EXTERIOR SQUARE L-FUNCTION FOR $GL(n)$

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1. Introduction. Let π be an irreducible generic representation of $GL(n)$ over a non-Archimedean local field k. By Bernstein and Zelevinsky [1,11] we know we can write

$$\pi = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)$$

with each σ_i quasi-square-integrable of the form

$$\sigma_i = \mathrm{Q}(\mathrm{Ind}(\rho_i \otimes \rho_i\nu \otimes \cdots \otimes \rho_i\nu^{k_i}))$$

where Q denotes the unique irreducible quotient and ν is the unramified character $\nu(g) = |\det(g)|$.

Question. *Can we express $L(\pi, \wedge^2, s)$ in terms of the L-functions of $\sigma_1, \dots, \sigma_r$? Can we express $L(\sigma_i, \wedge^2, s)$ in terms of L-functions of ρ_i?*

Here, by $L(\pi, \wedge^2, s)$ we mean the exterior square L-function one gets from the Rankin–Selberg integral representation of Jacquet and Shalika [8].

The real goal is to do this in the Archimedean case and compare $L(\pi, \wedge^2, s)$ as indicated by Jacquet and Shalika [8] with that predicted by the Langlands classification. So we want techniques that have a chance of generalizing to the Archimedean case.

It is the global $L(\pi, \wedge^2, s)$ that controls the poles of the Eisenstein series used in defining the twisted L-function for $SO(2n+1)$ [6,9,10]. The control of the poles of this L-function is the last step we need to achieve the global Langlands lifting from $SO(2n+1)$ to $GL(2n)$ for generic cusp forms via the Converse Theorem [3,5].

2. Derivatives and Whittaker models. The basic tool of [1,11] is to analyze the representations of $GL(n)$ by first analyzing the representations of P_n, the mirabolic subgroup: $P_n = GL_{n-1} \ltimes U_n$ where $GL_{n-1} \hookrightarrow GL_n$ by $h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ and $U_n \simeq k^{n-1} \hookrightarrow GL_n$ by $u \mapsto \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix}$.

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The representations of P_n are analyzed by the use of four functors

$$\mathrm{Rep}(P_{n-1}) \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \mathrm{Rep}(P_n) \underset{\Psi^-}{\overset{\Psi^+}{\leftrightarrows}} \mathrm{Rep}(GL_{n-1}).$$

Φ^+ and Ψ^+ are induction functors, while Φ^- and Ψ^- are localization functors or Jacquet functors. All are normalized. They are defined as follows:

(a) $\mathrm{Rep}(P_n) \underset{\Psi^-}{\overset{\Psi^+}{\leftrightarrows}} \mathrm{Rep}(GL_{n-1})$.

To define Ψ^- we consider the space of U_n covariants. We let (τ, V_τ) be a smooth representation of P_n and let

$$V_\tau(U_n, \mathbf{1}) = \langle \tau(u)v - v \mid v \in V_\tau,\ u \in U_n \rangle.$$

Then the space of $\Psi^-(\tau)$ is $V_\tau/V_\tau(U_n, \mathbf{1})$, the largest quotient of V_τ on which U_n acts trivially. Since GL_{n-1} preserves U_n, GL_{n-1} will stabilize $V_\tau(U_n, \mathbf{1})$ and we have the natural action of GL_{n-1} on $V_\tau/V_\tau(U_n, \mathbf{1})$. Letting σ denote $\Psi^-(\tau)$, then σ is the normalized action of GL_{n-1} on $V_\tau/V_\tau(U_n, \mathbf{1})$ given by

$$\sigma(g)(v + V_\tau(U_n, \mathbf{1})) = |\det(g)|^{-1/2}(\tau(g)v + V_\tau(U_n, \mathbf{1})).$$

The functor Ψ^+ is just induction, or in this case, normalized extension by the trivial representation. Given a smooth representation (σ, V_σ) of GL_{n-1} we let $\tau = \Psi^+(\sigma)$ be the representation of P_n on V_σ such that U_n acts trivially and GL_{n-1} acts by $\tau(g) = |\det(g)|^{1/2}\sigma(g)$.

