Artin factors for real groups, *Duke Math. J.* **52** (1985), 973–1007.
24. F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups, *Ann. Math.* **132** (1990), 273–330.
25. F. Shahidi, The notion of norm and the representation theory of orthogonal groups, *Invent. Math.* **119** (1995), 1–36.

26. F. Shahidi, Local coefficients as Mellin transforms of Bessel functions: towards a general stability, *Int. Math. Res. Not.* **39** (2002), 2075–2119.
27. F. Shahidi, On the Ramanujan conjecture for quasi-split groups, *Asian J. Math.* **8** (2004), 813–836 (part of a special volume in memory of Armand Borel).
28. T. A. Springer, *Linear algebraic groups*, 2nd edn (Birkhäuser, Boston, MA, 1998).
29. R. Sundaravaradhan, Some structural results for the stability of root numbers, preprint (2006).