# Lectures on $L$-functions, Converse Theorems, and Functoriality for $GL_n$

**J.W. Cogdell**

Department of Mathematics, Oklahoma State University, Stillwater OK 74078

# Contents

# Preface

These are the lecture notes that accompanied my lecture series at the Fields Institute in the Spring of 2003 as part of the Thematic Program on Automorphic Forms. The posted description of the course was the following.

"The theory of $L$-functions of automorphic forms (or modular forms) via integral representations has its origin in the paper of Riemann on the zeta-function. However the theory was really developed in the classical context of $L$-functions of modular forms for congruence subgroups of $SL(2,\mathbb{Z})$ by Hecke and his school. Much of our current theory is a direct outgrowth of Hecke's. $L$-functions of automorphic representations were first developed by Jacquet and Langlands for $GL(2)$. Their approach followed Hecke combined with the local-global techniques of Tate's thesis. The theory for $GL(n)$ was then developed along the same lines in a long series of papers by various combinations of Jacquet, Piatetski-Shapiro, and Shalika. In addition to associating an $L$-function to an automorphic form, Hecke also gave a criterion for a Dirichlet series to come from a modular form, the so called Converse Theorem of Hecke. In the context of automorphic representations, the Converse Theorem for $GL(2)$ was developed by Jacquet and Langlands, extended and significantly strengthened to $GL(3)$ by Jacquet, Piatetski-Shapiro, and Shalika, and then extended to $GL(n)$."

"In these lectures we hope to present a synopsis of this work and in doing so present the paradigm for the analysis of general automorphic $L$-functions via integral representations. We will begin with the classical theory of Hecke and then a description of its translation into automorphic representations of $GL(2)$ by Jacquet and Langlands. We will then turn to the theory of automorphic representations of $GL(n)$, particularly cuspidal representations. We will first develop the Fourier expansion of a cusp form and present results on Whittaker models since these are essential for defining Eulerian integrals. We will then develop integral representations for $L$-functions for $GL(n) \times GL(m)$ which have nice analytic properties (meromorphic continuation, boundedness in vertical strips, functional equations) and have Eulerian factorization into products of local integrals."

"We next turn to the local theory of $L$-functions for $GL(n)$, in both the archimedean and non-archimedean local contexts, which comes out of the Euler factors of the global integrals. We finally combine the global Eulerian integrals with the definition and analysis of the local $L$-functions to define the global $L$-function of an automorphic representation and derive their major analytic properties."

"We will then turn to the various Converse Theorems for $GL(n)$. We will begin with the simple inversion of the integral representation. Then we will show how to proceed from this to the proof of the basic Converse Theorems, those requiring twists by cuspidal representations of $GL(m)$ with $m$ at most $n-1$. We will then discuss how one can reduce the twisting to $m$ at most $n-2$. Finally we will consider what is conjecturally true about the amount of twisting necesssary for a Converse Theorem."

"We will end with a description of the applications of these Converse Theorems to new cases of Langlands Functoriality. We will discuss both the basic paradigm for using the Converse Theorem to establish liftings to $GL(n)$ and the specifics of the lifts from the split classical groups $SO(2n+1)$, $SO(2n)$, and $Sp(2n)$ to the appropriate $GL(N)$."

I have chosen to keep the informal format of the actual lectures; what follows are the texed versions of the notes that I lectured from. Other than making corrections they remain as they were when posted weekly on the web to accompany the recorded lectures. In particular, I have left each lecture with its individual references, but there are no citations within the body of the notes. For full details of the proofs, many of which are only sketched in the notes and many others omitted, the reader should consult the references for that section.

Of course, there will be some overlap with other surveys I have written on this subject, particularly my PCMI Lecture notes *L-functions and Converse Theorems for* $GL_n$. However there are several lectures, particularly among the early ones and later ones, that appear in survey form, at least by me, for the first time. I hope this more informal presentation of the material, in conjunction with the accompanying Lectures of Henry Kim and Ram Murty, add value to this contribution.

I would like to thank the staff of the Fields Institute, and particularly the program managers for our special program – Allison Conway and Sonia Houle – for taking such good care of us during the Thematic Program on Automorphic Forms.

LECTURE 1

# Modular Forms and Their $L$-functions

I want to begin by describing the classical theory of holomorphic modular forms and their $L$-functions more or less in the terms in which it was developed by Hecke.

Let $\mathfrak{H} = \{z = x + iy \mid y > 0\}$ denote the upper half plane. The group $PSL_2(\mathbb{R})$ or $PGL_2^+(\mathbb{R})$ acts on $\mathfrak{H}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

We will be interested in certain discrete groups of motions $\Gamma$ which have finite volume quotients $\Gamma\backslash\mathfrak{H}$. We will consider two main examples.

1. *The full modular group* $SL_2(\mathbb{Z})$. This group is generated by the two transformations $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It has the usual (closed) fundamental domain given by

$$\mathcal{F} = \{z = x + iy \mid \tfrac{-1}{2} \le x \le \tfrac{1}{2},\ |z| \ge 1\}.$$

Then the quotient $\Gamma\backslash\mathfrak{H} \simeq \mathbb{P}^1 - \{\infty\}$ is a once punctured sphere.

2. *The Hecke congruence groups* $\Gamma_0(N)$. These are defined by

$$\Gamma_0(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \Big| \quad c \equiv 0 \pmod{N} \right\}.$$

These groups preserve not only $\mathfrak{H}$ but also the rational points on the real line: $\mathbb{Q} \cup \{\infty\}$. So if we let $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ then $\Gamma$ acts on $\mathfrak{H}^*$ and $\Gamma\backslash\mathfrak{H}^*$ is a compact Riemann surface.

The cusps of $\Gamma$ are the ($\Gamma$–equivalence classes of) points of $\mathbb{Q} \cup \{\infty\}$. These are finite in number. If $a \in \mathbb{Q}$ then there is an element $\sigma_a \in SL_2(\mathbb{Q})$ such that $\sigma_a \cdot a = \infty$. Thus locally all cusps look like the cusp at infinity.

Modular forms for $\Gamma$ are a special class of function on $\mathfrak{H}$.

**Definition 1.1** *A (holomorphic) modular form of (integral) weight $k \ge 0$ for $\Gamma$ is a function $f : \mathfrak{H} \to \mathbb{C}$ satisfying*

(i) [modularity] *for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have the modular transformation law $f(\gamma z) = (cz+d)^k f(z)$;*

(ii) [regularity] $f$ *is holomorphic on* $\mathfrak{H}$;
(iii) [growth condition] $f$ *extends holomorphically to every cusp of* $\Gamma$.

Let us explain the condition (iii) for the cusp at infinity. The element $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$ and $T$ generates the stabilizer $\Gamma_\infty$ of the point $\infty$ in $\Gamma$. On modular forms $T$ act as

$$f(Tz) = f(z+1) = f(z)$$

so any modular form is periodic in $z \mapsto z+1$. $f(z)$ then defines a holomorphic function on $\Gamma_\infty \backslash \mathfrak{H}$ which can be viewed as either a cylinder or a punctured disk "centered at $\infty$". We can take as a local parameter on this disk $D$ the parameter $q = q_\infty = e^{2\pi i z}$. Then $z \mapsto q$ maps $\Gamma \backslash \mathfrak{H} \to D^\times = D - \{0\}$. Since $f$ is holomorphic on $D^\times$ we can write it in a Laurent expansion in the variable $q$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q^n.$$

For $f$ to be holomorphic at the cusp $\infty$ means that $a_n = 0$ for all $n < 0$, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

This expansion is called the *Fourier expansion* (or *$q$-expansion*) of $f(z)$ at the cusp $\infty$. There is a similar expansion at any cusp.

A modular form is called a *cusp form* if in fact $f(z)$ vanishes at each of the cusps of $\Gamma$. In the Fourier expansion of $f(z)$ at the cusp $\infty$ this takes the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

Traditionally one lets $M_k(\Gamma)$ denote the space of all holomorphic modular forms of weight $k$ for $\Gamma$ and $S_k(\Gamma)$ the subspace of cusp forms. It is a fundamental fact that the imposed conditions on modular forms are strong enough to give a basic finiteness result.

**Theorem 1.1** $\dim_{\mathbb{C}} M_k(\Gamma) < \infty$.

The proof in this context is essentially an application of Riemann–Roch to the powers of the canonical bundle on the compact Riemann surface $\Gamma \backslash \mathfrak{H}^*$.

## 1 Examples

Here are some well known examples of classical modular forms. Note the arithmetic nature of the Fourier coefficients in each case.

1. *Eisenstein series.* Let $k > 2$ be an even integer. Then

$$G_k(z) = \sum_{(m,n) \neq (0,0)} (mz+n)^{-k}$$

is a modular form of weight $k$ for $SL_2(\mathbb{Z})$. It has a Fourier expansion

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi)^k}{\Gamma(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}$$

where $\sigma_r(n) = \sum_{d|n} d^r$. The normalized Eisenstein series $E_k(z)$ is defined to have constant Fourier coefficient equal to 1 so that

$$G_k(z) = 2\zeta(k) E_k(z).$$

2. *The Discriminant function.*

$$\Delta(z) = e^{2\pi i z} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z})^{24} = \frac{1}{1728}(E_4(z)^3 - E_6(z)^2)$$

is the unique cusp of weight 12 for $SL_2(\mathbb{Z})$. It has the Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$$

where $\tau(n)$ is the Ramanujan $\tau$-function.

3. *Theta series.* Let $Q$ be a positive definite integral quadratic from in $2k$ variables. Then

$$\Theta_Q(z) = \sum_{\vec{m} \in \mathbb{Z}^{2k}} e^{2\pi i Q(\vec{m}) z} = 1 + \sum_{n=1}^{\infty} r_Q(n) e^{2\pi i n z}$$

is a modular form of weight $k$ for an appropriate congruence group $\Gamma$. Here the Fourier coefficients are the representation numbers for $Q$

$$r_Q(n) = \left| \{ \vec{m} \in \mathbb{Z}^{2k} \mid Q(\vec{m}) = n \} \right|.$$

## 2 Growth Estimates on Cusp Forms

As preliminaries to the definition of the $L$-function we look at two estimates on cusp forms. So let $f(z) \in S_k(\Gamma)$.

1. From the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

we obtain

$$|f(x+iy)| \ll e^{-2\pi y}$$

as $y \to \infty$, uniformly in $x$, with similar estimates at any cusp. So cusp forms are *rapidly decreasing* at all cusps.

2. Since $y^{k/2}|f(z)|$ is a bounded function on $\Gamma \backslash \mathfrak{H}$ we obtain

$$|f(x+iy)| \ll y^{-k/2}$$

as $y \to 0$, uniformly in $x$. Since

$$a_n e^{-2\pi i n y} = \int_0^1 f(x+iy) e^{-2\pi i n x}\, dx$$

then combining this with the above estimate and setting $y = \frac{1}{n}$ we obtain Hecke's estimate on the Fourier coefficients of a cups form

$$|a_n| \ll n^{k/2}.$$

## 3 The $L$-function of a Cusp Form

Hecke associated to the cusp form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

the Dirichlet series, or $L$-function, formed out of its Fourier coefficients

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which converges absolutely for $Re(s) > \frac{k}{2} + 1$ by his estimate on the Fourier coefficients. The $L$-function is analytically related to $f(z)$ by the Mellin transform

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^{\infty} f(iy) y^s \, d^{\times}y$$

giving an *integral representation* for the completed $L$-function $\Lambda(s,f)$. Through this integral representation Hecke was able to derive the analytic properties of $\Lambda(s,f)$ from those of of $f(z)$.

If we take $\Gamma = SL_2(\mathbb{Z})$, then $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$ and we have that

$$f(Sz) = f(-1/z) = z^k f(z) \quad \text{or} \quad f(i/y) = i^k y^k f(iy).$$

Using this transformation law in the integral representation gives

$$\begin{aligned} \Lambda(s,f) &= \int_1^{\infty} f(iy)y^s \, d^{\times}y + \int_0^1 f(iy)y^s \, d^{\times}y \\ &= \int_1^{\infty} f(iy)y^s \, d^{\times}y + \int_1^{\infty} f(i/y)y^{-s} \, d^{\times}y \\ &= \int_1^{\infty} f(iy)y^s \, d^{\times}y + i^k \int_1^{\infty} f(iy)y^{k-s} \, d^{\times}y \\ &= i^k \Lambda(k-s,f). \end{aligned}$$

Note that from the rapidly decrease of cusp forms, the integrals from 1 to $\infty$ are all absolutely convergent for all $s$ and bounded in vertical strips.

**Theorem 1.2** *The completed $L$-function $\Lambda(s,f)$ is* nice *i.e., it converges absolutely in a half-plane and*

(i) *extends to an entire function of $s$,*
(ii) *is bounded in vertical strips,*

(iii) *satisfies the functional equation* $\Lambda(s,f) = i^k \Lambda(k-s,f)$

Moreover, Hecke was able to invert the integral representation (via the Mellin inversion formula) and prove a *Converse* to this *Theorem*.

**Theorem 1.3** *Suppose* $D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *is absolutely convergent for* $Re(s) \gg 0$ *and, setting*

$$\Lambda(s) = (2\pi)^{-s}\Gamma(s)D(s),$$

*that* $\Lambda(s)$ *is* nice, *i.e., satisfies (i)–(iii) in Theorem 1.2. Then*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

*is a cusp form of weight* $k$ *for* $SL_2(\mathbb{Z})$.

*Proof:* The convergence of the Dirichlet series gives an estimate on the coefficients of the form $|a_n| \ll n^c$ which in turn gives the convergence and holomorphy of $f(z)$ as a function on $\mathfrak{H}$. Recall that $SL_2(\mathbb{Z})$ is generated by the two transformations

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By construction we have $f(Tz) = f(z+1) = f(z)$ so we need to prove the transformation law for $f(z)$ under $S$. Since we already know $f(z)$ is holomorphic it suffices to show $f(S \cdot iy) = f(i/y) = (iy)^k f(iy)$. But by using the Mellin inversion formula and the functional equation for $\Lambda(s)$ we have

$$\begin{aligned} f(iy) = \sum_{n=1}^{\infty} a_n e^{-2\pi n y} &= \frac{1}{2\pi i}\int_{Re(s)=\frac{k}{2}} \Lambda(s) y^{-s}\, ds \\ &= \frac{i^k}{2\pi i}\int_{Re(s)=\frac{k}{2}} \Lambda(k-s) y^{-s}\, ds = \frac{i^k}{2\pi i}\int_{Re(s)=\frac{k}{2}} \Lambda(s) y^{s-k}\, ds \\ &= \frac{i^k y^{-k}}{2\pi i}\int_{Re(s)=\frac{k}{2}} \Lambda(s) y^{s}\, ds = \left(\frac{i}{y}\right)^k \frac{1}{2\pi i}\int_{Re(s)=\frac{k}{2}} \Lambda(s)\left(\frac{1}{y}\right)^{-s} ds \\ &= \left(\frac{i}{y}\right)^k f\left(\frac{i}{y}\right). \end{aligned}$$

Note then that $f(z)$ is cuspidal from its Fourier expansion. □

For $\Gamma = \Gamma_0(N)$ the situation is more complicated. The functional equation for $\Lambda(s,f)$ now comes from the action of

$$S_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

which only normalizes $\Gamma_0(N)$. However if $f(z) \in S_k(\Gamma_0(N))$ then one can show that the function $g(z)$ obtained from the action of $S_N$ on $f(z)$, namely

$$g(z) = N^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right)$$

is also in $S_k(\Gamma_0(N))$ and the Mellin transform now leads to a functional equation of the form

$$\Lambda(s,f) = i^k N^{\frac{k}{2}-s}\Lambda(k-s,g)$$

and that this function extends to an entire function of $s$ which is bounded in vertical strips, i.e., is *nice*.

The converse to this result is due to Weil. One variant of Weil's statement is the following.

**Theorem 1.4** *Let* $D_1(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *and* $D_2(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ *be absolutely convergent in some right half-plane* $Re(s) \gg 0$. *For any primitive Dirichlet character* $\chi$ *set*

$$D_1(s,\chi) = \sum_{n-1}^{\infty} \frac{\chi(n)a_n}{n^s} \quad \text{and} \quad D_2(s,\chi) = \sum_{n-1}^{\infty} \frac{\chi(n)b_n}{n^s}$$

*and set*

$$\Lambda_i(s,\chi) = (2\pi)^{-s}\Gamma(s)D_i(s,\chi).$$

*Suppose that there exists an* $N$ *such that for all primitive characters* $\chi$ *of conductor* $q$ *prime to* $N$ *we have*

(i) *the* $\Lambda_i(s,\chi)$ *extend to entire functions of* $s$,
(ii) *the* $\Lambda_i(s,\chi)$ *are bounded in vertical strips,*
(iii) *we have the functional equation*

$$\Lambda_1(s,\chi) = i^k \epsilon(\chi) N^{\frac{k}{2}-s}\Lambda_2(k-s,\overline{\chi}),$$

*with* $\epsilon(\chi) = \frac{\tau(\chi)^2}{q}\chi(N)$.

