EXTERIOR SQUARE L-FUNCTION FOR $GL(n)$

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1. Introduction. Let $\pi$ be an irreducible generic representation of $GL(n)$ over a non-Archimedean local field $k$. By Bernstein and Zelevinsky [1,11] we know we can write

$$\pi = \text{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)$$

with each $\sigma_i$ quasi-square-integrable of the form

$$\sigma_i = Q(\text{Ind}(\rho_i \otimes \nu \otimes \cdots \otimes \nu^{k_i}))$$

where $Q$ denotes the unique irreducible quotient and $\nu$ is the unramified character $\nu(g) = |\det(g)|$.

**Question.** Can we express $L(\pi, \wedge^2, s)$ in terms of the $L$-functions of $\sigma_1, \ldots, \sigma_r$? Can we express $L(\sigma_i, \wedge^2, s)$ in terms of $L$-functions of $\rho_i$?

Here, by $L(\pi, \wedge^2, s)$ we mean the exterior square $L$-function one gets from the Rankin Selberg integral representation of Jacquet and Shalika [8].

The real goal is to do this in the Archimedean case and compare $L(\pi, \wedge^2, s)$ as indicated by Jacquet and Shalika [8] with that predicted by the Langlands classification. So we want techniques that have a chance of generalizing to the Archimedean case.

It is the global $L(\pi, \wedge^2, s)$ that controls the poles of the Eisenstein series used in defining the twisted $L$-function for $SO(2n+1)$ [6,9,10]. The control of the poles of this $L$-function is the last step we need to achieve the global Langlands lifting from $SO(2n+1)$ to $GL(2n)$ for generic cusp forms via the Converse Theorem [3,5].

2. Derivatives and Whittaker models. The basic tool of [1,11] is to analyze the representations of $GL(n)$ by first analyzing the representations of $P_n$, the mirabolic subgroup:

$$P_n = GL_{n-1} \ltimes U_n$$

where $GL_{n-1} \hookrightarrow GL_n$ by $h \mapsto \begin{pmatrix} h & \hline \end{pmatrix}$ and $U_n \simeq k^{n-1} \hookrightarrow GL_n$ by

$$u \mapsto \begin{pmatrix} I_{n-1} & u \\ \hline & 1 \end{pmatrix}.$$

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The representations of $P_n$ are analyzed by the use of four functors

$$
\begin{array}{ccc}
\text{Rep}(P_n) & \xrightarrow{\Phi^+} & \text{Rep}(GL_{n-1}) \\
\downarrow{\Phi^-} & & \downarrow{\Psi^-} \\
\text{Rep}(P_{n-1}) & \xleftarrow{\Psi^+} & \text{Rep}(P_n)
\end{array}
$$

$\Phi^+$ and $\Psi^+$ are induction functors, while $\Phi^-$ and $\Psi^-$ are localization functors or Jacquet functors. All are normalized. They are defined as follows:

(a) $\text{Rep}(P_n) \xrightarrow{\Psi^+} \text{Rep}(GL_{n-1})$.

To define $\Psi^-$ we consider the space of $U_n$ covariants. We let $(\tau, V_\tau)$ be a smooth representation of $P_n$ and let

$$V_\tau(U_n, 1) = \langle \tau(u)v - v \mid v \in V_\tau, u \in U_n \rangle.$$  

Then the space of $\Psi^-(\tau)$ is $V_\tau/V_\tau(U_n, 1)$, the largest quotient of $V_\tau$ on which $U_n$ acts trivially. Since $GL_{n-1}$ preserves $U_n$, $GL_{n-1}$ will stabilize $V_\tau(U_n, 1)$ and we have the natural action of $GL_{n-1}$ on $V_\tau/V_\tau(U_n, 1)$. Letting $\sigma$ denote $\Psi^-(\tau)$, then $\sigma$ is the normalized action of $GL_{n-1}$ on $V_\tau/V_\tau(U_n, 1)$ given by

$$\sigma(g)(v + V_\tau(U_n, 1)) = |\det(g)|^{-1/2}(\tau(g)v + V_\tau(U_n, 1)).$$

The functor $\Psi^+$ is just induction, or in this case, normalized extension by the trivial representation. Given a smooth representation $(\sigma, V_\sigma)$ of $GL_{n-1}$ we let $\tau = \Psi^+(\sigma)$ be the representation of $P_n$ on $V_\sigma$ such that $U_n$ acts trivially and $GL_{n-1}$ acts by $\tau(g) = |\det(g)|^{1/2}\sigma(g)$.