(b) $\mathrm{Rep}(P_{n-1}) \underset{\Phi^-}{\overset{\Phi^+}{\rightleftarrows}} \mathrm{Rep}(P_n)$.

Here we consider $P_{n-1} \hookrightarrow GL_{n-1} \hookrightarrow P_n$. If we fix a non-trivial additive character ψ of K, then ψ defines a character of U_n, which by abuse of notation we again denote by ψ, defined by $\psi(u) = \psi(u_{n-1,n})$. GL_{n-1} is the stabilizer of U_n and the stabilizer of this character in GL_{n-1} is exactlyP_{n-1}.

To construct Φ^-, let (τ, V_τ) be a smooth representation of P_n. We form the space of (U_n, ψ)-covariants by taking

$$V_\tau(U_n, \psi) = \langle \tau(u)v - \psi(u)v \mid u \in U_n,\ v \in V_\tau \rangle$$

and forming the quotient vector space $V_\tau/V_\tau(U_n, \psi)$. This is the largest quotient on which U_n acts by the character ψ. Then $\sigma = \Phi^-(\tau)$ is the normalized representation of P_{n-1} on $V_\tau/V_\tau(U_n, \psi)$ given by

$$\sigma(p)(v + V_\tau(U_n, \psi)) = |\det(p)|^{-1/2}(\tau(p)v + V_\tau(U_n, \psi)).$$

Φ^+ is the functor of normalized compactly supported induction. If (σ, V_σ) is a smooth representation of P_{n-1} we extend it to a representation of $P_{n-1}U_n$ by letting U_n act by the character ψ. Then

$$\tau = \Phi^+(\sigma) = \mathrm{ind}_{P_{n-1}U_n}^{P_n}(|\det|^{1/2}\sigma \otimes \psi)$$

where the induction ind is non-normalized using smooth functions of compact support modulo $P_{n-1}U_n$.

Facts:

(1) Any irreducible representation of τ of P_n is of the form $\tau \simeq (\Phi^+)^{k-1}\Psi^+(\rho)$ with ρ an irreducible representation of GL_{n-k}. The index k and the representation ρ are completely determined by τ.

(2) The derivatives: Let $\tau \in \mathrm{Rep}(P_n)$. For each $k = 1, 2, \dots, n$ there are representations $\tau_{(k)} \in \mathrm{Rep}(P_{n-k})$ and $\tau^{(k)} \in \mathrm{Rep}(GL_{n-k})$ associated to τ by

$$\tau_{(k)} = (\Phi^-)^k(\tau) \qquad \text{and} \qquad \tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau).$$

Diagrammatically:

$$\begin{tikzcd}[column sep=small]
 & & & \tau \arrow[dl] \arrow[dr] & \\
 & & \tau_{(1)} \arrow[dl] \arrow[dr] & & \tau^{(1)} \\
 & \tau_{(2)} \arrow[dl] \arrow[dr] & & \tau^{(2)} & \\
\iddots & & \tau^{(3)} & &
\end{tikzcd}$$

where all leftward arrows represent an application of Φ^- and the rightward arrows an application of Ψ^-. $\tau^{(k)}$ is called the k^{th} derivative of τ.

(3) We have a short exact sequence

$$1 \to \Phi^+\Phi^-(\tau) \to \tau \to \Psi^+\Psi^-(\tau) \to 1.$$

(4) There is a canonical filtration of τ by derivatives. Any $\tau \in \mathrm{Rep}(P_n)$ has a natural filtration by P_n submodules

$$0 \subset \tau_n \subset \tau_{n-1} \subset \cdots \subset \tau_2 \subset \tau_1 = \tau$$

such that $\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)$. The successive quotients are completely determined by the derivatives of τ since

$$\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+(\tau^{(k)}).$$

The proofs of these statements can be found in the work of Bernstein and Zelevinsky [1,11]

If π is an irreducible admissible representation of GL_n, its derivatives are defined by $\pi^{(0)} = \pi$, $\pi_{(0)} = \pi|_{P_n}$ and then $\pi^{(k)} = (\pi_{(0)})^{(k)}$, etc.