*Then both*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n e^{2\pi i n z}$$

*are cusp forms of weight* $k$ *for* $\Gamma_0(N)$ *and are related by*

$$g(z) = N^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right).$$

## 4 The Euler Product

One of Hecke's crowning achievements was to give conditions on a modular form $f(z)$ that would guarantee that its $L$-function would have an Euler product factorization. He did this via what are now known as the *Hecke operators* $T_n$ for $n \in \mathbb{N}$. In essence $T_n$ acts on a modular form by averaging it over integer matrices of determinant $n$.

To make this precise, introduce a weight $k$ action of $GL_2^+(\mathbb{R})$ on holomorphic functions on $\mathfrak{H}$ by

$$f|_k g(z) = \frac{\det(g)^{k/2}}{(cz+d)^k} f(gz) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

(This is the action of $S_N$ that we spoke of without defining in the previous section.) Then the condition of modularity of weight $k$ for $f(z)$ with respect to $\Gamma$ becomes simply $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

If we let

$$\mathcal{L}_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \Big| \ ad - bc = n \right\}$$

and

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathcal{L}_n \Big| \ ad = n \quad \text{and} \quad 0 \le b < d \right\}$$

then

$$\mathcal{L}_n = \coprod_{\delta \in \Delta_n} SL_2(\mathbb{Z})\delta.$$

We can then define the Hecke operator (or averaging operator) $T_n$ on $M_k(SL_2(\mathbb{Z}))$ by

$$T_n f(z) = n^{\frac{k}{2}-1} \sum_{\delta \in \Delta_n} f|_k \delta(z) = n^{k-1} \sum_{\delta \in \Delta_n} d^{-k} f\left(\frac{az+b}{d}\right).$$

Here are some basic facts about these Hecke operators for $\Gamma = SL_2(\mathbb{Z})$.

(i) $T_n : M_k(\Gamma) \to M_k(\Gamma)$ and preserves $S_k(\Gamma)$.
(ii) $T_n \cdot T_m = \sum_{d|(m,n)} d^{k-1} T_{\frac{nm}{d^2}} = T_m \cdot T_n$.

In particular

(iii) If $(n,m) = 1$ then $T_n \cdot T_m = T_{nm}$.
(iv) If $p$ is a prime then $T_p \cdot T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$.

Let $\mathcal{H}$ denote the $\mathbb{Z}$-algebra generated by the $T_n$. This is the *Hecke algebra*. It is commutative and generated by the $T_p$ for $p$ prime.

One can easily compute the action of the Hecke operators on modular forms in terms of their Fourier expansions. If we write

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \quad \text{and} \quad T_m f(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

then we find

$$b_n = \sum_{d|(m,n)} d^{k-1} a_{\frac{nm}{d^2}}$$

and in particular

$$b_0 = \sigma_{k-1}(m) a_0 \quad \text{and} \quad b_1 = a_m$$

thus showing that the $T_n$ preserve the space of cusp forms as claimed.

Suppose now that

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

is a cusp form of weight $k$ for $SL_2(\mathbb{Z})$ which is a simultaneous eigen-function for all the Hecke operators. If we set $T_n f = \lambda(n) f$ then we find that the Fourier coefficients are related to the Hecke eigenvalues by

$$\lambda(n) a_1 = a_n$$

coming from the computation of the first Fourier coefficient of $T_n f$ above. So if we normalize $f(z)$ by requiring $a_1 = 1$ then we have $\lambda(n) = a_n$ so that the Hecke eigen-values carry the same arithmetic information that the Fourier coefficients do. In addition, from the relations among the Hecke operators, and thus the Hecke eigen-values, we obtain the following recursions on the Fourier coefficients of $f$.

(i) If $(n, m) = 1$ then $a_n a_m = a_{nm}$.
(ii) If $p$ is a prime then $a_p a_{p^r} = a_{p^{r+1}} + p^{k-1} a_{p^{r-1}}$ or

$$a_{p^{r+1}} - a_p a_{p^r} + p^{k-1} a_{p^{r-1}} = 0.$$

If we see what these imply about the $L$-function associated to $f$ we find

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \left( \sum_{r=0}^{\infty} \frac{a_{p^r}}{p^{rs}} \right) = \prod_p \left(1 - a_p p^{-s} + p^{k-1} p^{-2s}\right)^{-1}.$$

**Theorem 1.5** *Let $f(z) \in S_k(SL_2(\mathbb{Z}))$ have $a_1 = 1$. Then $f$ is an eigen-function for all the Hecke operators iff*

$$L(s, f) = \prod_p \left(1 - a_p p^{-s} + p^{k-1} p^{-2s}\right)^{-1}$$

*or*

$$\Lambda(s, f) = (2\pi)^{-s} \Gamma(s) \prod_p \left(1 - a_p p^{-s} + p^{k-1} p^{-2s}\right)^{-1}.$$

## 5 References

[1] E. Hecke, *Über die Bestimmung Dirischletscher Reihen durch ihre Funktionalgleichung*. Math. Ann. **112** (1936), 664–699.

[2] E. Hecke, Mathematische Werke. Vandenhoeck & Ruprecht, Göttingen, 1959.

[3] H. Iwaniec, Topics in Classical Automorphic Forms. Graduate Studies in Mathematics, 17. American Mathematical Society, Providence, RI, 1997.

[4] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions. Princeton University Press, Princeton, NJ, 1994.

[5] A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen.* Math. Ann. **168** (1967), 149–156.

# Automorphic Forms

In this lecture I want to begin the passage from classical modular forms $f$ to automorphic forms $\varphi$ and finally to automorphic representations $\pi$. This will entail a change of tools from the theory of one complex variable to the use of non-abelian harmonic analysis, that is, representation theory.

## 1 Automorphic Forms on $GL_2$

We begin with a classical modular form $f \in M_m(\Gamma)$ for $\Gamma = SL_2(\mathbb{Z})$. So $f : \mathfrak{H} \to \mathbb{C}$. The upper half plane $\mathfrak{H}$ is a symmetric space for $GL_2^+(\mathbb{R})$ acting by linear fractional transformations. If we take $i \in \mathfrak{H}$ as a base point then $\mathfrak{H} = GL_2^+(\mathbb{R}) \cdot i$. The stabilizer of $i$ in $GL_2^+(\mathbb{R})$ is $Z \cdot K_\infty^+$ where $Z = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \right\}$ is the center of $GL_2^+(\mathbb{R})$ and $K_\infty^+ = SO(2)$ is the maximal compact subgroup of $GL_2^+(\mathbb{R})$. We can lift $f$ to a function $F$ on $GL_2^+(\mathbb{R})$ by

$$\begin{array}{ccc} GL_2^+(\mathbb{R}) & & \\ \downarrow & \overset{F}{\searrow} & \\ \mathfrak{H} & \xrightarrow{f} & \mathbb{C} \end{array} .$$

Then $F$ is defined by $F(g) = f(g \cdot i)$ for $g \in GL_2^+(\mathbb{R})$ and it satisfies

$$F(zgk) = F(g) \quad \text{for} \quad z \in Z,\ k \in K_\infty^+.$$

$F$ still has a modular transformation law under $\Gamma = SL_2(\mathbb{Z}) = GL_2^+(\mathbb{Z})$. We would like to work with $\Gamma$-invariant functions. To this end, set

$$j(g; z) = \det(g)^{-1/2}(cz + d) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}),\ z \in \mathfrak{H}.$$

Then modularity for $f(z)$ becomes $f(\gamma z) = j(\gamma; z)^m f(z)$ for all $\gamma \in \Gamma$. If we set

$$\varphi(g) = j(g; i)^{-m} F(g) = j(g; i)^{-m} f(g \cdot i)$$

then one easily checks that $\varphi(g)$ satisfies

(i) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$ and $g \in GL_2^+(\mathbb{R})$
(ii) $\varphi(zg) = \varphi(g)$ for all $z \in Z$
(iii) $\varphi(gk_\theta) = e^{\pi i m \theta} \varphi(g)$ for $k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in K_\infty^+$

We have interchanged the properties of $\Gamma$-modularity and $K_\infty^+$-invariance for $\Gamma$-invariance and a $K_\infty^+$-transformation law.

Now let $\mathfrak{g}$ denote the complexified Lie algebra of $GL_2(\mathbb{R})$. Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$ and $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. $\mathcal{Z}$ is the space of invariant differential operators on $GL_2(\mathbb{R})$. Then the holomorphy of $f(z)$ can be expressed in terms of these operators as

(iv) $\varphi(g)$ is an eigen-function for $\mathcal{Z}$.

Finally, one can express the growth condition of $f(z)$ being holomorphic at infinity as

(v) $\varphi(g)$ is of moderate growth on $GL_2^+(\mathbb{R})$, i.e., for any norm $\|\ \|$ on $GL_2^+(\mathbb{R})$ there exists a positive integer $r$ such that

$$|\varphi(g)| \leq C\|g\|^r.$$

For the norm we can take $\|g\| = (tr(g^t g) + tr((g^{-1})^t g^{-1}))^{1/2}$.

Note that we could do the same passage for a holomorphic modular form for some $\Gamma_0(N)$ or for a Maass form.

Functions on $GL_2^+(\mathbb{R})$ that satisfy (i) – (v) are examples of automorphic forms.

For our purposes it will be more convenient to work with automorphic forms on $GL_2(\mathbb{A})$ where $\mathbb{A}$ is a ring of adeles. Recall that $\mathbb{Q}$ has several completions, namely $\mathbb{R} = \mathbb{Q}_\infty$ and the various $\mathbb{Q}_p$ for primes $p$. The ring of adeles $\mathbb{A}$ of $\mathbb{Q}$ is then the restricted product of these completions

$$\mathbb{A} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p = \prod_v{}' \mathbb{Q}_v \subset \prod_v \mathbb{Q}_v$$

with respect to the compact open subrings $\mathbb{Z}_p \subset \mathbb{Q}_p$. More precisely if we let $S_f$ run over all finite sets of primes then $\mathbb{A}$ is the union, or inductive limit,

$$\mathbb{A} = \varinjlim_{S_f} \left( \mathbb{R} \times \prod_{p \in S_f} \mathbb{Q}_p \times \prod_{p \notin S_f} \mathbb{Z}_p \right).$$

Each $\mathbb{R} \times \prod_{p\in S_f} \mathbb{Q}_p \times \prod_{p \notin S_f} \mathbb{Z}_p$ receives the product topology and $\mathbb{A}$ the inductive limit topology. Then $\mathbb{Q} \hookrightarrow \mathbb{A}$ diagonally as a canonical discrete subgroup and the quotient $\mathbb{Q}\backslash\mathbb{A}$ is compact. Note that $\mathbb{Z} = \mathbb{Q} \cap (\mathbb{R} \times \prod \mathbb{Z}_p)$.

Accordingly, one has

$$\begin{aligned} GL_2(\mathbb{A}) &= GL_2(\mathbb{R}) \times \prod_p{}' GL_2(\mathbb{Q}_p) = \prod_v{}' GL_2(\mathbb{Q}_v) \\ &= \varinjlim_{S_f} \left( GL_2(\mathbb{R}) \times \prod_{p\in S_f} GL_2(\mathbb{Q}_p) \times \prod_{p \notin S_f} GL_2(\mathbb{Z}_p) \right) \end{aligned}$$

a restricted product with respect to the maximal open compact subgroups $GL_2(\mathbb{Z}_p)$. Once again $GL_2(\mathbb{Q}) \hookrightarrow GL_2(\mathbb{A})$ diagonally as a canonical discrete subgroup having finite co-volume modulo the center.

Let us now set $G = GL_2$ and let

$$\begin{aligned} G_\infty &= G(\mathbb{R}) \supset K = O(2) \\ G_f &= \prod_p{}' G(\mathbb{Q}_p) \supset K_f = \prod_p \mathbb{Z}_p \\ K &= K_\infty K_f \subset G_\infty G_f = G(\mathbb{A}). \end{aligned}$$

The groups $K_\infty$, $K_f$ and $K$ are all maximal compact subgroups and $K_f$ is open in $G_f$. Then Strong Approximation for $SL_2$ combined with the fact that $\mathbb{Q}$ has class number one lets us write

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot G^+(\mathbb{R}) K_f$$

and since

$$\Gamma = GL_2^+(\mathbb{Z}) = GL_2(\mathbb{Q}) \cap (G^+(\mathbb{R}) K_f)$$

we have

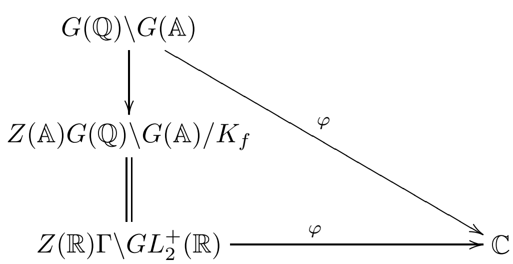
$$\Gamma \backslash GL_2^+(\mathbb{R}) = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f$$

and

$$Z(\mathbb{R}) \Gamma \backslash GL_2^+(\mathbb{R}) = Z(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f.$$

To carry out this process for $\Gamma = \Gamma_0(N)$ we would replace $K_f$ by an appropriate open compact subgroup $L \subset K_f$ which would no longer be maximal.

If we return to our automorphic form $\varphi$ on $Z(\mathbb{R})\Gamma\backslash GL_2^+(\mathbb{R})$ we can further lift it to a function, still denoted by $\varphi$, on $G(\mathbb{Q})\backslash G(\mathbb{A})$ by




The function $\varphi(g)$ on $G(\mathbb{A})$ which we construct in this way will be a *smooth* function in the following sense. If we write $g \in G(\mathbb{A}) = G_\infty \cdot G_f$ as $g = (g_\infty, g_f)$ then $\varphi(g) = \varphi(g_\infty, g_f)$ will be $C^\infty$ in the archimedean $g_\infty$ variable and locally constant in the non-archimedean $g_f$ variables. Moreover it will satisfy:

(i) [automorphy] $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Q})$;
(ii) [$K$-finite] $\varphi(g k_\theta k_f) = e^{im\theta}\varphi(g)$ for $k_\theta \in K_\infty^+$ and $k_f \in K_f$, or, more generally, the space $\langle \varphi(gk) \mid k \in K \rangle$ is finite dimensional;
(iii) [$\mathcal{Z}$-finite] there exists an ideal $\mathcal{J} \subset \mathcal{Z}$ of finite co-dimension such that $\mathcal{J} \cdot \varphi = 0$, or equivalently, the space $\langle X\varphi(g) \mid X \in \mathcal{Z} \rangle$ is finite dimensional;

(iv) [moderate growth] for any norm $\| \ \|$ on $G(\mathbb{A})$ there exists a positive integer $r$ such that

$$|\varphi(g)| \leq C\|g\|^r.$$

For an adelic norm on $G(\mathbb{A})$ we can take

$$\|g\| = \prod_v \left( \max_{i,j} \{|g_{i,j}|_v, |(g^{-1})_{i,j}|_v\} \right).$$

**Definition 2.1** *A smooth function $\varphi : GL_2(\mathbb{A}) \to \mathbb{C}$ satisfying conditions (i) – (iv) is called a ($K$-finite) automorphic form on $GL_2(\mathbb{A})$.*

We let $\mathcal{A} = \mathcal{A}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}))$ denote the space of automorphic forms on $GL_2$. If we wish to specify a behavior under the center $Z(\mathbb{A})$ then for any continuous character $\omega : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ we let

$$\mathcal{A}(\omega) = \{\varphi \in \mathcal{A} \mid \varphi(zg) = \omega(z)\varphi(g) \quad \text{for} \quad z \in Z(\mathbb{A})\}.$$

With this generality in the conditions (i)–(iv), the space $\mathcal{A}$ will contain the lifts of all holomorphic modular forms and all Maass forms for all $\Gamma_0(N)$ as well.

## 2 Automorphic Forms on $GL_n$

It should be clear how to define automorphic forms on $GL_n(k)\backslash GL_n(\mathbb{A})$ for $\mathbb{A}$ the ring of adeles for any global field $k$. For our purposes, we will stick to $\mathbb{A}$ being the ring of adeles of a *number field* $k$. Let $\mathcal{O}$ denote the ring of integers of $k$.

The ring $\mathbb{A}$ is then the restricted product of the completions $k_v$ of $k$ with respect to the maximal compact subrings $\mathcal{O}_v \subset k_v$ for non-archimedean places $v < \infty$.

$$\mathbb{A} = \prod_v{}' k_v = \varinjlim_S \left( \prod_{v\in S} k_v \times \prod_{v\notin S} \mathcal{O}_v \right)$$

where now we have taken $S$ to run through all finite sets of places of $k$ such that $S$ contains $\mathcal{V}_\infty = \{v \mid v|\infty\}$, the set of archimedean places of $k$. Then we can write $\mathbb{A} = k_\infty \mathbb{A}_f$ where

$$k_\infty = \prod_{v|\infty} k_v \quad \text{and} \quad \mathbb{A}_f = \prod_{v<\infty}{}' k_v.$$

If $k$ has $r_1$ real embeddings and $r_2$ pairs of complex embeddings, then $k_\infty = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

Then

$$GL_n(\mathbb{A}) = \prod_v{}' GL_n(k_v) = \varinjlim_S \left( \prod_{v\in S} GL_n(k_v) \times \prod_{v\notin S} GL_n(\mathcal{O}_v) \right)$$

is the restricted product with respect to the maximal open compact subgroups $K_v = GL_n(\mathcal{O}_v) \subset GL_n(k_v)$ for the non-archimedean places. If we agree to now let $G = GL_n$ then as before we have

$$G(\mathbb{A}) = G_\infty \cdot G_f \supset K = K_\infty \cdot K_f$$

where

$$G_\infty = GL_n(\mathbb{R})^{r_1} \times GL_n(\mathbb{C})^{r_2} \supset K_\infty = O(n)^{r_1} \times U(n)^{r_2}$$
$$G_f = GL_n(\mathbb{A}_f) = \prod_{v<\infty}{}' GL_n(k_v) \supset K_f = \prod_{v<\infty} GL_n(\mathcal{O}_v).$$

Again, $G(k) \hookrightarrow G(\mathbb{A})$ diagonally as a canonical discrete subgroup with finite co-volume modulo the center $Z(\mathbb{A})$.