(b) $\text{Rep}(P_{n-1}) \xleftarrow{\Phi^-} \text{Rep}(P_n)$.

Here we consider $P_{n-1} \hookrightarrow GL_{n-1} \hookrightarrow P_n$. If we fix a non-trivial additive character $\psi$ of $K$, then $\psi$ defines a character of $U_n$, which by abuse of notation we again denote by $\psi$, defined by $\psi(u) = \psi(u_{n-1,n})$. $GL_{n-1}$ is the stabilizer of $U_n$ and the stabilizer of this character in $GL_{n-1}$ is exactly $P_{n-1}$.

To construct $\Phi^-$, let $(\tau, V_\tau)$ be a smooth representation of $P_n$. We form the space of $(U_n, \psi)$-covariants by taking

$$V_\tau(U_n, \psi) = \langle \tau(u)v - \psi(u)v \mid u \in U_n, v \in V_\tau \rangle$$

and forming the quotient vector space $V_\tau/V_\tau(U_n, \psi)$. This is the largest quotient on which $U_n$ acts by the character $\psi$. Then $\sigma = \Phi^-(\tau)$ is the normalized representation of $P_{n-1}$ on $V_\tau/V_\tau(U_n, \psi)$ given by

$$\sigma(p)(v + V_\tau(U_n, \psi)) = |\det(p)|^{-1/2}(\tau(p)v + V_\tau(U_n, \psi)).$$
\( \Phi^+ \) is the functor of normalized compactly supported induction. If \((\sigma, V_\sigma)\) is a smooth representation of \(P_{n-1}\) we extend it to a representation of \(P_{n-1}U_n\) by letting \(U_n\) act by the character \(\psi\). Then

\[
\tau = \Phi^+(\sigma) = \text{ind}_{P_{n-1}U_n}^{P_n}(|\det|^{1/2}\sigma \otimes \psi)
\]

where the induction \(\text{ind}\) is non-normalized using smooth functions of compact support modulo \(P_{n-1}U_n\).

**Facts:**

1. Any irreducible representation of \(\tau\) of \(P_n\) is of the form \(\tau \simeq (\Phi^+)^{k-1}(\Psi^+)(\rho)\) with \(\rho\) an irreducible representation of \(GL_{n-k}\). The index \(k\) and the representation \(\rho\) are completely determined by \(\tau\).

2. The derivatives: Let \(\tau \in \text{Rep}(P_n)\). For each \(k = 1, 2, \ldots, n\) there are representations \(\tau_{(k)} \in \text{Rep}(P_{n-k})\) and \(\tau^{(k)} \in \text{Rep}(GL_{n-k})\) associated to \(\tau\) by

\[
\tau_{(k)} = (\Phi^-)^k(\tau) \quad \text{and} \quad \tau^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau).
\]

Diagrammatically:

\[
\begin{array}{cccccc}
\tau & \leftarrow & \tau_{(1)} & \downarrow & \tau^{(1)} & \\
\tau_{(2)} & \leftarrow & \tau^{(2)} & \downarrow & \\
\vdots & & & & \\
\tau^{(3)} & \end{array}
\]

where all leftward arrows represent an application of \(\Phi^-\) and the rightward arrows an application of \(\Psi^-\). \(\tau^{(k)}\) is called the \(k^{th}\) derivative of \(\tau\).

3. We have a short exact sequence

\[
1 \rightarrow \Phi^+\Phi^- (\tau) \rightarrow \tau \rightarrow \Psi^+\Psi^- (\tau) \rightarrow 1.
\]

4. There is a canonical filtration of \(\tau\) by derivatives. Any \(\tau \in \text{Rep}(P_n)\) has a natural filtration by \(P_n\) submodules

\[
0 \subset \tau_n \subset \tau_{n-1} \subset \cdots \subset \tau_2 \subset \tau_1 = \tau
\]

such that \(\tau_k = (\Phi^+)^{k-1}(\Phi^-)^{k-1}(\tau)\). The successive quotients are completely determined by the derivatives of \(\tau\) since

\[
\tau_k/\tau_{k+1} = (\Phi^+)^{k-1}\Psi^+(\tau^{(k)}).
\]
The proofs of these statements can be found in the work of Bernstein and Zelevinsky [1,11].