Now suppose that π is an irreducible generic representation of GL_n. This is the case iff $\pi^{(n)} = 1$. Then this structure can easily be seen in its Whittaker model $\mathcal{W}(\pi, \psi)$ [4]. Namely:

(1) $\left\{ W \begin{pmatrix} g_m & \\ & I_{n-m} \end{pmatrix} \mid W \in \mathcal{W}(\pi, \psi) \right\}$ is a model for $\pi_{(n-m-1)}$.

(2) $\Phi^+\Phi^-(\pi_{(n-m-1)})$ is realized as the subspace of those $W \begin{pmatrix} g_m & \\ & I_{n-m} \end{pmatrix}$ such that there exists an N such that $\max_i |g_{m,i}| < q^{-N}$ implies $W \begin{pmatrix} g_m & \\ & I_{n-m} \end{pmatrix} = 0$.

(3) If $\pi^{(n-m)}$ is either irreducible or at most completely reducible and $v \in \pi_{(n-m-1)}$ projects to $p(v) \in \tau$ an irreducible constituent of $\pi^{(n-m)}$ then τ is generic with Whittaker model given by $W_{p(v)}(g_m) = \lim_{a \to 0} \omega_\tau(a) W_v \begin{pmatrix} ag_m & \\ & I_{n-m} \end{pmatrix}$, with the limit existing in the stable sense.

Note that the functions $W \begin{pmatrix} g_m & \\ & I_{n-m} \end{pmatrix}$ often arise in Rankin–Selberg integrals for GL_n.

3. The integrals of Jacquet and Shalika. Let π be an irreducible representation of GL_{2n} and $W \in \mathcal{W}(\pi, \psi)$ a function in its Whittaker model. Let $\Phi \in \mathcal{S}(k^n)$. The Jacquet and Shalika [8] indicate that $L(\pi, \wedge^2, s)$ should be computed by the following family of integrals:

$$J(W, \Phi, s) = \int_{N_n \backslash GL_n} \int_{V_n \backslash M_n} W\left(\sigma \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix}\right) \psi^{-1}(\operatorname{tr} X) dx \ \Phi(e_n g) |\det(g)|^s dg$$

where M_n is the space of $n \times n$ matrices, V_n is the subspace of strictly upper triangular matrices in M_n, and σ is the permutation given by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n & | & n+1 & n+2 & \cdots & 2n \\ 1 & 3 & \cdots & 2n-1 & | & 2 & 4 & \cdots & 2n \end{pmatrix}$$

Theorem. *(i) These integrals converge for* $\operatorname{Re}(s) >> 0$ *and define rational functions in* q^{-s}.

(ii) $\mathcal{I} = \{J(W, \Phi, s)\}$ *forms a fractional* $\mathbb{C}[q^s, q^{-s}]$ *ideal with generator of the form* $P(q^{-s})^{-1}$ *with* $P(x) \in \mathbb{C}[X]$ *satisfying* $P(0) = 1$.

Definition. *We set* $L(\pi, \wedge^2, s) = P(q^{-s})^{-1}$.

If π is unramified, Jacquet and Shalika showed that this definition agrees with what is predicted by Langlands [8].

One analyses $P(q^{-s})$ by analyzing the poles of the integrals $J(W,\Phi,s)$. Suppose that $J(W,\Phi,s)$ has a pole at $s=s_0$. Its Laurent expansion at that point will be of the form

$$J(W,\Phi,s)=\frac{B(W,\Phi)}{(q^s-q^{s_0})^k}+\cdots$$

with $B(W,\Phi)$ a bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{S}(k^n)$ with certain equivariance properties.