Let $\mathcal{Z} = \mathcal{Z}(\mathfrak{g})$ denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g}$ of $G_\infty$.

**Definition 2.2** *A smooth function $\varphi : GL_n(\mathbb{A}) \to \mathbb{C}$ is called a ($K$-finite) automorphic form if it satisfies:*

(i) *[automorphy] $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Q})$;*
(ii) *[$K$-finite] the space $\langle \varphi(gk) \mid k \in K \rangle$ is finite dimensional;*
(iii) *[$\mathcal{Z}$-finite] the space $\langle X\varphi(g) \mid X \in \mathcal{Z} \rangle$ is finite dimensional;*
(iv) *[moderate growth] for any norm $\| \ \|$ on $G(\mathbb{A})$ there exists a positive integer $r$ such that*

$$|\varphi(g)| \le C\|g\|^r.$$

We again denote this space by $\mathcal{A} = \mathcal{A}(GL_n(k)\backslash GL_n(\mathbb{A}))$.

As in the classical case, the conditions defining automorphic forms imply strong finiteness results.

**Theorem 2.1 (Harish-Chandra)** *If we fix $\delta$ be a finite dimensional representation of $K_\infty$, $L \subset K_f$ a compact open subgroup, $\mathcal{J} \subset \mathcal{Z}$ an ideal of finite co-dimension, and $\omega : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ a central character and let $\mathcal{A}(\delta, L, \mathcal{J}, \omega)$ denote the set of $\varphi \in \mathcal{A}(\omega)$ such that*

(i) *$\varphi$ transforms by $\delta$ under $K_\infty$,*
(ii) *$\varphi(g\ell) = \varphi(g)$ for all $\ell \in L$,*
(iii) *$\mathcal{J} \cdot \varphi = 0$.*

*then* $\dim_{\mathbb{C}} \mathcal{A}(\delta, L, \mathcal{J}, \omega) < \infty$.

## 3 Smooth Automorphic Forms

One would hope to be able to analyze $\mathcal{A}$ as a representation of $GL_n(\mathbb{A})$ acting by right translation. Unfortunately, this is not possible since condition (ii) in the definition of automorphic forms is not preserved under right translation. More specifically, it is being $K_\infty$-finite that is not preserved under right translation by $G_\infty$.

[To make this more precise, consider $\varphi(g) \in \mathcal{A}$ and set

$$\varphi'(g) = R(g')\varphi(g) = \varphi(gg').$$

Then $\varphi(g)$ is right $K$-finite and $\varphi'(g)$ is naturally right $K'$-finite where $K' = g'K(g')^{-1}$ is a conjugate of $K$. Now, at the finite places, since $K_f$ and $K'_f$ are both compact and open, the intersection $K_f \cap K'_f$ is of finite index in both. So there is no difference between $K_f$-finiteness and $K'_f$-finiteness. On the other hand at the archimedean places there is no reason for $K_\infty$ and $K'_\infty$ to have anything more in common than the identity. This is quite apparent when considering $GL_2$ or $SL_2$ where the maximal compacts are one dimensional. So while the notion of being $K_f$-finite is really independent of the choice of maximal compact, the notion of $K_\infty$-finiteness is dependent on the choice of $K_\infty$.]

There are two ways to remedy this: (i) settle for representations of something smaller – namely the Hecke algebra $\mathcal{H}$; or (ii) enlarge the space of automorphic forms. We will address the Hecke algebra in the next lecture. The most natural enlargement is the space of *smooth* automorphic forms, in which the condition of $K_\infty$-finiteness is weakened to a condition of *uniform moderate growth*.

**Definition 2.3** *A smooth function* $\varphi : GL_n(\mathbb{A}) \to \mathbb{C}$ *is called a* smooth *automorphic form if it satisfies:*

(i) *[automorphy]* $\varphi(\gamma g) = \varphi(g)$ *for all* $\gamma \in G(\mathbb{Q})$*;*
(ii) *[$K_f$-finite] there is a compact open subgroup* $L \subset K_f$ *such that* $\varphi(g\ell) = \varphi(g)$ *for all* $\ell \in L$*;*
(iii) *[$\mathcal{Z}$-finite] there exists an ideal* $\mathcal{J} \subset \mathcal{Z}$ *of finite co-dimension such that* $\mathcal{J}\varphi = 0$*;*
(iv) *[uniform moderate growth] there exists a positive integer $r$ such that for all differential operators* $X \in \mathcal{U}(\mathfrak{g})$
$$|X\varphi(g)| \leq C_X \|g\|^r.$$

We will denote the space of smooth automorphic forms by
$$\mathcal{A}^\infty = \mathcal{A}^\infty(GL_n(k)\backslash GL_n(\mathbb{A})).$$
Now $GL_n(\mathbb{A})$ does act on $\mathcal{A}^\infty$ by right translation. Moreover $\mathcal{A}^\infty$ will carry a limit Fréchet topology coming from the uniform moderate growth semi-norms. $\mathcal{A}^\infty$ is not that far removed from $\mathcal{A}$. By a theorem of Harish-Chandra we know that $K_\infty$-finiteness implies uniform moderate growth, so that $\mathcal{A} \subset \mathcal{A}^\infty$ and $\mathcal{A}$ is precisely the space of $\varphi \in \mathcal{A}^\infty$ that are $K$-finite, and that in fact $\mathcal{A}$ is dense in $\mathcal{A}^\infty$ in this natural topology.

## 4 $L^2$-automorphic Forms

Another natural class of automorphic forms are the $L^2$-automorphic forms. To define these we must fix a *unitary* central character $\omega : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^1$. Then
$$L^2(\omega) = L^2(GL_n(k)\backslash GL_n(\mathbb{A}); \omega)$$
is the space of all measurable $\varphi : GL_n(k)\backslash GL_n(\mathbb{A}) \to \mathbb{C}$ such that $\varphi(zg) = \omega(z)\varphi(g)$ for $z \in Z(\mathbb{A})$ and
$$\int_{Z(\mathbb{A})GL_n(k)\backslash GL_n(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty.$$

This is a Hilbert space and the group $GL_n(\mathbb{A})$ acts by right translation on this space preserving the norm; hence $L^2(\omega)$ affords a unitary representation of $GL_n(\mathbb{A})$.

## 5 Cusp Forms

As in the classical case, the cusp forms will play a special role for us. Recall that if $f(z)$ is a classical modular form for $\Gamma = SL_2(\mathbb{Z})$ then $f$ is a cusp form if

$$0 = a_0 = \int_0^1 f(x+iy)\ dx = \int_0^1 f\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\cdot iy\right)\ dx.$$

If $\varphi(g)$ is an automorphic form on $GL_2$, then the group of translations is

$$N_2 = \left\{ n = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right\} \subset GL_2.$$

The analogous integral would be over $N_2(k)\backslash N_2(\mathbb{A}) \simeq k\backslash\mathbb{A}$, which is compact. Thus we have the following definition.

**Definition 2.4** *An automorphic form $\varphi(g)$ on $GL_2(\mathbb{A})$ is a cusp form iff*

$$\int_{N_2(k)\backslash N_2(\mathbb{A})} \varphi(ng)\ dn = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\ dx = 0.$$

This is the same no matter whether $\varphi \in \mathcal{A}$, $\mathcal{A}^\infty$, or $L^2$.

For $GL_n$ there are many translation subgroups. They are given by the unipotent radicals of (rational) parabolic subgroups. For $GL_n$ the parabolic subgroups are parameterized (up to conjugation) by partitions $n = n_1 + \cdots + n_r$ of $n$. The associated parabolic is

$$P = \left\{ \begin{pmatrix} g_1 & * & \cdots & * \\ 0 & g_2 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & g_r \end{pmatrix} \Big| g_i \in GL_{n_i} \right\} = M \cdot U$$

where $M \simeq GL_{n_1} \times \cdots \times GL_{n_r}$ is the Levi subgroup and

$$U = \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & I_{n_r} \end{pmatrix} \right\}$$

is the unipotent radical of $P$. $P$ is called proper if the partition is non-trivial. As for $GL_2$ for any of these unipotent radicals, $U(k)\backslash U(\mathbb{A})$ is compact.

**Definition 2.5** *An automorphic form $\varphi(g)$ on $GL_n(\mathbb{A})$ is a cusp form iff*

$$\int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\ du = 0$$

*for all unipotent radicals of all proper parabolic subgroups of $GL_n$.*

Since all parabolics are (rationally) conjugate to a standard (upper triangular) one, it suffices to only consider integrals over the standard unipotent radicals. (This is often referred to by saying that when one works adelically $GL_n$ has only one cusp.) In fact, it suffices to consider only unipotent radicals of maximal parabolic subgroups, i.e., those associated to partitions with only two terms $n = n_1 + n_2$, so

$$P = \left\{ \begin{pmatrix} g_1 & X \\ & g_2 \end{pmatrix} \right\} \supset U = \left\{ \begin{pmatrix} I_{n_1} & X \\ & I_{n_2} \end{pmatrix} \right\} \simeq M_{n_1 \times n_2},$$

since at least one of these unipotent groups will occur as a normal subgroup in any other standard unipotent radical.

As in the classical case we have that cusp forms are rapidly decreasing.

**Theorem 2.2 (Gelfand and Piatetski-Shapiro)** *If $\varphi(g)$ is a cusp form, then it is rapidly decreasing modulo the center on a fundamental domain $\mathcal{F}$ for $GL_n(k)\backslash GL_n(\mathbb{A})$, that is, for some integer $r$*

$$|g(zg)| \leq C|z|^r \|g\|^{-N} \quad \textit{for all} \quad N \in \mathbb{N}$$

*where we restrict $g$ to lie in $\mathcal{F} \cap SL_n(\mathbb{A})$.*

We should note that exact fundamental domains for $GL_n(\mathbb{A})$ are rather unwieldy and instead one usually replaces $\mathcal{F}$ with a slightly bigger set $\mathfrak{S}$, called a Siegel set, which is easier to construct.

We will denote the subspaces of cusp forms by $\mathcal{A}_0$, $\mathcal{A}_0^\infty$, and $L_0^2$. Note that if we fix a unitary central character $\omega$ then one consequence of this rapid decay is the containment $\mathcal{A}_0(\omega) \subset \mathcal{A}_0^\infty(\omega) \subset L_0^2(\omega)$.

## 6 References

[1] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations.* Proc. Sympos. Pure Math. **33**, part 1, (1979), 189–207.

[2] S. Gelbart, Automorphic Forms on Adele Groups. Annals of Math. Studies **83**, Princeton University Press, Princeton, 1975.

[3] I.M. Gelfand, M.I. Graev, and I.I. Piatetski-Shapiro, Representation Theory and Automorphic Functions. Saunders, Philadelphia, 1968.

[4] Harish-Chandra, Automorphic forms on Semisimple Lie Groups. Lecture Notes in Mathematics, No. 62 Springer-Verlag, Berlin-New York 1968.

[5] N.R. Wallach, *$C^\infty$ vectors.* Representations of Lie Groups and Quantum Groups, Pitman Res. Notes Math. Ser. **311**, Longman Sci. Tech., Harlow, 1994, 205–270.

LECTURE 3

# Automorphic Representations

We have defined our spaces of automorphic forms. Now we turn to our tools. We will analyze $\mathcal{A}$, $\mathcal{A}^\infty$, or $L^2(\omega)$ as representation spaces for certain algebras or groups. Throughout we will let $G = GL_n$, although the results remain true for any reductive algebraic group $G$, let $k$ be a number field, and retain all notations from before.

## 1 ($K$-finite) automorphic representations

As we have noted the space $\mathcal{A}$ of ($K$-finite) automorphic forms does not give a representation of $G(\mathbb{A})$. It will be a representation space for the global Hecke algebra $\mathcal{H}$.

1.0.1 *The Hecke algebra.* The global Hecke algebra $\mathcal{H}$ will be a restricted tensor product of local Hecke algebras: $\mathcal{H} = \otimes' \mathcal{H}_v$. $\mathcal{H}$ and each $\mathcal{H}_v$ will be idempotented algebras under convolution. So there will be a directed family of fundamental idempotents $\{\xi_i\}$ such that

$$\mathcal{H} = \varinjlim_i \xi_i * \mathcal{H} * \xi_i = \bigcup_i \xi_i * \mathcal{H} * \xi_i$$

and

$$\mathcal{H}_v = \varinjlim_i \xi_{i,v} * \mathcal{H}_v * \xi_{i,v} = \bigcup_i \xi_{i,v} * \mathcal{H} * \xi_{i,v}.$$

Neither $\mathcal{H}$ nor any $\mathcal{H}_v$ will have an identity, but for each $\xi_i$ the subalgebra $\xi_i * \mathcal{H} * \xi_i$ will have $\xi_i$ as an identity.

(i) If $v < \infty$ is an non-archimedean place of $k$ then $\mathcal{H}_v = C_c^\infty(G(k_v))$ is the algebra of smooth (locally compact) compactly supported functions on $G_v = G(k_v)$. It is naturally an algebra under convolution. For each compact open subgroup $L_v \subset G_v$ there is a fundamental idempotent

$$\xi_{L_v} = \frac{1}{Vol(L_v)} \mathfrak{X}_{L_v}$$

where $\mathfrak{X}_{L_v}$ is the characteristic function of $L_v$. Then $\xi_{L_v} * \mathcal{H}_v * \xi_{L_v} = \mathcal{H}(G_v // L_v)$ is the algebra of $L_v$-bi-invariant compactly supported functions on $G_v$. In any representation of $\mathcal{H}_v$ the idempotent $\xi_{L_v}$ will act as a projection onto the $L_v$-fixed vectors. We will let $\xi_v^\circ$ denote the fundamental idempotent associated to the maximal compact subgroup $K_v$. Note that if $k = \mathbb{Q}$, $G = GL_2$, and $L_p = K_p = GL_2(\mathbb{Z}_p)$ then $\xi_p^\circ * \mathcal{H}_p * \xi_p^\circ = \mathcal{H}(GL_2(\mathbb{Q}_p)//GL_2(\mathbb{Z}_p))$ is isomorphic to the complex algebra spanned by the classical Hecke operators $\langle T_{p^r} \rangle$.

(ii) If $v|\infty$ is an archimedean place of $k$ then $\mathcal{H}_v$ is the convolution algebra of bi-$K_v$-finite distributions on $G_v$ with support in $K_v$. Then $\mathcal{H}_v$ contains both

$$\mathcal{U}(\mathfrak{g}):\ \text{distributions supported at the identity}$$

and

$$A(K_v):\ \text{finite measures on } K_v$$

and in fact

$$\mathcal{H}_v = \mathcal{U}(\mathfrak{g}_v) \otimes_{\mathcal{U}(\mathfrak{k}_v)} A(K_v).$$

For each finite dimensional representation $\delta_v$ of $K_v$ we have a fundamental idempotent

$$\xi_{\delta_v} = \frac{1}{\deg(\delta_v)} \Theta_{\delta_v} dk_v$$

where $\deg(\delta_v)$ is the degree and $\Theta_{\delta_v}$ is the character of $\delta_v$ and $dk_v$ is the normalized Haar measure on $K_v$ . In any representation $\delta_v$ should act as the projection onto the $\delta_v$-isotypic component.

(iii) The global Hecke algebra $\mathcal{H}$ is then the restricted tensor product of the local algebras $\mathcal{H}_v$ with respect to the idempotents $\{\xi_v^\circ\}$ at the non-archimedean places., i.e.,

$$\mathcal{H} = \otimes_v' \mathcal{H}_v = \varinjlim_{S} ((\otimes_{v\in S}\mathcal{H}_v) \otimes (\otimes_{v\notin S}\xi_v^\circ))$$

as $S$ runs over finite sets of places of $k$ which contain all archimedean places $\mathcal{V}_\infty$. Let us write $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f$ where, as usual,

$$\mathcal{H}_\infty = \otimes_{v|\infty}\mathcal{H}_v \quad \text{and} \quad \mathcal{H}_f = \otimes_{v<\infty}' \mathcal{H}_v.$$

Then the fundamental idempotents in $\mathcal{H}$ are of the form $\xi = \xi_\infty \otimes \xi_f$ where

$$\xi_\infty = \xi_\delta = \otimes_{v|\infty}\xi_{\delta_v} \in \mathcal{H}_\infty$$

is associated to a finite dimensional representation $\delta = \otimes\delta_v$ of $K_\infty$ and

$$\xi_f = \xi_L = \otimes_{v<\infty}\xi_{L_v} \in \mathcal{H}_f$$

is associated to a compact open subgroup $L = \prod L_v$ of $G_f$ (so for almost all places $L_v = K_v$ and $\xi_{L_v} = \xi_v^\circ$).