If \( \pi \) is an irreducible admissible representation of \( GL_n \), its derivatives are defined by \( \pi^{(0)} = \pi, \pi^{(0)} = \pi|_{P_n} \) and then \( \pi^{(k)} = (\pi^{(0)})^{(k)} \), etc.

Now suppose that \( \pi \) is an irreducible generic representation of \( GL_n \). This is the case iff \( \pi^{(n)} = 1 \). Then this structure can easily be seen in its Whittaker model \( \mathcal{W}(\pi, \psi) \) [4]. Namely:

1. \( \left\{ W \begin{pmatrix} g_m & I_{n-m} \\ & & \end{pmatrix} \mid W \in \mathcal{W}(\pi, \psi) \right\} \) is a model for \( \pi_{(n-m)} \).

2. \( \Phi^+ \Phi^- (\pi_{(n-m)}) \) is realized as the subspace of those \( W \begin{pmatrix} g_m & I_{n-m} \\ & & \end{pmatrix} \) such that there exists an \( N \) such that \( \max_i |g_{m,i}| < q^{-N} \) implies \( W \begin{pmatrix} g_m & I_{n-m} \\ & & \end{pmatrix} = 0 \).

3. If \( \pi^{(n-m)} \) is either irreducible or at most completely reducible and \( v \in \pi_{(n-m)} \) projects to \( p(v) \in \tau \) an irreducible constituent of \( \pi^{(n-m)} \) then \( \tau \) is generic with Whittaker model given by \( W_{p(v)}(g_m) = \lim_{a \to 0} \omega_\tau(a) W_\nu \begin{pmatrix} a g_m & I_{n-m} \\ & & \end{pmatrix} \), with the limit existing in the stable sense.

Note that the functions \( W \begin{pmatrix} g_m & I_{n-m} \\ & & \end{pmatrix} \) often arise in Rankin Selberg integrals for \( GL_n \).

3. The integrals of Jacquet and Shalika. Let \( \pi \) be an irreducible representation of \( GL_{2n} \) and \( W \in \mathcal{W}(\pi, \psi) \) a function in its Whittaker model. Let \( \Phi \in S(k^n) \). The Jacquet and Shalika [8] indicate that \( L(\pi, \wedge^2, s) \) should be computed by the following family of integrals:

\[
J(W, \Phi, s) = \int_{N_n \backslash GL_n} \int_{V_n \backslash M_n} W \left( \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix} \right) \psi^{-1}(\text{tr} X) dx \Phi(e_ng)|\det(g)|^s dg
\]

where \( M_n \) is the space of \( n \times n \) matrices, \( V_n \) is the subspace of strictly upper triangular matrices in \( M_n \), and \( \sigma \) is the permutation given by

\[
\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 3 & \cdots & 2n-1 \\ 2 & 4 & \cdots & 2n \end{pmatrix}
\]

**Theorem.**

(i) These integrals converge for \( \text{Re}(s) >> 0 \) and define rational functions in \( q^{-s} \).

(ii) \( I = \{ J(W, \Phi, s) \} \) forms a fractional \( \mathbb{C}[q^s, q^{-s}] \) ideal with generator of the form \( P(q^{-s})^{-1} \) with \( P(x) \in \mathbb{C}[X] \) satisfying \( P(0) = 1 \).

**Definition.** We set \( L(\pi, \wedge^2, s) = P(q^{-s})^{-1} \).

If \( \pi \) is unramified, Jacquet and Shalika showed that this definition agrees with what is predicted by Langlands [8].
One analyses $P(q^{-s})$ by analyzing the poles of the integrals $J(W, \Phi, s)$. Suppose that $J(W, \Phi, s)$ has a pole at $s = s_0$. Its Laurent expansion at that point will be of the form

$$J(W, \Phi, s) = \frac{B(W, \Phi)}{(q^s - q^{-s})^k} + \cdots$$

with $B(W, \Phi)$ a bilinear form on $\mathcal{W}(\pi, \psi) \times \mathcal{S}(k^n)$ with certain equivariance properties.