We now split the poles into two families. $\mathcal{S}(k^n)$ has a small filtration $\{0\}\subset\mathcal{S}_0(k^n)\subset\mathcal{S}(k^n)$ with $\mathcal{S}_0(k^n)=\{\Phi\mid\Phi(0)=0\}$. Suppose that $B(W,\Phi)$ is trivial on $\mathcal{S}_0(k^n)$. Then we can write $B(W,\Phi)=\Lambda(W)\Phi(0)$ where Λ is a so-called (twisted) *Shalika functional* [7] on π. If we let

$$R_{2n}=\left\{r=\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\middle| X\in M_n,\ g\in GL_n\right\}$$

then $\Lambda(\pi(r)W)=\psi(\mathrm{tr}(X))|\det(g)|^{s_0}\Lambda(W)$. We call such poles *exceptional* and we let $L_{ex}(\pi,\wedge^2,s)$ denote their contribution to $L(\pi,\wedge^2,s)$.

In the other case, $B(W,\Phi)$ remains non-zero upon restriction to $\mathcal{W}(\pi,\psi)\times\mathcal{S}_0(k^n)$. Then we can find $\Phi_0\in\mathcal{S}_0(k^n)$ which is responsible for this pole. Using the support of Φ_0 one can reduce the integral $J(W,\Phi_0,s)$ to a finite sum of integrals of the form

$$J'(W,s)=\int_{N_n\backslash P_n}\int_{V_n\backslash M_n} W\left(\sigma\begin{pmatrix} I & \\ 0 & I\end{pmatrix}\begin{pmatrix} p & \\ & p\end{pmatrix}\right)\psi^{-1}(\mathrm{tr}\,X)dX\ |\det(p)|^s dp.$$

Now, the argument of W lies in P_{2n}, that is, the integral depends on $\pi_{(0)}$, the restriction of π to P_{2n}. We can now use the theory of derivatives. Still the pole at $s=s_0$ looks like

$$J'(W,s)=\frac{\Lambda(W)}{(q^{-s}-q^{-s})^k}+\cdots$$

where Λ is now a (twisted) Shalika functional on $\pi_{(0)}$ with respect to

$$R'_{2n}=\left\{r=\begin{pmatrix} I & X\\ 0 & I\end{pmatrix}\begin{pmatrix} p & \\ & p\end{pmatrix}\middle| X\in M_n\ p\in P_n\right\}.$$

4. Results on Shalika functionals. First consider the Shalika functionals on representations of GL_n. These are mainly due to Jacquet and Rallis [7].

Theorem [7]**.** *Let π be an irreducible representation of GL_n. Then there is at most a one dimensional space of Shalika functionals on π. If π has a Shalika functional (untwisted), then π must be self–contragredient.*

It is not hard to establish the following result.

Proposition. *An irreducible representation of the form* $\mathrm{Ind}(\sigma\otimes\tilde{\sigma})$ *with σ square integrable has a Shalika functional.*

Shalika functionals on P_n seem to be easier to analyze. The results are as follows.

Theorem. *(i) If $\tau = (\Phi^+)^{n-k-1}\Psi^+(\rho)$ with ρ an irreducible representation of GL_k, then τ has no Shalika functional unless $k = 2m$ is even. In this case there is at most a one dimensional space of Shalika functionals on τ.*

(ii) If $k = 0$ above, then τ has a Shalika functional.

Note that (ii) is responsible for the local functional equation for the exterior square L-function, while (i) ensures that this method works equally well for the Jacquet–Shalika integrals for the exterior square for GL_{2n+1}.

5. Consequences of the existence of Shalika functionals for P_n. Using the results on Shalika functionals on P_n above, the canonical filtration by derivatives for $\pi_{(0)}$ from Section 2, and the properties of the Whittaker functions relative to this filtration, one can prove the following result.