1.0.2 *The representation on automorphic forms.* The space $\mathcal{A}$ of $K$-finite automorphic forms is naturally an $\mathcal{H}$-module by right convolution. For $\xi \in \mathcal{H}$ and $\varphi \in \mathcal{A}$ set

$$R(\xi)\varphi(g) = \varphi * \check{\xi}(g) = \int_{G(\mathbb{A})} \varphi(gh)\xi(h)\ dh$$

where $\check{\xi}(g) = \xi(g^{-1})$. Note that with this action the $K$-finiteness condition on $\varphi \in \mathcal{A}$ can now be stated as: there exists a fundamental idempotent $\xi = \xi_\infty \otimes \xi_f = \xi_\delta \otimes \xi_L$ such that $R(\xi)\varphi = \varphi$.

The representations that we will be most interested in will be *admissible* representations of $\mathcal{H}$.

**Definition 3.1** *A representation $(\pi_v, V_v)$ of a local Hecke algebra $\mathcal{H}_v$ is admissible if for every fundamental idempotent $\xi_v$ we have*

$$\dim_{\mathbb{C}}(\pi_v(\xi_v)V_v) < \infty.$$

*Similarly a representation $(\pi, V)$ of the global Hecke algebra $\mathcal{H}$ is admissible if for every global fundamental idempotent $\xi \in \mathcal{H}$ the subspace $\pi(\xi)V$ is finite dimensional.*

One consequence of admissibility, which we state in the global case, is that as a representation of $K$ the space $V$ decomposes into a direct sum of irreducibles with finite multiplicities:

$$V = \bigoplus_{\tau \in \hat{K}} m(\tau, V) V_\tau.$$

The reason for our interest in admissible representations is the following fundamental result of Harish-Chandra (probably first due to Jacquet and Langlands for $GL_2$).

**Theorem 3.1** *Suppose $\varphi \in \mathcal{A}$. Then the $\mathcal{H}$-module generated by $\varphi$, namely*

$$V_\varphi = R(\mathcal{H})\varphi = \varphi * \mathcal{H} \subset \mathcal{A},$$

*is an admissible $\mathcal{H}$-module.*

This makes the following definition reasonable.

**Definition 3.2** *An automorphic representation $(\pi, V)$ of $\mathcal{H}$ is an irreducible (hence admissible) sub-quotient of $\mathcal{A}(G(k)\backslash G(\mathbb{A}))$.*

There is a canonical way to construct admissible representations of $\mathcal{H}$ abstractly using the restricted tensor product structure $\mathcal{H} = \otimes' \mathcal{H}_v$. Suppose we have a collection $\{(\pi_v, V_v)\}$ of admissible representations of the local Hecke algebras $\mathcal{H}_v$ such that for almost all finite places the representation $V_v$ contains a (fixed) $K_v$-invariant vector, say $u_v^\circ$. Then we can define the restricted tensor product of these representations with respect to the $\{u_v^\circ\}$ in the (by now) usual manner:

$$V = \otimes_v' V_v = \varinjlim_S \left( (\otimes_{v \in S} V_v) \otimes (\otimes_{v \notin S} u_v^\circ) \right).$$

Note that since $u_v^\circ$ is $K_v$-fixed, then $\pi_v(\xi_v^\circ) u_v^\circ = u_v^\circ$ so this space does carry a natural representation of $\mathcal{H}$, coming from its restricted tensor product decomposition, which we will denote by $\pi = \otimes' \pi_v$. We leave it as an exercise to verify that if each of the $(\pi_v, V_v)$ is admissible then so is $(\pi, V)$ and if each $(\pi_v, V_v)$ is irreducible, then so is $(\pi, V)$.

An important fact for us, which is a purely algebraic fact about $\mathcal{H}$-modules, is the converse to this construction.

**Theorem 3.2 (Decomposition Theorem)** *If $(\pi, V)$ is an irreducible admissible representation of $\mathcal{H}$ then for each place $v$ of $k$ there exists an irreducible admissible representation $(\pi_v, V_v)$ of $\mathcal{H}_v$, having a $K_v$-fixed vector for almost all $v$, such that $\pi = \otimes' \pi_v$.*

Therefore in the context of automorphic representations of $\mathcal{H}$ we have the following corollary.

**Corollary 3.2.1** *If $(\pi, V)$ is an automorphic representation, then $\pi$ decomposes into a restricted tensor product of local irreducible admissible representations: $\pi = \otimes' \pi_v$.*

Note that the decomposition given in this corollary is an abstract decomposition. It does not give a factorization of automorphic forms into a product of functions on the local groups $G(k_v)$.

## 2 Smooth automorphic representations

Now things are more straight forward on the one hand, since $G(\mathbb{A})$ acts in $\mathcal{A}^\infty(G(k)\backslash G(\mathbb{A}))$ by right translation. However the representation theory is now a bit more complicated. More precisely, for every compact open subgroup $L \subset K_f$ the space of $L$-invariant functions $(\mathcal{A}^\infty)^L$ in $\mathcal{A}^\infty$, namely

$$(\mathcal{A}^\infty)^L = \{\varphi \in \mathcal{A}^\infty \mid \varphi(g\ell) = \varphi(g) \quad \text{for} \quad \ell \in L\},$$

is a representation for $G_\infty$. The spaces $(\mathcal{A}^\infty)^L$ all carry compatible limits of smooth Fréchet topologies coming from the uniform moderate growth semi-norms on $\mathcal{A}^\infty$ and the representation of $G_\infty$ on these spaces are limits of smooth Fréchet representation of moderate growth. More precisely, if we let

$$\mathcal{A}_r^\infty = \{\varphi \in \mathcal{A}^\infty \mid \sup_{g \in G(\mathbb{A})} (\|g\|^{-r} |X\varphi(g)|) < \infty \text{ for all } X \in \mathcal{U}(\mathfrak{g})\}$$

then for any open compact subgroup $L \subset K_f$ the space of $L$-fixed vectors $(\mathcal{A}_r^\infty)^L$ in $\mathcal{A}_r^\infty$ is a smooth Frechet representation of moderate growth for $G_\infty$ defined by the natural seminorms

$$q_{X,r}(\varphi) = \sup_g (\|g\|^{-r} |X\varphi(g)|) \quad \text{for} \quad X \in \mathcal{U}(\mathfrak{g}).$$

Then as topological representations both

$$(\mathcal{A}^\infty)^L = \varinjlim_r (\mathcal{A}_r^\infty)^L \quad \text{and} \quad \mathcal{A}^\infty = \varinjlim_L (\mathcal{A}^\infty)^L$$

carry a limit-Fréchet topology. Without going into details on such representations, let us state the results we will need analogous to those for representations of $\mathcal{H}$.

**Theorem 3.3 (Harish-Chandra; Wallach)** *If $\varphi \in \mathcal{A}^\infty$ is a smooth automorphic form then the (closed) sub-representation generated by $\varphi$, namely*

$$V_\varphi = \overline{R(G(\mathbb{A}))\varphi} \subset \mathcal{A}^\infty,$$

*is* admissible *in the sense that its (dense) subspace of $K$-finite vectors $(V_\varphi)_K$ is admissible as an $\mathcal{H}$-module.*

Then we can make the following definition.

**Definition 3.3** *A smooth automorphic representation $(\pi, V)$ of $G(\mathbb{A})$ is a (closed) irreducible sub-quotient of $\mathcal{A}^\infty(G(k)\backslash G(\mathbb{A}))$.*

Note that the smooth automorphic representations are automatically admissible in the above sense. We still have a version of the Decomposition Theorem, which we state as follows.

**Theorem 3.4 (Decomposition Theorem)** *If $(\pi, V)$ is a smooth automorphic representation of $G(\mathbb{A})$ then there exist irreducible admissible smooth representations $(\pi_v, V_v)$ of $G(k_v)$, which are smooth Fréchet representations of moderate growth if $v|\infty$, such that $\pi = \pi_\infty \otimes \pi_f$ where*

$$\pi_\infty = \widehat{\otimes}_{v|\infty}\pi_v$$

*is the topological tensor product of smooth Fréchet representations and*

$$\pi_f = \otimes'_{v<\infty}\pi_v$$

*is the restricted tensor product of smooth representations of the $G(k_v)$. Moreover, if $(\pi_K, V_K)$ is the associated irreducible $\mathcal{H}$-module of $K$-finite vectors in $V$ then in the decomposition $\pi_K = \otimes'(\pi_K)_v$ we have $\pi_v = (\pi_K)_v$ for $v < \infty$ while for $v|\infty$ we have $(\pi_v)_K = (\pi_K)_v$ and $\pi_v = \widehat{(\pi_K)_v}$ is the Casselman-Wallach canonical completion of the $\mathcal{H}_v$-module $(\pi_K)_v$.*

Even though the theory of smooth automorphic representations is topological, according to Wallach it is also quite algebraic. These representations will be algebraically irreducible as representations of the global *Schwartz algebra* $\mathcal{S} = \mathcal{S}(G(\mathbb{A}))$. This is a restricted tensor product of the local Schwatrz algebras $\mathcal{S}_v = \mathcal{S}(G(k_v))$. For archimedean places $v|\infty$ then $\mathcal{S}_v$ is the usual space of smooth (infinitely differentiable) rapidly decreasing functions on $G(k_v)$. At the non-archmiedean places rapidly decreasing is interpreted as having compact support, so $\mathcal{S}_v$ is the space of smooth (locally constant) compactly supported supported functions on $G(k_v)$, that is, $\mathcal{S}_v = \mathcal{H}_v$. Then $\mathcal{S} = \mathcal{S}_\infty \otimes \mathcal{S}_f$ where now

$$\mathcal{S}_\infty = \mathcal{S}(G_\infty) = \widehat{\otimes}_{v|\infty}\mathcal{S}_v \quad \text{and} \quad \mathcal{S}_f = \otimes'_{v<\infty}\mathcal{S}_v = \mathcal{H}_f.$$

## 3 $L^2$-automorphic representations

If we now fix a unitary central character $\omega : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ and consider the associated space of $L^2$-automorphic forms $L^2(G(k)\backslash G(\mathbb{A}); \omega)$ then this space is a Hilbert space and affords a unitary representation representation of $G(\mathbb{A})$ acting by right translation. In some sense this is the easiest situation to be in.

**Theorem 3.5 (Harish-Chandra)** *If $\varphi \in L^2(\omega)$ then*

$$V_\varphi = \overline{R(G(\mathbb{A}))\varphi} \subset L^2(\omega)$$

*is an admissible sub-representation in the sense that the (dense) sub-space $(V_\varphi)_K$ of $K$-finite vectors is admissible as as $\mathcal{H}$-module.*

**Definition 3.4** *An $L^2$-automorphic representation $(\pi, V)$ is an irreducible constituent in the $L^2$-decomposition of some $L^2(\omega)$.*

In the context of $L^2$-automorphic representations, the Decomposition Theorem predates the algebraic one and is due to Gelfand and Piatetski-Shapiro.

**Theorem 3.6** *If $(\pi, V)$ is an $L^2$-automorphic representation then there exist irreducible unitary representations $(\pi_v, V_v)$ of $G(k_v)$ such that $\pi = \widehat{\otimes}'\pi_v$ is a restricted Hilbert tensor product of local representations.*

## 4 Cuspidal representations

Since the cuspidality condition is defined by the vanishing of a left unipotent integration

$$\int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\, du = 0,$$

which is a closed condition, and our actions of $\mathcal{H}$ or $G(\mathbb{A})$ on the spaces of automorphic forms are by right convolution or right translations we see that the spaces of cusp forms $\mathcal{A}_0$, $\mathcal{A}_0^\infty$, or $L_0^2(\omega)$ are all (closed) sub-representations of the relevant spaces of automorphic forms.

A fundamental result of the space of $L^2$-cusp forms is the following result of Gelfand and Piatetski-Shapiro.

**Theorem 3.7** *The space $L_0^2(\omega)$ of $L^2$-cusp forms decomposes into a discrete Hilbert direct sum with finite multiplicities of irreducible unitary sub-representations*

$$L_0^2(\omega) = \oplus m(\pi) V_\pi \quad \textit{with} \quad m(\pi) < \infty.$$

We can then make the following definition.

**Definition 3.5** *The irreducible constituents $(\pi, V_\pi)$ of the various $L_0^2(\omega)$ are the $L^2$-cuspidal representations.*

Recall that for a fixed unitary central character $\omega$ we have, as a consequence of the rapid decrease of cusp forms, the inclusions

$$\mathcal{A}_0(\omega) \subset \mathcal{A}_0^\infty(\omega) \subset L_0^2(\omega)$$

and in fact upon passing to smooth vectors and then $K$-finite vectors we have

$$\mathcal{A}_0^\infty(\omega) = L_0^2(\omega)^\infty \quad \text{and} \quad \mathcal{A}_0(\omega) = \mathcal{A}_0^\infty(\omega)_K = L_0^2(\omega)_K$$

so we can deduce the decompositions

$$\mathcal{A}_0^\infty(\omega) = \oplus m(\pi) V_\pi^\infty \quad \text{and} \quad \mathcal{A}_0(\omega) = \oplus m(\pi)(V_\pi)_K.$$

**Definition 3.6** *The irreducible constituents of $\mathcal{A}_0(\omega)$ are the unitary ($K$-finite) cuspidal representations of $G(\mathbb{A})$ and the irreducible constituents of $\mathcal{A}^\infty(\omega)$ are the unitary smooth cuspidal representations of $G(\mathbb{A})$.*

Note that if $(\pi, V_\pi)$ is a cuspidal representation (in any context) then the elements of $V_\pi$ are indeed cusp forms, that is, $V_\pi \subset \mathcal{A}_0$ as a subspace not a more general sub-quotient.

In general any irreducible subrepresentation of $\mathcal{A}_0$ or $\mathcal{A}_0^\infty$ will be called a cuspidal representation. Due to the rapid decrease of cusp forms, any cuspidal representation $(\pi, V_\pi)$ will be an unramified twist of a unitary cuspidal representation, that is, if we define for any character $\chi : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ the twisted representation $\pi\otimes\chi$ as the representation by right translation on the space $V \otimes \chi = \{\varphi(g)\chi(\det g) \mid \varphi \in V_\pi\}$, then one can always find an unramified character $\chi$ such that $\pi\otimes\chi$ is a unitary cuspidal representation as above. Some choose to call the non-unitary cuspidal representations quasi-cuspidal.

## 5 Connections with classical forms

Suppose we return to a classical cusp form $f$ for $SL_2(\mathbb{Z})$ of weight $m$. If we follow our passage $f \mapsto \varphi \mapsto (\pi_\varphi, V_\varphi)$ then $(\pi_\varphi, V_\varphi)$ is an admissible subspace of the space of cuspidal automorphic forms. It need not be irreducible. However, if in addition $f$ is a simultaneous eigen-function for all the classical Hecke operators, then $(\pi_\varphi, V_\varphi)$ is irreducible and hence a cuspidal representation. Then the Decomposition Theorem lets us decompose $\pi_\varphi$ as $\pi_\varphi = \pi_\infty \otimes (\otimes' \pi_p)$. In this decomposition

(i) $\pi_\infty$ is completely determined by the weight $m$ of $f$
(ii) $\pi_p$ is completely determined by the Hecke eigen-value $\lambda(p)$ of $T_p$ acting on $f$.

In fact, as we shall see, the Decomposition Theorem for $\pi_\varphi$ is equivalent to the Euler product factorization for the completed $L$-function $\Lambda(s, f)$.

## 6 References

[1] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*. Proc. Sympos. Pure Math. **33**, part 1, (1979), 189–207.

[2] D. Flath, *Decomposition of representations into tensor products*. Proc. Sympos. Pure Math. **33**, part 1, (1979), 179–183.

[3] I.M. Gelfand, M.I. Graev, and I.I. Piatetski-Shapiro, Representation Theory and Automorphic Functions. Saunders, Philadelphia, 1968.

[4] Harish-Chandra, Automorphic forms on Semisimple Lie Groups. Lecture Notes in Mathematics, No. 62 Springer-Verlag, Berlin-New York 1968.

[5] H. Jacquet and R.P. Langlands, Automorphic Forms on $GL(2)$. Springer Lecture Notes in Mathematics, No. 114, Springer-Verlag, Berlin-New York, 1971.

[6] N.R. Wallach, $C^\infty$ *vectors*. Representations of Lie Groups and Quantum Groups, Pitman Res. Notes Math. Ser. **311**, Longman Sci. Tech., Harlow, 1994, 205–270.

LECTURE 4

# Fourier Expansions and Multiplicity One Theorems

We now start with results which are often $GL_n$ *specific*. So we let $G = GL_n$ (however one should also keep in mind $G = GL_n \times GL_m$) and still take $k$ to be a number field.