We now split the poles into two families. $\mathcal{S}(k^n)$ has a small filtration $\{0\} \subset \mathcal{S}_0(k^n) \subset \mathcal{S}(k^n)$ with $\mathcal{S}_0(k^n) = \{ \Phi \mid \Phi(0) = 0 \}$. Suppose that $B(W, \Phi)$ is trivial on $\mathcal{S}_0(k^n)$. Then we can write $B(W, \Phi) = \Lambda(W)\Phi(0)$ where $\Lambda$ is a so-called (twisted) Shalika functional [7] on $\pi$. If we let

$$R_{2n} = \left\{ r = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} g & X \\ g & I \end{pmatrix} \mid X \in M_n, \ g \in GL_n \right\}$$

then $\Lambda(\pi(r)W) = \psi(\text{tr}(X))|\text{det}(g)|^{s_0}\Lambda(W)$. We call such poles exceptional and we let $L_{ex}(\pi, \lambda^2, s)$ denote their contribution to $L(\pi, \lambda^2, s)$.

In the other case, $B(W, \Phi)$ remains non-zero upon restriction to $\mathcal{W}(\pi, \psi) \times \mathcal{S}_0(k^n)$. Then we can find $\Phi_0 \in \mathcal{S}_0(k^n)$ which is responsible for this pole. Using the support of $\Phi_0$ one can reduce the integral $J(W, \Phi_0, s)$ to a finite sum of integrals of the form

$$J'(W, s) = \int_{N_n \setminus P_n} \int_{V_n \setminus M_n} W \left( \sigma \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} p & X \\ p & p \end{pmatrix} \right) \psi^{-1}(\text{tr}(X))dX \ |\text{det}(p)|^s dp.$$

Now, the argument of $W$ lies in $P_{2n}$, that is, the integral depends on $\pi_{(0)}$, the restriction of $\pi$ to $P_{2n}$. We can now use the theory of derivatives. Still the pole at $s = s_0$ looks like

$$J'(W, s) = \frac{\Lambda(W)}{(q^{-s} - q^{-s})^k} + \cdots$$

where $\Lambda$ is now a (twisted) Shalika functional on $\pi_{(0)}$ with respect to

$$R'_{2n} = \left\{ r = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} p & X \\ p & p \end{pmatrix} \mid X \in M_n, \ p \in P_n \right\}.$$

4. **Results on Shalika functionals.** First consider the Shalika functionals on representations of $GL_n$. These are mainly due to Jacquet and Rallis [7].

**Theorem** [7]. Let $\pi$ be an irreducible representation of $GL_n$. Then there is at most a one dimensional space of Shalika functionals on $\pi$. If $\pi$ has a Shalika functional (untwisted), then $\pi$ must be self contragredient.

It is not hard to establish the following result.

**Proposition.** An irreducible representation of the form $\text{Ind}(\sigma \otimes \overline{\sigma})$ with $\sigma$ square integrable has a Shalika functional.

Shalika functionals on $P_n$ seem to be easier to analyze. The results are as follows.
Theorem. (i) If \( \tau = (\Phi^+)^{n-k-1} \Psi^+ (\rho) \) with \( \rho \) an irreducible representation of \( GL_k \), then \( \tau \) has no Shalika functional unless \( k = 2m \) is even. In this case there is at most a one dimensional space of Shalika functionals on \( \tau \).

(ii) If \( k = 0 \) above, then \( \tau \) has a Shalika functional.

Note that (ii) is responsible for the local functional equation for the exterior square L-function, while (i) ensures that this method works equally well for the Jacquet Shalika integrals for the exterior square for \( GL_{2n+1} \).

5. Consequences of the existence of Shalika functionals for \( P_n \). Using the results on Shalika functionals on \( P_n \) above, the canonical filtration by derivatives for \( \pi(0) \) from Section 2, and the properties of the Whittaker functions relative to this filtration, one can prove the following result.

Theorem. If all derivatives of \( \pi \) are completely reducible, then the non-exceptional poles of \( L(\pi, \wedge^2, s) \) are exactly the exceptional poles of the even derivatives of \( \pi \), i.e., \( L(\pi, \wedge^2, s) \) is completely determine by the \( L_{ex}(\pi^{(2m)}, \wedge^2, s) \) for \( m = 0, \ldots, n \).