Theorem. *If all derivatives of π are completely reducible, then the non-exceptional poles of $L(\pi, \wedge^2, s)$ are exactly the exceptional poles of the even derivatives of π, i.e., $L(\pi, \wedge^2, s)$ is completely determine by the $L_{ex}(\pi^{(2m)}, \wedge^2, s)$ for $m = 0, \ldots, n$.*

6. Deformation and specialization. Now take $\pi = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ generic with each σ_i quasi-square-integrable. We introduce complex parameters by setting $z = (z_1, \ldots, z_r)$, $\sigma_i(z_i) = \sigma_i \nu(z_i)$ and $\pi(z) = \mathrm{Ind}(\sigma_1(z_1) \otimes \cdots \otimes \sigma_r(z_r))$. Consider $\mathcal{I} = \{J(W_z, \Phi, s) \mid W_z \in \mathcal{W}(\pi(z), \psi), \Phi \in \mathcal{S}(k^n)\}$. These functions will define rational functions in $q^{\pm z_i}$ and $q^{\pm s}$ and will form a fractional $\mathbb{C}[q^{\pm s}, q^{\pm z_i}]$ ideal. It may no longer be principal. But for z in general position, $\pi(z)$ will be irreducible and its derivatives will be completely reducible [2]. Our method then computes a polynomial $P(q^{-s}, q^{-z})$ which controls the poles as a rational function in $r + 1$ variables. These poles will come from $L_{ex}(\pi(z)^{(2m)}, \wedge^2, s)$.

The existence of a (twisted) Shalika functional for $\pi(z)^{(2m)}$ gives a certain number of linear conditions on the variables $z_1, \ldots, z_r$ and s from the (twisted) self-contragrediance requirement. Since a function of several complex variables cannot have isolated singularities of co-dimension greater than or equal to two by Hartog's Theorem, then we really need only consider those $L_{ex}(\pi(z)^{(2m)}, \wedge^2, s)$ which contribute the co-dimension one poles. These derivatives will all be either of the form $\sigma_i(z_i)^{(k_i)}$ or of the form $\mathrm{Ind}(\sigma_i(z_i)^{(k_i)} \otimes \sigma_j(z_j)^{(k_j)})$ with $\tilde{\sigma}_i = \sigma_j$ up to a twist. The first type will reassemble to give a contribution of $\prod L(\sigma_i(z_i), \wedge^2, s) = \prod L(\sigma_i, \wedge^2, s + 2z_i)$. Since we understand the Shalika functionals on the second type of representation, these should contribute $\prod L(\sigma_i \times \sigma_j, s + z_i + z_j)$. Hence we would arrive at

$$P(q^{-s}, q^{-z})^{-1} = \prod_i L(\sigma_i, \wedge^2, s + 2z_i) \prod_{i<j} L(\sigma_i \times \sigma_j, s + z_i + z_j).$$

If we then let $z \to 0$ we get the expected equality

$$L(\pi, \wedge^2, s) = \prod_i L(\sigma_i, \wedge^2, s) \prod_{i<j} L(\sigma_i \times \sigma_j, s).$$

7. A computation for square-integrables. Suppose now that σ is the simplest non-cuspidal quasi-square-integrable:

$$\sigma = \mathrm{Q}(\mathrm{Ind}(\rho \otimes \rho\nu))$$

with ρ a cuspidal representation of GL_k. The the derivatives of σ are $\sigma^{(0)} = \sigma$ and $\sigma^{(k)} = \rho\nu$. Then the above gives

$$L(\sigma, \wedge^2, s) = L_{ex}(\sigma, \wedge^2, s) L_{ex}(\rho\nu, \wedge^2, s) = L_{ex}(\sigma, \wedge^2, s) L(\rho, \wedge^2, s+2).$$

On the other hand, if I have computed correctly, the predicted L-function should be

$$L(\sigma, \wedge^2, s) = L(\rho, \mathrm{Sym}^2, s) L(\rho, \wedge^2, s+2).$$

This leads quickly to the following conjecture.

Conjecture.

$$L_{ex}(\mathrm{Q}(\mathrm{Ind}(\rho \otimes \cdots \otimes \rho\nu^k)), \wedge^2, s) = \begin{cases} L(\rho, \wedge^2, s) & \text{if } k \text{ is even} \\ L(\rho, \mathrm{Sym}^2, s) & \text{if } k \text{ is odd} \end{cases}.$$

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