## 1 The Fourier expansion of a cusp form

Let $(\pi, V_\pi)$ be a smooth cuspidal representation, so $V_\pi \subset \mathcal{A}_0^\infty$. Let $\varphi \in V_\pi$ be a smooth cusp form.

We begin with $G = GL_2$. Our translation subgroup is

$$N = N_2 = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}.$$

For any $g \in G(\mathbb{A})$ the function

$$x \mapsto \varphi\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right)$$

is a smooth function of $x \in \mathbb{A}$ which is periodic under $k$. Since $k\backslash\mathbb{A}$ is a compact abelian group we will have an abelian Fourier expansion of this function.

For each continuous character $\psi : k\backslash\mathbb{A} \to \mathbb{C}$ we define a $\psi$-Fourier coefficient, or $\psi$-Whittaker function, of $\varphi$ by

$$W_{\varphi,\psi}(g) = \int_{k\backslash\mathbb{A}} \varphi\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi^{-1}(x)\, dx.$$

This function satisfies

$$W_{\varphi,\psi}\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi(x) W_{\varphi,\psi}(g).$$

Then by standard abelian Fourier analysis we have

$$\varphi\left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \sum_{\psi \in \widehat{k\backslash\mathbb{A}}} W_{\varphi,\psi}(g)\psi(x)$$

or

$$\varphi(g) = \sum_{\psi} W_{\varphi,\psi}(g).$$

By standard duality theory, $\widehat{k\backslash\mathbb{A}} \simeq k$ and if we *fix* one non-trivial character $\psi$ then any other is of the form $\psi_\gamma(x) = \psi(\gamma x)$ for $\gamma \in k$, so

$$\varphi(g) = \sum_{\gamma \in k} W_{\varphi,\psi_\gamma}(g).$$

Since $\varphi$ is cuspidal, for $\gamma = 0$ we have

$$W_{\varphi,\psi_0}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\, dx = 0$$

and for $\gamma \neq 0$ it is an easy change of variables to see that

$$W_{\varphi,\psi_\gamma}(g) = W_{\varphi,\psi}\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

which gives for our Fourier expansion for $GL_2$

$$\varphi(g) = \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

where we have set $W_{\varphi,\psi} = W_\varphi$.

Now consider $G = GL_n$. The role of the translations is played by the full maximal unipotent subgroup

$$N = N_n = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_{n-1,n} \\ 0 & & & 1 \end{pmatrix} \right\}$$

which is now non-abelian. If we retain out fixed additive character $\psi$ of $k\backslash\mathbb{A}$ from before, then $\psi$ defines a (continuous) character of $N(k)\backslash N(\mathbb{A})$ by

$$\psi(n) = \psi\left(\begin{pmatrix} 1 & x_{1,2} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_{n-1,n} \\ 0 & & & 1 \end{pmatrix}\right) = \psi(x_{1,2} + \cdots + x_{n-1,n}).$$

The associated $\psi$-Whittaker function of $\varphi$ is now

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{N(k)\backslash N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)\, dn$$

which again satisfies $W_\varphi(ng) = \psi(n)W_\varphi(g)$ for all $n \in N(\mathbb{A})$. The Fourier expansion of $\varphi$ which is useful is

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

This is not hard to prove. It is essentially an induction based on the above argument, begun by expanding about the last column of $N$, which is abelian.

For $G = GL_3$ one would begin with

$$\varphi\left(\begin{pmatrix}1 & x_1 & x_2\\ & 1 & x_3\\ & & 1\end{pmatrix} g\right) = \varphi\left(\begin{pmatrix}1 & & x_2\\ & 1 & x_3\\ & & 1\end{pmatrix}\begin{pmatrix}1 & x_1 & \\ & 1 & \\ & & 1\end{pmatrix} g\right)$$

and expand this as a function of $\begin{pmatrix}x_2\\ x_3\end{pmatrix} \in (k\backslash\mathbb{A})^2$. Remember that $((k\backslash\mathbb{A})^2)^\wedge \simeq k^2$ and that $GL_2(k)$ acts on $k^2$ with two orbits: $\{0\}$ and an open orbit $(0,1)\cdot GL_2(k)$. The $\{0\}$ orbit contributes 0 by cuspidality and the open orbit can be parameterized by $P_2(k)\backslash GL_2(k)$ where $P_2 = \left\{\begin{pmatrix}a & b\\ 0 & 1\end{pmatrix}\right\} = Stab((0,1))$. One then expands the resulting terms as functions of $x_1$ as before.

As I said, the proof is not hard. The difficult thing, if there is one, is in recognizing that this is what one needs. This was recognized independently by Piatetski-Shapiro and Shalika.

## 2 Whittaker models

Consider now the functions $W = W_\varphi$ which appear in the Fourier expansion of our cusp forms $\varphi \in V_\pi$. These are smooth functions on $G(\mathbb{A})$ satisfying $W(ng) = \psi(n)W(g)$ for all $n \in N(\mathbb{A})$. Let

$$\mathcal{W}(\pi,\psi) = \{W_\varphi \mid \varphi \in V_\pi\}.$$

The group $G(\mathbb{A})$ acts in this space by right translation and the map

$$\varphi \mapsto W_\varphi \quad \text{intertwines} \quad V_\pi \xrightarrow{\sim} \mathcal{W}(\pi,\psi).$$

Note that since we can recover $\varphi$ from $W_\varphi$ through its Fourier expansion we are guaranteed that $W_\varphi \neq 0$ for all $\varphi \neq 0$. The space $\mathcal{W}(\pi,\psi)$ is called the *Whittaker model of* $\pi$.

The idea of a Whittaker model makes sense over a local field (and even a finite field). If we let $k_v$ be a local field (a completion of our global field $k$) and let $\psi_v$ be a non-trivial (continuous) additive character of $k_v$ then as before $\psi_v$ defines a character of the local translations $N(k_v)$. Let $\mathcal{W}(\psi_v)$ denote the full space of smooth functions $W : G(k_v) \to \mathbb{C}$ which satisfy $W(ng) = \psi_v(n)W(g)$ for all $n \in N(k_v)$. This is the space of smooth Whittaker functions on $G(k_v)$ and $G(k_v)$ acts on it by right translation.

If $(\pi_v, V_{\pi_v})$ is a smooth irreducible admissible representation of $G(k_v)$, then an intertwining

$$V_{\pi_v} \hookrightarrow \mathcal{W}(\psi_v) \quad \text{given by} \quad \xi_v \mapsto W_{\xi_v}$$

gives a Whittaker model $\mathcal{W}(\pi_v,\psi_v)$ of $\pi_v$.

For a representation $(\pi_v, V_{\pi_v})$ to have a Whittaker model it is necessary and sufficient for $V_{\pi_v}$ to have a non-trivial (continuous) Whittaker functional, that is , a continuous functional $\Lambda_v : V_{\pi_v} \to \mathbb{C}$ satisfying

$$\Lambda_v(\pi_v(n)\xi_v) = \psi_v(n)\Lambda_v(\xi_v)$$

for all $n \in N(k_v)$ and $\xi_v \in V_{\pi_v}$. A model $\xi_v \mapsto W_{\xi_v}$ gives a functional by

$$\Lambda_v(\xi_v) = W_{\xi_v}(e)$$

and a functional $\Lambda_v$ gives a model by setting

$$W_{\xi_v}(g) = \Lambda_v(\pi_v(g)\xi_v).$$

The fundamental result on local Whittaker models is due to Gelfand and Kazhdan ($v < \infty$) and Shalika ($v|\infty$).

**Theorem 4.1 (Local Uniqueness)** *Given $(\pi_v, V_{\pi_v})$ an irreducible admissible smooth representation of $G(k_v)$ the space of (continuous) Whittaker functionals is at most one dimensional, that is, and $\pi_v$ has at most one Whittaker model.*

*Remarks.* (i) One proves this by showing that the space of Whittaker functions $\mathcal{W}(\psi_v)$ is multiplicity free as a representation of $G(k_v)$. Writing $\mathcal{W}(\psi_v) = Ind(\psi_v)^\infty$ one shows that the intertwining algebra of Bessel distributions, that is, distributions $B$ satisfying $B(n_1 g n_2) = \psi_v(n_1)B(g)\psi_v(n_2)$, is commutative by exhibiting an anti-involution of the algebra that stabilizes the individual distributions.
(ii) When $v|\infty$, if we worked simply with irreducible admissible representations of the Hecke algebra $\mathcal{H}_v$ then the space of (algebraic) Whittaker functionals on $(V_{\pi_v})_K$ would have dimension $n!$, but only one extends continuously to $V_{\pi_v}$ with its (smooth moderate growth) Fréchet topology.
(iii) Simultaneously, one shows that if $\pi_v$ has a Whittaker model, then so does its contragredient $\widetilde{\pi}_v$ and in fact

$$\mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1}) = \{\widetilde{W}(g) = W(w_n\, {}^t g^{-1}) \mid W \in \mathcal{W}(\pi_v, \psi_v)\}$$

where $w_n$ denotes the long Weyl element $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

**Definition 4.1** *A representation $(\pi_v, V_{\pi_v})$ having a Whittaker model is called generic.*

Of course, the same definition applies in the global situation. Note that for $G = GL_n$ this notion is independent of the choice of (non-trivial) $\psi_v$ or $\psi$.

Now return to our smooth cuspidal representation $(\pi, V_\pi)$, or in fact any irreducible admissible smooth representation of $G(\mathbb{A})$. If we factor $\pi$ into its local components

$$\pi \simeq \otimes' \pi_v \quad \text{with} \quad V_\pi \simeq \otimes' V_{\pi_v}$$

then any Whittaker functional $\Lambda$ on $V_\pi$ determines a family of compatible Whittaker functionals $\Lambda_v$ on the $V_{\pi_v}$ by

$$\Lambda_v : V_{\pi_v} \hookrightarrow \otimes' V_{\pi_v} \xrightarrow{\sim} V_\pi \xrightarrow{\Lambda} \mathbb{C}$$

such that $\Lambda = \otimes \Lambda_v$. Similarly, any suitable family $\{\Lambda_v\}$ of Whittaker functionals on the $V_{\pi_v}$, where suitable means $\Lambda_v(\xi_v^\circ) = 1$ for our distinguished $K_v$-fixed vectors $\xi_v^\circ$ giving the restricted tensor product, determines a global Whittaker functional $\Lambda = \otimes \Lambda_v$ on $V_\pi = \otimes' V_{\pi_v}$.

The Local Uniqueness Theorem then has the following consequences.

**Corollary 4.1.1 (Global Uniqueness)** *If $\pi = \otimes' \pi_v$ is any irreducible admissible smooth representation of $G(\mathbb{A})$ then the space of Whittaker functionals of $V_\pi$ is at most one dimensional, that is, $\pi$ has a unique Whittaker model.*

If $(\pi, V_\pi)$ is our cuspidal representation then we have seen that $V_\pi$ has a global Whittaker functional given by

$$\Lambda(\varphi) = W_\varphi(e) = \int_{N(k)\backslash N(\mathbb{A})} \varphi(n)\psi^{-1}(n)\, dn.$$

**Corollary 4.1.2** *If $(\pi, V_\pi)$ is cuspidal with $\pi \simeq \otimes' \pi_v$ then $\pi$ and each of its local components $\pi_v$ are generic.*

A most important consequence for our purposes is:

**Corollary 4.1.3 (Factorization of Whittaker Functions)** *If $(\pi, V_\pi)$ is a cuspidal representation with $\pi \simeq \otimes' \pi_v$ and $\varphi \in V_\pi$ such that under the isomorphism $V_\pi \simeq \otimes' V_{\pi_v}$ we have $\varphi \mapsto \otimes \xi_v$ (so $\varphi$ is decomposable) then*

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v).$$

The proof is essentially the following simple computation:

$$\begin{aligned} W_\varphi(g) = \Lambda(\pi(g)\varphi) &= (\otimes \Lambda_v)(\otimes \pi_v(g_v)\xi_v) \\ &= \prod_v \Lambda_v(\pi_v(g_v)\xi_v) = \prod_v W_{\xi_v}(g_v). \end{aligned}$$

Note once again that the cusp form $\varphi(g)$ itself does not factor. The $G(k)$-invariance mixes the various places together. Only $W_\varphi$ factors for decomposable $\varphi$. If $f \in S_m(SL_2(\mathbb{Z}))$ is a classical Hecke eigen-form of weight $m$ for $SL_2(\mathbb{Z})$, with its usual Fourier expansion $f(z) = \sum a_n e^{2\pi i n z}$, and $f \mapsto \varphi$ is our lifted automorphic form, then $\varphi$ is decomposable and the Whittaker function $W_\varphi$ factors. If we write $W_\varphi = W_\infty W_f$ then

$$W_\infty \begin{pmatrix} ny & \\ & 1 \end{pmatrix} = (ny)^{m/2} e^{-2\pi n y} \quad \text{and} \quad W_f \begin{pmatrix} n & \\ & 1 \end{pmatrix} = a_n.$$

## 3 Multiplicity One for $GL_n$

The uniqueness of the Whittaker model is the key to the following result.

**Theorem 4.2 (Multiplicity One)** *Let $(\pi, V_\pi)$ be a smooth irreducible admissible (unitary) representation of $GL_n(\mathbb{A})$. Then its multiplicity $m(\pi)$ in the space of cusp forms is at most one.*

This result was proven independently by Piatetski-Shapiro and Shalika, based on the Fourier expansion and the global uniqueness of Whittaker models. Suppose we have two realizations of $\pi$ in the space of cusp forms:

$$V_\pi \hookrightarrow V_{\pi,i} \subset \mathcal{A}_0^\infty \quad \text{for} \quad i = 1, 2.$$

For $\xi \in V_\pi$ let $\varphi_1$ and $\varphi_2$ be the corresponding cusp forms. Then the maps

$$\xi \mapsto \varphi_i \mapsto W_{\varphi_i}(e) = \Lambda_i(\xi)$$

give two Whittaker functionals on $V_\pi$. By uniqueness, there exists $c \neq 0$ such that $\Lambda_1 = c\Lambda_2$. Then

$$W_{\varphi_1}(g) = \Lambda_1(\pi(g)\xi) = c\Lambda_2(\pi(g)\xi) = cW_{\varphi_2}(g)$$

so that

$$\varphi_1(g) = \sum_\gamma W_{\varphi_1}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right) = c\sum_\gamma W_{\varphi_2}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right) = c\varphi_2(g).$$

But then $V_{\pi,1} \cap V_{\pi,2} \neq \{0\}$. So by irreducibility $V_{\pi,1} = V_{\pi,2}$, that is, $m(\pi) = 1$.

## 4 Strong Multiplicity One for $GL_n$

Strong Multiplicity One was originally due to Piatetski-Shapiro. His proof, which we will sketch here, is a variant of the proof of Multiplicity One. We will give a second proof due to Jacquet and Shalika based on $L$-functions later.

**Theorem 4.3 (Strong Multiplicity One)** *Let $(\pi_1, V_{\pi_1})$ and $(\pi_2, V_{\pi_2})$ be two cuspidal representations of $GL_n(\mathbb{A})$. Decompose them as $\pi_1 \simeq \otimes'\pi_{1,v}$ and $\pi_1 \simeq \otimes'\pi_{2,v}$. Suppose that there is a finite set of places $S$ such that $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$. Then $(\pi_1, V_{\pi_1}) = (\pi_2, V_{\pi_2})$.*

In place of the Whittaker model, Piatetski-Shapiro used a variant known as the *Kirillov model*. To define this, let

$$P = P_n = \left\{\begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1\end{pmatrix}\right\} = \mathit{Stab}\ ((0, \ldots, 0, 1))$$

denote the *mirabolic subgroup* of $GL_n$. If we let $(\pi_v, V_{\pi_v})$ be an irreducible admissible generic representation of $G(k_v)$ with Whittaker model $\mathcal{W}(\pi_v, \psi_v)$ then we can consider the restrictions $W_v(p_v)$ of the functions $W_v \in \mathcal{W}(\pi_v, \psi_v)$ to $P_v = P(k_v)$. The first surprising fact is:

**Theorem 4.4** *The map $W_v \mapsto W_v|_{P_v}$ is injective, that is, if $W_v \neq 0$ then $W_v(p_v) \not\equiv 0$.*

This is due to Bernstein and Zelevinsky if $v < \infty$ and Jacquet and Shalika if $v|\infty$.

**Definition 4.2** *The (local) Kirillov model of a generic $(\pi_v, V_{\pi_v})$ is the space of functions on $P_v$ defined by*

$$\mathcal{K}(\pi_v, \psi_v) = \{W_v(p_v) \mid W_v \in \mathcal{W}(\pi_v, \psi_v),\ p_v \in P_v\}.$$

A second surprising fact is that no matter what the generic representation $(\pi_v, V_{\pi_v})$ we begin with, the Kirillov models all have a common $P_v$ sub-module, namely

$$\tau(\psi_v) = \begin{cases} ind_{N_v}^{P_v}(\psi_v) & v < \infty \\ \\ Ind_{N_v}^{P_v}(\psi_v)^{\infty} & v|\infty \end{cases}.$$

This is a canonical space of functions on $P_v$. The result for $v < \infty$ is of course due to Bernstein and Zelevinsky and for $v|\infty$ it is due to Jacquet and Shalika.