6. Deformation and specialization. Now take \( \pi = \text{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r) \) generic with each \( \sigma_i \) quasi-square-integrable. We introduce complex parameters by setting \( z = (z_1, \ldots, z_r) \), \( \sigma_i(z_i) = \sigma_i \nu(z_i) \) and \( \pi(z) = \text{Ind}(\sigma_1(z_1) \otimes \cdots \otimes \sigma_r(z_r)) \). Consider \( I = \{ J(W_z, \Phi, s) \mid W_z \in \mathcal{W}(\pi(z), \psi), \Phi \in \mathcal{S}(k^n) \} \). These functions will define rational functions in \( q^{\pm z_i} \) and \( q^{\pm z} \) and will form a fractional \( \mathbb{C}[q^{\pm s}, q^{\pm z_i}] \) ideal. It may no longer be principal. But for \( z \) in general position, \( \pi(z) \) will be irreducible and its derivatives will be completely reducible [2]. Our method then computes a polynomial \( P(q^{-s}, q^{-z}) \) which controls the poles as a rational function in \( r + 1 \) variables. These poles will come from \( L_{ex}(\pi(z)^{(2m)}, \wedge^2, s) \).

The existence of a (twisted) Shalika functional for \( \pi(z)^{(2m)} \) gives a certain number of linear conditions on the variables \( z_1, \ldots, z_r \) and \( s \) from the (twisted) self-congruence requirement. Since a function of several complex variables cannot have isolated singularities of co-dimension greater than or equal to two by Hartog's Theorem, then we really need only consider those \( L_{ex}(\pi(z)^{(2m)}, \wedge^2, s) \) which contribute the co-dimension one poles. These derivatives will all be either of the form \( \sigma_i(z_i) (k_i) \) or of the form \( \text{Ind}(\sigma_i(z_i) (k_i) \otimes \sigma_j(z_j) (k_j)) \) with \( \sigma_i = \sigma_j \) up to a twist. The first type will resemble to give a contribution of \( \prod L(\sigma_i(z_i), \wedge^2, s) = \prod L(\sigma_i, \wedge^2, s + 2z_i) \). Since we understand the Shalika functionals on the second type of representation, these should contribute \( \prod L(\sigma_i \times \sigma_j, s + z_i + z_j) \). Hence we would arrive at

\[
P(q^{-s}, q^{-z})^{-1} = \prod_i L(\sigma_i, \wedge^2, s + 2z_i) \prod_{i < j} L(\sigma_i \times \sigma_j, s + z_i + z_j).
\]

If we then let \( z \to 0 \) we get the expected equality

\[
L(\pi, \wedge^2, s) = \prod_i L(\sigma_i, \wedge^2, s) \prod_{i < j} L(\sigma_i \times \sigma_j, s).
\]
7. A computation for square-integrables. Suppose now that \( \sigma \) is the simplest non-cuspidal quasi-square-integrable:

\[
\sigma = Q(\text{Ind}(\rho \otimes \rho^k))
\]

with \( \rho \) a cuspidal representation of \( GL_k \). The the derivatives of \( \sigma \) are \( \sigma^{(0)} = \sigma \) and \( \sigma^{(k)} = \rho^k \). Then the above gives

\[
L(\sigma, \Lambda^2, s) = L_{ex}(\sigma, \Lambda^2, s)L_{ex}(\rho^k, \Lambda^2, s) = L_{ex}(\sigma, \Lambda^2, s)L(\rho, \Lambda^2, s + 2).
\]

On the other hand, if I have computed correctly, the predicted \( L \)-function should be

\[
L(\sigma, \Lambda^2, s) = L(\rho, \text{Sym}^2, s)L(\rho, \Lambda^2, s + 2).
\]

This leads quickly to the following conjecture.

Conjecture.

\[
L_{ex}(Q(\text{Ind}(\rho \otimes \cdots \otimes \rho^k))), \Lambda^2, s) = \begin{cases} 
L(\rho, \Lambda^2, s) & \text{if } k \text{ is even} \\
L(\rho, \text{Sym}^2, s) & \text{if } k \text{ is odd}
\end{cases}
\]

References


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