We may now sketch the proof of the Strong Multiplicity One Theorem. Let $\pi_1$, $\pi_2$, and $S$ be as in the statement of the theorem. As before, our goal is to produce a common non-zero cusp form $\varphi \in V_{\pi_1} \cap V_{\pi_2}$.

(i) Let $P' = P'_n = P_n Z_n$ be the $(n-1, 1)$ parabolic subgroup of $GL_n$ (here $Z_n$ is still the center of $GL_n$). Then $P'(k)\backslash P'(\mathbb{A})$ is dense in $GL_n(k)\backslash GL_n(\mathbb{A})$. [This follows from the fact that $P'\backslash GL_n \simeq \mathbb{P}^{n-1}$ and $\mathbb{P}^{n-1}(k)$ is dense in $\mathbb{P}^{n-1}(\mathbb{A})$.] So it suffices to find $\varphi_i \in V_{\pi_i}$ such that $\varphi_1(p') = \varphi_2(p')$ for all $p' \in P'(\mathbb{A})$.

(ii) Utilizing the Fourier expansion as before, it suffices to find non-zero $W_i \in \mathcal{W}(\pi_i, \psi)$ such that $W_1(p') = W_2(p')$.

(iii) Since $\omega = \omega_{\pi_1} = \omega_{\pi_2}$ (by weak approximation) it suffices to find non-zero $W_i$ such that $W_1(p) = W_2(p)$ for all $p \in P(\mathbb{A})$, that is, to find non-zero

$$W = W_1 = W_2 \in \mathcal{K}(\pi_1, \psi) \cap \mathcal{K}(\pi_2, \psi).$$

(iv) At $v \notin S$ we have $\pi_{1,v} \simeq \pi_{2,v}$ so that

$$\mathcal{K}(\pi_{1,v}, \psi_v) = \mathcal{K}(\pi_{2,v}, \psi_v) \quad \text{for} \quad v \notin S.$$

At $v \in S$ we have

$$\tau(\psi_v) \subset \mathcal{K}(\pi_{1,v}, \psi_v) \cap \mathcal{K}(\pi_{2,v}, \psi_v) \quad \text{for} \quad v \in S$$

which is a quite large intersection. So we may simply take any

$$W \in \prod_{v \in S} \tau(\psi_v) \prod_{v \notin S}{}' \mathcal{K}(\pi_{i,v}, \psi_v) \subset \mathcal{K}(\pi_1, \psi) \cap \mathcal{K}(\pi_2, \psi)$$

which is non-zero.

Now retrace the steps to obtain $\varphi \in V_{\pi_1} \cap V_{\pi_2}$ forcing $V_{\pi_1} = V_{\pi_2}$ as before.

*Remark.* In Piatetski-Shapiro's original proof, he had to require that the set $S$ consisted of finite places, since the result of Jacquet and Shalika was not available at that time. Once it became available his proof worked for general finite set $S$ as well.

## 5 References

[1] J. Bernstein and A. Zelevinsky, *Representations of $GL(n, F)$ where $F$ is a non-archimedean local field.* Russian Math. Surveys **31** (1976), 1–68.

[2] J.W. Cogdell, *L-functions and Converse Theoresm for* $GL_n$. IAS/PCMI Lecture Notes, to appear. Available at *www.math.okstate.edu/∼cogdell* .

[3] I.M. Gelfand and D.A. Kazhdan, *Representations of* $GL(n, K)$ *where* $K$ *is a local field*. Lie Groups and Their Representations (I.M. Gelfand, ed.) Halsted, New York, 1975, 95–118.

[4] H. Jacquet and R.P. Langlands, Automorphic Forms on $GL(2)$. Springer Lecture Notes in Mathematics, No. 114, Springer-Verlag, Berlin-New York, 1970.

[5] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations*, I & II. Amer. J. Math. **103** (1981), 499–588 & 777–815.

[6] I.I. Piatetski-Shapiro, *Euler Subgroups*. Lie Groups and Their Representations (I.M. Gelfand, ed.) Halsted, New York, 1975, 597–620.

[7] I.I. Piatetski-Shapiro, *Multiplicity one theorems*. Proc. Sympos. Pure Math. **33**, Part 1 (1979), 209–212.

[8] J. Shalika, *The multiplicity one theorem for* $GL(n)$. Ann. Math. **100** (1974), 171–193.

LECTURE 5

# Eulerian Integral Representations

We now turn to the integral representations for the $L$-functions for $GL_n$. We will be interested not only in the integral representations for $L(s,\pi)$, with $\pi$ a cuspidal representation of $GL_n(\mathbb{A})$, but also the twisted $L$-functions $L(s,\pi\times\pi')$ with $\pi'$ a cuspidal representation of $GL_m(\mathbb{A})$. We have seen these in Weil's Converse Theorem and in the lectures of Ram Murty.

Two points before we begin. (1) The space $V_\pi$ of cusp forms of $\pi$ is an infinite dimensional space. The integral representations will involve the $\varphi\in V_\pi$, but we eventually will want a single $L(s,\pi)$. (2) In contrast to Hecke, we will need to see the Euler factorization already at the individual integral level. This is reasonable in view of the Decomposition Theorem since our representations are irreducible. This is very much in the spirit of Tate's thesis.

We still take $k$ to be a number field. We will take $(\pi,V_\pi)$ to always be a smooth unitary cuspidal automorphic representation of $GL_n(\mathbb{A})$, so $V_\pi\subset\mathcal{A}_0^\infty(\omega)$ with unitary central character $\omega=\omega_\pi$. Similarly, $(\pi',V_{\pi'})$ will be a smooth unitary representation of $GL_m(\mathbb{A})$ with unitary central character $\omega'=\omega_{\pi'}$. The condition of unitarity is not restrictive, but allows for convenient normalizations.

## 1 $GL_2\times GL_1$

We begin with $GL_2$ where we can follow Hecke's lead. So $(\pi,V_\pi)$ is a smooth cuspidal representation of $GL_2(\mathbb{A})$. Let $\chi : k^\times\backslash\mathbb{A}^\times\to\mathbb{C}$ be a unitary idele class character, that is a (cuspidal) automorphic representation of $GL_1(\mathbb{A})$.

Just as Hecke set

$$L(s,f)=\int_0^\infty f(iy)y^s\,d^\times y=\int_0^\infty f\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\cdot i\right)y^s\,d^\times y,$$

for $\varphi\in V_\pi$ we set

$$I(s,\varphi,\chi)=\int_{k^\times\backslash\mathbb{A}^\times}\varphi\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\chi(a)|a|^{s-\frac{1}{2}}\,d^\times a.$$

Utilizing the rapid decrease of cusp forms we quickly arrive at the following result.

**Proposition 5.1** *(i) $I(s,\varphi,\chi)$ is absolutely convergent for all $s\in\mathbb{C}$, hence entire.*

*(ii)* $I(s,\varphi,\chi)$ *is bounded in vertical strips.*
*(iii)* $I(s,\varphi,\chi)$ *satisfies the functional equation*

$$I(s,\varphi,\chi) = I(1-s,\widetilde{\varphi},\chi^{-1})$$

*where* $\widetilde{\varphi}(g) = \varphi({}^tg^{-1})$.

Statement (i) follows from the rapid decrease of cusp forms and (ii) follows from (i). The functional equation follows from the change of variable $a \mapsto a^{-1}$ in the integral. The function $\widetilde{\varphi}$ is again a cusp form. If we set $\widetilde{\pi}(g) = \pi({}^tg^{-1})$ then $\widetilde{\pi}$ is the *contragredient* representation of $\pi$ and $\widetilde{\varphi} \in V_{\widetilde{\pi}}$. The map $g \mapsto g^\iota = {}^tg^{-1}$ is the outer automorphism of $GL_n$ and will be responsible for all of our functional equations.

So this family of integrals, as $\varphi$ varies over $V_\pi$, is *nice*. To see that the integrals are Eulerian we first replace $\varphi$ by its Fourier expansion.

$$\begin{aligned} I(s,\varphi,\chi) &= \int_{k^\times\backslash\mathbb{A}^\times} \varphi\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}}\, d^\times a \\ &= \int_{k^\times\backslash\mathbb{A}^\times} \sum_{\gamma\in k^\times} W_\varphi \begin{pmatrix} \gamma a & 0 \\ 0 & 1\end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}}\, d^\times a \\ &= \int_{\mathbb{A}^\times} W_\varphi \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} \chi(a)|a|^{s-\frac{1}{2}}\, d^\times a \qquad \text{for} \quad Re(s) > 1. \end{aligned}$$

Since $\pi \simeq \otimes'\pi_v$, if $\varphi \in V_\pi$ is decomposable, say $\varphi \simeq \otimes\xi_v \in \otimes' V_{\pi_v}$, then we have seen from the uniqueness of the Whittaker model that

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v).$$

Since $\chi(a) = \prod \chi_v(a_v)$ and $|a| = \prod |a_v|_v$ we have

$$\begin{aligned} I(s,\varphi,\chi) &= \prod_v \int_{k_v^\times} W_{\xi_v} \begin{pmatrix} a_v & 0 \\ 0 & 1\end{pmatrix} \chi_v(a_v)|a_v|_v^{s-\frac{1}{2}}\, d^\times a_v \\ &= \prod_v \Psi_v(s, W_{\xi_v}, \chi_v) \qquad \text{for} \quad Re(s) > 1. \end{aligned}$$

This gives a factorization of our global integral into a product of local integrals.

## 2 $GL_n \times GL_m$ with $m < n$

Now take $\varphi \in V_\pi \subset \mathcal{A}_0^\infty(GL_n)$ and $\varphi' \in V_{\pi'} \subset \mathcal{A}_0^\infty(GL_m)$. The natural extension of the above would seemingly be to consider

$$\int_{GL_m(k)\backslash GL_m(\mathbb{A})} \varphi\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix} \varphi'(h)|\det h|^{s-*}\, dh.$$

This family of integrals is indeed *nice* but will not be Eulerian unless $m = n-1$. In the case $m = n-1$ we can indeed proceed as before. In general, families of Eulerian integrals are very difficult to find. For $GL_n \times GL_m$ the situation is relatively simple.

For each pair $(n, m)$ we introduce a "projection" of $\varphi$ to a space of cuspidal functions on $P_{m+1}$, the usual mirabolic subgroup inside $GL_{m+1}$, by taking a partial Whittaker transform. Let

$$Y = Y_{n,m} = \left\{ \begin{pmatrix} I_{m+1} & * \\ 0 & x \end{pmatrix} \mid x \in N_{n-m-1} \right\} \subset N = N_n.$$

Then $Y$ is the unipotent radical of the parabolic subgroup $Q$ of $GL_n$ associated to the partition $(m+1, 1, \ldots, 1)$ of $n$. We have our usual additive character $\psi : N(k)\backslash N(\mathbb{A}) \to \mathbb{C}$ and we may restrict this to $Y$. Note that if $M \simeq GL_{m+1} \times GL_1 \times \cdots \times GL_1$ is the Levi subgroup of $Q$ then $Stab_M(Y, \psi) = P_{m+1} \subset GL_{m+1} \subset M$.

For $\varphi \in V_\pi$ and $p \in P_{m+1}(\mathbb{A})$ set

$$\mathbb{P}\varphi(p) = |\det p|^{-\frac{n-m-1}{2}} \int_{Y(k)\backslash Y(\mathbb{A})} \varphi\left(y \begin{pmatrix} p & 0 \\ 0 & I_{n-m-1} \end{pmatrix}\right) \psi^{-1}(y)\, dy.$$

The following proposition is then fairly routine.

**Proposition 5.2** *(i)* $\mathbb{P}\varphi(p)$ *is left invariant under* $P_{m+1}(k)$ *and cuspidal on* $P_{m+1}(\mathbb{A})$ *in the sense that all relevant unipotent integrals are zero.*
*(ii) Setting* $p = \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$ *with* $h \in GL_m(\mathbb{A})$ *we have the Fourier expansion*

$$\mathbb{P}\varphi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} = |\det h|^{-\frac{n-m-1}{2}} \sum_{\gamma \in N_m(k)\backslash GL_m(k)} W_\varphi \begin{pmatrix} \gamma h & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

We view $\mathbb{P}$ as projecting the cusp forms on $GL_n(\mathbb{A})$ to cusp forms on $P_{m+1}(\mathbb{A})$. If $n = m+1$, so there is no integration, then $\mathbb{P}$ is simply the restriction of functions from $GL_n(\mathbb{A})$ to $P_n(\mathbb{A})$.

We are now in the same situation as we were for $GL_2 \times GL_1$ or $GL_n \times GL_{n-1}$. For $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ we set

$$I(s, \varphi, \varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}\varphi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}}\, dh.$$

As before, but with a bit more work, we have:

**Proposition 5.3** *(i)* $I(s, \varphi, \varphi')$ *is absolutely convergent for all* $s \in \mathbb{C}$, *hence entire.*
*(ii)* $I(s, \varphi, \varphi')$ *is bounded in vertical strips.*
*(iii)* $I(s, \varphi, \varphi')$ *satisfies the functional equation*

$$I(s, \varphi, \varphi') = \widetilde{I}(1-s, \widetilde{\varphi}, \widetilde{\varphi}')$$

*where* $\widetilde{\varphi}(g) = \varphi({}^t g^{-1}) = \varphi(g^\iota) \in V_{\widetilde{\pi}}$ *and* $\widetilde{\varphi}'(h) = \varphi'(h^\iota) \in V_{\widetilde{\pi}'}$ *as before and the integral appearing in the right hand side is*

$$\widetilde{I}(s, \varphi, \varphi') = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \widetilde{\mathbb{P}}\varphi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}}\, dh$$

*with* $\widetilde{\mathbb{P}} = \iota \circ \mathbb{P} \circ \iota$.

So once again our family of integrals is *nice*. Moreover they are now Eulerian just as before. We first insert the Fourier expansion of $\mathbb{P}\varphi$ to obtain

$$\begin{aligned}
I(s,\varphi,\varphi') &= \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}\varphi\begin{pmatrix} h & 0 \\ 0 & 1\end{pmatrix} \varphi'(h)|\det h|^{s-\frac{1}{2}}\ dh \\
&= \int \sum_{\gamma\in N_m(k)\backslash GL_m(k)} W_\varphi\begin{pmatrix} \gamma h & 0 \\ 0 & I_{n-m}\end{pmatrix} \varphi'(h)|\det h|^{s-\frac{n-m}{2}}\ dh \\
&= \int_{N_m(k)\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix} \varphi'(h)|\det h|^{s-\frac{n-m}{2}}\ dh \\
&= \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix} \times \\
&\qquad \times \left(\int_{N_m(k)\backslash N_m(\mathbb{A})} \varphi'(nh)\psi(n)\ dn\right)|\det h|^{s-\frac{n-m}{2}}\ dh \\
&= \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi\begin{pmatrix} h & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'}(h)|\det h|^{s-\frac{n-m}{2}}\ dh.
\end{aligned}$$

where this is again convergent only for $Re(s) > 1$ once we unfold the Fourier expansion. The fact that the boundary of convergence is $Re(s) > 1$ depends on the fact that $\pi$ and $\pi'$ are taken to be unitary and the estimates on Whittaker functions that we will present in future lectures. Note that $W'_{\varphi'} \in \mathcal{W}(\pi', \psi^{-1})$. If we now take $\varphi$ and $\varphi'$ decomposable, say $\varphi \simeq \otimes\xi_v \in \otimes' V_{\pi_v}$ and $\varphi' \simeq \otimes\xi'_v \in \otimes' V_{\pi'_v}$, then we again have

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v) \qquad \text{and} \qquad W'_{\varphi'}(h) = \prod_v W'_{\xi'_v}(h_v)$$

from the uniqueness of the Whittaker model and the integral now factors as

$$I(s,\varphi,\varphi') = \prod_v \Psi_v(s, W_{\xi_v}, W'_{\xi'_v}) \qquad \text{for} \qquad Re(s) > 1$$

with the local integrals given by

$$\begin{aligned}
&\Psi_v(s, W_v, W'_v) = \\
&\quad = \int_{N_m(k_v)\backslash GL_m(k_v)} W_v\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_v(h_v)|\det h_v|_v^{s-\frac{n-m}{2}}\ dh_v.
\end{aligned}$$

Hence our family is once again Eulerian.

For this family of integrals the two sides of the functional equation involve slightly different integrals. With a little (?) more work, one shows that the integrals occurring in the right hand side of the functional equation are also Eulerian, with

$$\widetilde{I}(1-s,\widetilde{\varphi},\widetilde{\varphi}') = \prod_v \widetilde{\Psi}_v(1-s, R(w_{n,m})\widetilde{W}_{\xi_v}, \widetilde{W}'_{\xi'_v}) \quad \text{for} \quad Re(s) < 0$$

where $R$ is simply right translation,

$$w_{n,m} = \begin{pmatrix} I_m & \\ & w_{n-m}\end{pmatrix}$$

where as always $w_{n-m}$ is the long Weyl element, with ones along the skew diagonal, in $GL_{n-m}$, $\widetilde{W}_v(g) = W_v(w_n g^\iota) \in \mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1})$, $\widetilde{W}'_v(h) = W'_v(w_m h^\iota) \in \mathcal{W}(\widetilde{\pi}'_v, \psi_v)$, and finally

$$\widetilde{\Psi}_v(s, W_v, W_v') =$$
$$= \iint W_v \begin{pmatrix} h_v & & \\ x_v & I_{n-m-1} & \\ & & 1 \end{pmatrix} dx_v W_v'(h_v) |\det h_v|^{s - \frac{n-m}{2}} dh_v$$

where the inner $x_v$ integration is over the matrix space $M_{n-m-1,m}(k_v)$ and the outer $h_v$ integration is over $N_m(k_v)\backslash GL_m(k_v)$ as usual. The lower unipotent integration over $M_{n-m-1,m}(k_v)$ is the remnant of $\widetilde{\mathbb{P}}$.

## 3 $GL_n \times GL_n$

In the $m = n$ case the paradigm comes not from Hecke but from the classical work of Rankin and Selberg, again with a bit of Tate's thesis mixed in. One might be first tempted to try

$$\int_{GL_n(k)\backslash GL_n(\mathbb{A})} \varphi(g)\varphi'(g)|\det g|^s \, dg$$

for $\varphi \in V_\pi \subset \mathcal{A}_0^\infty(GL_n)$ and $\varphi' \in V_{\pi'} \subset \mathcal{A}_0^\infty(GL_n)$, but this, convergence issues aside, would give an invariant pairing and would be zero unless $\widetilde{\pi} \simeq \pi' \otimes |\det|^s$. (In fact, such integrals will arise as residues of poles of our family of Eulerian integrals.) Instead, we will consider integrals of the form

$$\int \varphi(g)\varphi'(g)E(g,s) \, dg$$

where $E(g,s)$ is an appropriate Eisenstein series. Murty wrote down the classical version of this in his lectures.

To construct our Eisenstein series we begin with a Schwartz-Bruhat function $\Phi \in \mathcal{S}(\mathbb{A}^n)$. Recall that $\mathcal{S}(\mathbb{A}^n)$ is a restricted tensor product (topological at the archimedean places), $\mathcal{S}(\mathbb{A}^n) = \otimes' \mathcal{S}(k_v^n)$, where for $v|\infty$ we have $\mathcal{S}(k_v^n) = \mathcal{S}(\mathbb{R}^n)$ or $\mathcal{S}(\mathbb{C}^n)$ is the usual Schwartz space of infinitely differentiable rapidly decreasing functions and for $v < \infty$ we have $\mathcal{S}(k_v^n) = C_c^\infty(k_v^n)$ is Bruhat's space of smooth (i.e., locally constant) compactly supported functions.

For each $\Phi \in \mathcal{S}(\mathbb{A}^n)$ we may form a $\Theta$-series

$$\Theta_\Phi(a,g) = \sum_{\xi \in k^n} \Phi(a\xi g) \quad \text{with} \quad a \in \mathbb{A}^\times, \quad g \in GL_n(\mathbb{A}).$$

Our Eisenstein series is essentially the Mellin transform of $\Theta$. More precisely, for $\eta \in (k^\times\backslash\mathbb{A}^\times)^\wedge$ a unitary idele class character, set

$$E(g,s) = E(g,s;\Phi,\eta) = |\det g|^s \int_{k^\times\backslash\mathbb{A}^\times} \Theta_\Phi'(a,g)\eta(a)|a|^{ns} \, d^\times a$$

where $\Theta_\Phi'(a,g) = \Theta_\Phi(a,g) - \Phi(0)$. This converges for $Re(s) > 1$. The basic analytic properties of this Eisenstein series are the following.

**Proposition 5.4** *(i) $E(g,s) \in \mathcal{A}^\infty(\eta^{-1})$, that is, $E(g,s)$ is a smooth automorphic form on $GL_n(\mathbb{A})$ transforming by $\eta^{-1}$ under the center.*
*(ii) $E(g,s)$ extends to a meromorphic function of $s$ and away from its poles is*

*bounded in vertical strips. The extension still satisfies (i).*
*(iii)* $E(g,s)$ *has possible simple poles at* $s = i\sigma$ *and* $s = 1 + i\sigma$ *with* $\sigma \in \mathbb{R}$ *such that* $\eta(a) = |a|^{-in\sigma}$ *and no others.*
*(iv)* $E(g,s)$ *satisfies a functional equation*

$$E(g,s;\Phi,\eta) = E(g^\iota, 1-s; \widehat{\Phi}, \eta^{-1}),$$

*where* $\widehat{\Phi}$ *is the Fourier transform on* $\mathcal{S}(\mathbb{A}^n)$.

The proof of this result follows Hecke's proof of the analytic continuation of $L$-functions of classical modular forms, utilizing the Poisson summation formula for the $\Theta$-series.

There is a second construction of $E(g,s)$ which will be essential for us. Using the same data as in $E(g,s)$, set

$$F(g,s) = F(g,s;\Phi,\eta) = |\det g|^s \int_{\mathbb{A}^\times} \Phi(a e_n g)\eta(a)|a|^{ns}\ d^\times a$$

where $e_n = (0,\dots,0,1) \in k^n$. This is convergent for $Re(s) > \frac{1}{n}$. Recall that the mirabolic subgroup $P_n$ is the stabilizer of $e_n$. If $P'_n = Z_n P_n$ is the full $(n-1,1)$ parabolic subgroup of $GL_n$ then

$$F(g,s) \in Ind_{P'_n(\mathbb{A})}^{GL_n(\mathbb{A})}\left(\delta_{P'}^{s-\frac{1}{2}}\eta^{-1}\right),$$

that is,

$$F\left(\begin{pmatrix} h & y \\ 0 & d\end{pmatrix} g, s\right) = |\det h|^s |d|^{-(n-1)s}\eta^{-1}(d)F(g,s)$$

for $h \in GL_{n-1}(\mathbb{A})$ and $d \in \mathbb{A}^\times$. Here $\delta_{P'}$ is the modulus character for the parabolic $P'_n$. Then we may also write

$$E(g,s) = \sum_{\gamma \in P'_n(k)\backslash GL_n(k)} F(\gamma g, s)$$

which is the general form of an Eisenstein series from the representation theoretic point of view. This again converges for $Re(s) > 1$.

We now return to our family of Eulerian integrals. For $\varphi \in V_\pi$ , $\varphi' \in V_{\pi'}$, and $\Phi \in \mathcal{S}(\mathbb{A}^n)$ we set

$$I(s,\varphi,\varphi',\Phi) = \int_{Z_n(\mathbb{A})GL_n(k)\backslash GL_n(\mathbb{A})} \varphi(g)\varphi'(g)E(g,s;\Phi,\omega\omega')\ dg$$

where $\omega = \omega_\pi$ and $\omega' = \omega_{\pi'}$ are the central characters. As the cusp forms $\varphi$ and $\varphi'$ are rapidly decreasing on $Z_n(\mathbb{A})GL_n(k)\backslash GL_n(\mathbb{A})$, from our first expression for $E(g,s)$ we find that the integrals are (relatively) *nice*.

**Proposition 5.5** *(i)* $I(s,\varphi,\varphi',\Phi)$ *is a meromorphic function of* $s \in \mathbb{C}$ *with at most simple poles at* $s = i\sigma$ *and* $s = 1 + i\sigma$ *for* $\sigma \in \mathbb{R}$ *such that* $\widetilde{\pi} \simeq \pi' \otimes |\det|^{i\sigma}$.
*(ii)*$I(s,\varphi,\varphi',\Phi)$ *is bounded in vertical strips away from its poles.*
*(iii)* $I(s,\varphi,\varphi',\Phi)$ *satisfies the functional equation*

$$I(s,\varphi,\varphi',\Phi) = I(1-s,\widetilde{\varphi},\widetilde{\varphi}',\widehat{\Phi}).$$

On the other hand, if we replace $E(g,s)$ by its second expression $E(g,s) = \sum F(\gamma g, s)$ and unfold this sum, replace $\varphi$ by its Fourier expansion in the resulting expression, and proceed as before we find that our integrals are Eulerian. If $\varphi$, $\varphi'$, and $\Phi$ are all decomposable, say $\varphi \simeq \otimes\xi_v \in \otimes' V_{\pi_v}$, $\varphi' \simeq \otimes\xi'_v \in \otimes' V_{\pi'_v}$, and $\Phi \simeq \otimes\Phi_v \in \otimes'\mathcal{S}(k_v^n)$ then

$$I(s,\varphi,\varphi'\Phi) = \prod_v \Psi_v(s, W_{\xi_v}, W'_{\xi'_v}, \Phi_v) \quad \text{for} \quad Re(s) > 1$$

where the local integrals are given by

$$\Psi_v(s, W_v, W'_v, \Phi_v) = \int_{N_n(k_v)\backslash GL_n(k_v)} W_v(g_v)W'_v(g_v)\Phi(e_n g_v)|\det g_v|_v^s \, dg_v$$

with $W_v \in \mathcal{W}(\pi_v, \psi_v)$, $W'_v \in \mathcal{W}(\pi'_v, \psi_v^{-1})$ and $e_n = (0, \ldots, 0, 1) \in k^n$.

## 4 Summary

For each pair of cuspidal representations $\pi \simeq \otimes'\pi_v$ and $\pi' \simeq \otimes'\pi'_v$ we have associated:

1. A family of global Eulerian integrals

$$\{I(s,\varphi,\varphi')\} \quad \text{or} \quad \{I(s,\varphi,\varphi',\Phi)\}$$

for $\varphi \in V_\pi$, $\varphi' \in V_{\pi'}$, and if necessary $\Phi \in \mathcal{S}(\mathbb{A}^n)$.

2. For each place $v$ of $k$ a family of local integrals

$$\{\Psi_v(s, W_v, W'_v)\} \quad \text{or} \quad \{\Psi_v(s, W_v, W'_v, \Phi_v)\}$$

with $W_v \in \mathcal{W}(\pi_v, \psi_v)$, $W'_v \in \mathcal{W}(\pi'_v, \psi_v^{-1})$, and if necessary $\Phi_v \in \mathcal{S}(k_v^n)$.

Next we will first sift the local $L$-functions $L(s, \pi_v \times \pi'_v)$ from our families of local integrals. We then put these together to form the global $L$-function $L(s, \pi \times \pi')$. Then we relate this global $L$-function back to the family of global integrals and deduce its analytic properties from theirs. We will do this over the next 4 lectures.

## 5 References

[1] J.W. Cogdell, *L-functions and Converse Theorems for $GL_n$*. IAS/PCMI Lecture Notes, to appear. Available at *www.math.okstate.edu/∼cogdell* .

[2] H. Jacquet, Automorphic Forms on $GL(2)$, II, Springer Lecture Notes in Mathematics, No.278 , Springer-Verlag, Berlin, 1972.

[3] H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on* $GL(3)$, I & II. Ann. Math. **109** (1979), 169–258.

[4] H. Jacquet and R.P. Langlands, Automorphic Forms on $GL(2)$. Springer Lecture Notes in Mathematics, No. 114, Springer-Verlag, Berlin-New York, 1971.

[5] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations*, I & II. Amer. J. Math. **103** (1981), 499–588 & 777–815.

[6] R. Murty, *Symmetric Power L-functions and Applications to Analytic Number Theory.* This volume.

[7] J. Tate, *Fourier Analysis in Number Fields and Hecke's Zeta-Functions* [Princeton Thesis, 1950]. Algebraic Number Theory (J.W.S. Cassels and A. Frolich, eds.), Academic Press, London, 1967, 305–347.

LECTURE 6

# Local $L$-functions: the Non-Archimedean Case

For this lecture and the next $k$ will be a non-archimedean local field, $\mathcal{O}$ its ring of integers, $\mathfrak{p}$ its maximal ideal. We will let $\varpi$ denote a uniformizer, so $\mathfrak{p} = (\varpi)$. We normalize the absolute value by

$$|\varpi|^{-1} = |\mathcal{O}/\mathfrak{p}| = q.$$

$\psi$ will denote a non-trivial additive character of $k$.

Let $(\pi, V_\pi)$ be an irreducible admissible smooth unitary generic representation of $GL_n(k)$ and $(\pi', V_{\pi'})$ a representation of $GL_m(k)$ satisfying the same hypotheses.

[*Note*: Even if $\pi$ and $\pi'$ are not generic, they are Langlands quotients of induced representations $\Xi$ and $\Xi'$ which do have full Whittaker models. In what follows, for non-generic $\pi$ and/or $\pi'$ one uses $\mathcal{W}(\Xi, \psi)$ and $\mathcal{W}(\Xi', \psi^{-1})$ in place of $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \psi^{-1})$ to obtain a theory of local $L$-functions for non-generic representations.]

From the factorization of our global integrals we have defined families of local integrals

$$\{\Psi(s, W, W')\} \qquad \text{or} \qquad \{\Psi(s, W, W', \Phi)\}$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in \mathcal{S}(k^n)$. For convenience we will concentrate on the case $m < n$. The $m = n$ theory is analogous.

For each $j$ with $0 \le j \le n-m-1$ and $W$ and $W'$ as above we will also consider briefly the family of integrals

$$\Psi_j(s, W, W') = \iint W \begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} dx\ W'(h) |\det h|^{s - \frac{n-m}{2}} dh$$

where the inner $x$ integration is over the matrix space $M_{j,m}(k)$ and the outer $h$ integration is over $N_m(k)\backslash GL_m(k)$ as usual. In these terms, the integrals appearing in our Euler factorizations are $\Psi(s, W, W') = \Psi_0(s, W, W')$ and $\widetilde{\Psi}(s, W, W') = \Psi_{n-m-1}(s, W, W')$. As $j$ varies these families interpolate between those two.

## 1 Whittaker functions

To analyze any of these integrals we need to know some basic facts about smooth Whittaker functions $W$. Recall that they satisfy

(i) $W(ng) = \psi(n) W(g)$ for $n \in N(k)$ and $g \in GL_n(k)$.

(ii) $W(gk') = W(g)$ for $k' \in K'$ for some $K' \subset GL_n(\mathcal{O}) = K$.

So they are essentially controlled by their behavior on

$$A = A_n = \left\{ a = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix} \;\middle|\; a_i \in k^\times \right\}.$$

For $a \in A$ let $\alpha_i(a) = a_i/a_{i+1}$ be the simple roots, $1 \le i \le n-1$. The basic analytic properties of the Whittaker functions are given in the following proposition.

**Proposition 6.1** *There is a finite set of $A$-finite functions on $A$, say $X(\pi) = \{\chi_i\}$, depending only on $\pi$, such that for every $W \in \mathcal{W}(\pi,\psi)$ there exist Schwartz-Bruhat functions $\phi_i \in \mathcal{S}(k^{n-1})$ such that for $a \in A$ with $a_n = 1$ we have*

$$W\begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{n-1} & \\ & & & 1 \end{pmatrix} = \sum_{X(\pi)} \chi_i(a)\phi_i(\alpha_1(a), \ldots, \alpha_{n-1}(a)).$$

*Remarks*: 1. We can determine the behavior in $a_n$ by using the central character.
2. Since the $\phi_i$ are compactly supported on $k^{n-1}$, this says that $W(a)$ vanishes as the $\alpha_i(a) \to \infty$. [This fact is quite easy to see. If for example we take $G = GL_2$ and $\pi$ unramified and let $W$ be the $GL_2(\mathcal{O})$-invariant Whittaker function, then for $x \in \mathcal{O}$ we have

$$W\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} = W\left(\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \psi(a_1x)W\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that

$$0 = (1 - \psi(a_1x))W\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence if $\psi$ is also unramified this gives

$$W\begin{pmatrix} a_1 & 0 \\ 0 & 1 \end{pmatrix} = 0 \qquad \text{if} \quad |a_1| > 1.$$

A similar argument works in general.]
3. The collection of $A$-finite functions $X(\pi)$ comes from the Jacquet module of $\pi$. If we let $V_{\pi,N}$ denote the largest quotient of $V_\pi$ on which $N$ acts trivially, so

$$\begin{aligned} V_{\pi,N} &= V_\pi/\langle v - \pi(n)v \mid v \in V_\pi,\ n \in N\rangle \\ &= \mathcal{W}(\pi,\psi)/\langle W - \pi(n)W\rangle \\ &\simeq \{W(a)\}/\langle W(a) - \psi(ana^{-1})W(a)\rangle \end{aligned}$$

then $V_{\pi,N}$ is a finite dimensional representation of $A$. It may not be completely reducible – there may be Jordan blocks. The functions occurring in these blocks are essentially characters of $A$ or characters times powers of the *ord* function (the non-archimedean *log*). These functions make up $X(\pi)$. As the last form for $V_{\pi,N}$ given above indicates, they determine the asymptotics of $W(a)$ as the simple roots $\alpha_i(a) \to 0$. They depend only on $\pi$ and not the choice of $W$.

## 2 The local $L$-function ($m < n$)

If we take these properties of the Whittaker functions and utilize them in our family of local integrals, we quickly arrive at the following basic result.

**Proposition 6.2** *(i) Each local integral $\Psi_j(s,W,W')$ converges for $Re(s) \geq 1$.*
*(ii) Each $\Psi_j(s,W,W') \in \mathbb{C}(q^{-s})$ is a rational function of $q^{-s}$ and hence extends meromorphically to all of $\mathbb{C}$.*
*(iii) Each $\Psi_j(s,W,W')$ can be written with a common denominator determined by $X(\pi)$ and $X(\pi')$. Hence the family has "bounded denominators".*

To prove this proposition, we reduce the integrals to finite sums of integrals over $A$. Then part (i) follows from the fact that $\pi$ and $\pi'$ were taken to be unitary and what that implies about the functions occurring in $X(\pi)$ and $X(\pi')$. Part (ii) and (iii) follow from the fact that the $W(a)$ vanish as the $\alpha_i(a) \to \infty$ and have common summable asymptotics as the $\alpha_i(a) \to 0$ (essentially geometric series).

Now let us set

$$\mathcal{I}_j(\pi \times \pi') = \langle \Psi_j(s,W,W') \mid W \in \mathcal{W}(\pi,\psi),\ W' \in \mathcal{W}(\pi',\psi^{-1}) \rangle.$$

This is a subspace of $\mathbb{C}(q^{-s})$ having bounded denominators. Since in addition we have

$$\Psi_j\left(s, \pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix} W, \pi'(h)W'\right) = |\det h|^{-s-j+\frac{n-m}{2}}\Phi_j(s,W,W')$$

with $|\det h| = q^r$ for any $r \in \mathbb{Z}$ we see that each $\mathcal{I}_j(\pi \times \pi')$ is in fact a fractional ideal in $\mathbb{C}(q^{-s})$. A rather involved manipulation of the integrals guarantees that $\mathcal{I}_j(\pi \times \pi') = \mathcal{I}_{j+1}(\pi \times \pi')$ for each $j$ and hence

$$\mathcal{I}(\pi \times \pi') = \mathcal{I}_j(\pi \times \pi') \quad \text{is independent of } j.$$

Moreover, $W$ occurs in the integral as $W\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}$ and this lies in $K(\pi,\psi)$, the Kirillov model of $\pi$. Since $K(\pi,\psi) \supset \tau(\psi) = ind_N^P(\psi)$ we can show that in fact $1 \in \mathcal{I}(\pi \times \pi')$. Putting this together we have the following.

**Theorem 6.1** *The family of local integrals $\mathcal{I}(\pi \times \pi') = \langle \Psi(s,W,W') \rangle$ is a $\mathbb{C}[q^s,q^{-s}]$–fractional ideal of $\mathbb{C}(q^{-s})$ containing the constant $1$.*

Since the ring $\mathbb{C}[q^s,q^{-s}]$ is a principal ideal domain, the fractional ideal $\mathcal{I}(\pi\times\pi')$ has a generator. Since $1 \in \mathcal{I}(\pi \times \pi')$ we can take a generator having numerator 1 and normalized (up to units) to be of the form $P(q^{-s})^{-1}$ with $P(X) \in \mathbb{C}[X]$ having $P(0) = 1$.

**Definition 6.1** *The local $L$-function $L(s,\pi\times\pi') = P(q^{-s})^{-1}$ is the normalized generator of the fractional ideal $\mathcal{I}(\pi \times \pi')$ spanned by the local integrals. We set $L(s,\pi) = L(s,\pi \times \chi_0)$ where $\chi_0$ is the trivial character of $GL_1(k)$.*

Another useful way of viewing the local $L$-function is the following. $L(s,\pi\times\pi')$ is the minimal inverse polynomial $P(q^{-s})^{-1}$ such that the ratios

$$e(s,W,W') = \frac{\Psi(s,W,W')}{L(s,\pi\times\pi')} \in \mathbb{C}[q^s,q^{-s}]$$

are polynomials in $q^s$ and $q^{-s}$ and so are *entire* for all choices $W \in \mathcal{W}(\pi,\psi)$ and $W' \in \mathcal{W}(\pi',\psi^{-1})$.

Since $L(s,\pi\times\pi')$ is the generator of a fractional ideal, we find:

**Proposition 6.3** *There exist finite collections $\{W_i\}$ and $\{W_i'\}$ such that*

$$L(s,\pi\times\pi') = \sum_i \Psi(s,W_i,W_i').$$

For the same family we have $1 = \sum_i e(s,W_i,W_i')$. From this and the definition of the $e(s,W,W')$ we have the following corollary.

**Corollary 6.3.1** *The ratios $e(s,W,W')$ are entire, bounded in vertical strips, and for each $s_0 \in \mathbb{C}$ there is a choice of $W$ and $W'$ such that $e(s_0,W,W') \neq 0$.*

## 3 The local functional equation

From the functional equation of the global integrals we would expect a relation between $\Psi(s,W,W')$ and $\widetilde{\Psi}(1-s,R(w_{n,m})\widetilde{W},\widetilde{W'})$. This will follow from interpreting these integrals as giving quasi-invariant functionals on $V_\pi \times V_{\pi'}$ or $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$ and then invoking a uniqueness principle.

As for the quasi-invariance, note that the integrals $\Psi(s,W,W')$ naturally satisfy

$$\Psi\left(s,\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}W,\pi'(h)W'\right) = |\det h|^{-s+\frac{n-m}{2}}\Psi(s,W,W')$$

for all $h \in GL_m(k)$ and

$$\Psi(s,\pi(y)W,W') = \psi(y)\Psi(s,W,W')$$

for all $y \in Y_{n,m}(k) = \left\{\begin{pmatrix} I_{m+1} & * \\ 0 & x\end{pmatrix} \middle| x \in N_{n-m-1}(k)\right\}$. One can check (somewhat painfully) that $(W,W') \to \widetilde{\Psi}(1-s,R(w_{n,m})\widetilde{W},\widetilde{W'})$ satisfies the same quasi-invariance properties. Then one invokes the following uniqueness principle.

**Proposition 6.4** *Except for a finite number of exceptional values of $q^{-s}$, there is a unique (up to scalars) bilinear form $B_s$ on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$ satisfying*

(i) $B_s\left(\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}W,\pi'(h)W'\right) = |\det h|^{-s+\frac{n-m}{2}}B_s(W,W')$
(ii) $B_s(\pi(y)W,W') = \psi(y)B_s(W,W')$

*for all $h \in GL_m(k)$ and $y \in Y_{n,m}(k)$.*

The proof of this uniqueness principle takes place in the Kirillov models and involves the representation theory of the mirabolic subgroup $P$ and Bruhat theory. As a result we obtain our local functional equation.

**Theorem 6.2 (Local Functional Equation)** *There exists a rational function $\gamma(s,\pi\times\pi',\psi) \in \mathbb{C}(q^{-s})$ such that*

$$\widetilde{\Psi}(1-s,R(w_{n,m})\widetilde{W},\widetilde{W'}) = \omega_{\pi'}(-1)^{n-1}\gamma(s,\pi\times\pi',\psi)\Psi(s,W,W')$$

*for all* $W \in \mathcal{W}(\pi,\psi)$ *and* $W' \in \mathcal{W}(\pi',\psi^{-1})$.

This defines the local $\gamma$–factor $\gamma(s,\pi\times\pi',\psi)$. An equally important local factor is the local $\varepsilon$-factor

$$\varepsilon(s,\pi\times\pi',\psi) = \frac{\gamma(s,\pi\times\pi',\psi)L(s,\pi\times\pi')}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}$$

with which the local functional equation becomes

$$\frac{\widetilde{\Psi}(1-s,R(w_{n,m})\widetilde{W},\widetilde{W}')}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')} = \omega_{\pi'}(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)\frac{\Psi(s,W,W')}{L(s,\pi\times\pi')}$$

or

$$\widetilde{e}(1-s,R(w_{n,m})\widetilde{W},\widetilde{W}') = \omega_{\pi'}(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)e(s,W,W').$$

Recall that we have finite collections $\{W_i\}$ and $\{W_i'\}$ such that

$$1 = \sum_i e(s,W_i,W_i') = \sum_i \frac{\Psi(s,W_i,W_i')}{L(s,\pi\times\pi')}.$$

If we combine this with the local functional equation we find

$$\varepsilon(s,\pi\times\pi',\psi) = \omega_{\pi'}(-1)^{n-1}\sum_i \widetilde{e}(1-s,R(w_{n,m})\widetilde{W}_i.\widetilde{W}_i') \in \mathbb{C}[q^s,q^{-s}].$$

Applying the local functional equation twice then gives

$$\varepsilon(s,\pi\times\pi,\psi)\varepsilon(1-s,\widetilde{\pi}\times\widetilde{\pi}',\psi^{-1}) = 1$$

so $\varepsilon(s,\pi\times\pi',\psi)$ is a unit in $\mathbb{C}[q^s,q^{-s}]$. Thus:

**Proposition 6.5** *The local $\varepsilon$–factor is a monomial function of the form*

$$\varepsilon(s,\pi\times\pi',\psi) = cq^{-fs}.$$

## 4 The conductor of $\pi$

Let me explain a bit about the information contained in the $\varepsilon$–factor. Take $\pi' = \chi_0$ the trivial character of $GL_1(k)$ and write $\varepsilon(s,\pi,\psi) = \varepsilon(s,\pi\times\chi_0,\psi)$. Furthermore assume $\psi$ is unramified, that is, $\psi$ is trivial on $\mathcal{O}$ but $\psi(\varpi^{-1}) \neq 1$. Let us write

$$\varepsilon(s,\pi,\psi) = \varepsilon(\tfrac{1}{2},\pi,\psi)q^{-f(\pi)(s-1/2)}.$$

Then $|\varepsilon(\frac{1}{2},\pi,\psi)| = 1$. The "sign" $\varepsilon(\frac{1}{2},\pi,\psi)$ is the so called "local root number" of $\pi$ and can contain subtle information. On the other hand, the integer $f(\pi)$ occurring in the exponent contains the following basic information.

**Theorem 6.3** *(i)* $f(\pi) \geq 0$.
*(ii)* $f(\pi) = 0$ *iff* $\pi$ *is unramified, that is,* $V_\pi$ *has a non-zero vector fixed by* $K = GL_n(\mathcal{O})$.
*(iii) Let*

$$K_1(\mathfrak{p}^t) = \left\{ k \in GL_n(\mathcal{O}) \middle| k \equiv \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}^t} \right\}.$$

*Then if* $0 \leq t < f(\pi)$ *then* $V_\pi$ *has no non-zero* $K_1(\mathfrak{p}^t)$ *fixed vectors. If* $t = f(\pi)$ *then the space of* $K_1(\mathfrak{p}^{f(\pi)})$*–fixed vectors has dimension one.*

The integer $f(\pi)$ or the ideal $\mathfrak{p}^{f(\pi)}$ is the *conductor of* $\pi$. The one dimensional space of $K_1(\mathfrak{p}^{f(\pi)})$-fixed vectors is the space of "local new vectors" or "local new forms".

## 5 Multiplicativity and Stability of $\gamma$-factors

I would like to end with two facts about the local $\gamma$-factors. Their proofs would take us too far afield, but we will need them for later applications to liftings.

Suppose that $\pi$ is an induced representation. Then there is a maximal parabolic subgroup $Q = MN$ of $GL_n$ with $M \simeq GL_{r_1} \times GL_{r_2}$ and smooth irreducible admissible generic representations $\pi_1$ and $\pi_2$ such that

$$\pi \simeq Ind_{Q(k)}^{GL_n(k)}(\pi_1 \otimes \pi_2).$$

Then there is a relation between $\mathcal{W}(\pi,\psi)$ and the spaces $\mathcal{W}(\pi_1,\psi)$ and $\mathcal{W}(\pi_2,\psi)$ that can be exploited to show:

**Proposition 6.6 (Multiplicativity of $\gamma$)** *In the above situation*

$$\gamma(s,\pi\times\pi',\psi) = \gamma(s,\pi_1\times\pi',\psi)\gamma(s,\pi_2\times\pi',\psi).$$

In this case, one also has a divisibility among the $L$-functions

$$L(s,\pi\times\pi')^{-1}\Big|\ [L(s,\pi_1\times\pi')L(s,\pi_2\times\pi')]^{-1}.$$

There is also a similar multiplicativity and divisibility in the second variable, that is, for $\pi'$ induced.

Our second result is an instance of a common phenomenon: if one twists a situation by a highly ramified character then things become "standard", that is, one twists away all interesting information.

**Proposition 6.7 (Stability of $\gamma$)** *Let* $\pi_1$ *and* $\pi_2$ *be two irreducible admissible smooth generic representations of* $GL_n(k)$ *having the same central character. Then for every sufficiently highly ramified character* $\eta$ *of* $k^\times$ *we have*

$$\gamma(s,\pi_1\times\eta,\psi) = \gamma(s,\pi_2\times\eta,\psi).$$

In this situation, one also has that the $L$-functions stabilize

$$L(s,\pi_1\times\eta) = L(s,\pi_2\times\eta) \equiv 1$$

so that the $\varepsilon(s,\pi_i\times\eta,\psi)$ stabilize as well. More generally, we also have

$$\gamma(s,(\pi_1\otimes\eta)\times\pi',\psi) = \gamma(s,(\pi_2\otimes\eta)\times\pi',\psi)$$

and

$$L(s,(\pi_1\otimes\eta)\times\pi') = L(s,(\pi_2\otimes\eta)\times\pi') \equiv 1$$

for all sufficiently highly ramified characters $\eta$.

Note that if we combine multiplicativity and stability we can actually compute the stable form of $\gamma$ or $\varepsilon$. This is useful for comparing these local factors on different groups.

For example, let $\pi$ be an irreducible admissible generic representation with central character $\omega$. Let $\mu_1, \ldots, \mu_n$ be characters of $GL_1(k)$ such that $\mu_1 \cdots \mu_n = \omega$. Then taking $\pi_1 = \pi$ and $\pi_2 = Ind(\mu_1 \otimes \cdots \otimes \mu_n)$ we find that for sufficiently highly ramified $\eta$ we have

$$\begin{aligned}\gamma(s, \pi \times \eta, \psi) &= \gamma(s, Ind(\mu_1 \otimes \cdots \otimes \mu_n) \times \eta, \psi) \\ &= \prod_{i=1}^{n} \gamma(s, \mu_i \eta, \psi) = \prod_{i=1}^{n} \varepsilon(s, \mu_i \eta, \psi)\end{aligned}$$

expressing $\gamma(s, \pi \times \eta, \psi)$ as a product of standard abelian $\varepsilon$-factors.

## 6 References

[1] J.W. Cogdell, *L-functions and Converse Theorems for $GL_n$*. IAS/PCMI Lecture Notes, to appear. Available at *www.math.okstate.edu/∼cogdell* .

[2] H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, *Conducteur des représentations du groupe linéaire*. Math. Ann. **256** (1981), 199–214.

[3] H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, *Rankin–Selberg convolutions*. Amer. J. Math. **105** (1983), 367–464.

[4] H. Jacquet and J. Shalika, *A lemma on highly ramified $\epsilon$-factors*. Math. Ann. **271** (1985), 319–332.

LECTURE 7

# The Unramified Calculation

In this lecture I would like to calculate $L(s, \pi \times \pi')$ when both $\pi$ and $\pi'$ are unramified, that is, they both have vectors fixed under their respective maximal compact subgroups $GL_n(\mathcal{O})$ and $GL_m(\mathcal{O})$. We will do this by explicitly computing the local integral $\Psi(s, W^\circ, W'^\circ)$ for $W^\circ$ and $W'^\circ$ the normalized $K$-fixed Whittaker functions. This calculation is similar to and motivated by the calculation of the $p$-Euler factor for $L(s, f)$ for $f$ a classical cusp form. Recall that in the classical case of $f \in S_k(SL_2(\mathbb{Z}))$ to be able to compute the $p$-Euler factor for $L(s, f)$ we needed to know two things:

(i) that $f$ was an eigen-function for all Hecke operators $T_p$ or $T_n$;
(ii) the recursion among the $T_{p^r}$ for a fixed $p$.

Now again let $k$ be a non-archimedean local field of characteristic 0 with ring of integers $\mathcal{O}$, maximal ideal $\mathfrak{p}$ and uniformizer $\varpi$. Let

$$\mathcal{H}_K = \mathcal{H}(GL_n(k)//K) = C_c^\infty(GL_n(k)//GL_2(\mathcal{O}))$$

be the spherical Hecke algebra for $GL_n(k)$ consisting of compactly supported functions on $GL_n(k)$ which are bi-$K$-invariant, as in Lecture 3. This plays the role of the classical Hecke algebra in this context. It is a convolution algebra as before. For each $i$, $0 \le i \le n$, let $\Phi_i$ be the characteristic function

$$\Phi_i = Char\left(GL_n(\mathcal{O})\begin{pmatrix}\varpi I_i & \\ & I_{n-i}\end{pmatrix}GL_n(\mathcal{O})\right)$$

so that $\varpi$ occurs in the first $i$ diagonal entries. (For $G = GL_2$ and $k = \mathbb{Q}_p$, $\Phi_1$ is the avatar of the classical Hecke operator $T_p$.) Then a standard fact is:

**Proposition 7.1** *The spherical Hecke algebra $\mathcal{H}_K$ is a commutative algebra and is generated by the $\Phi_i$ for $1 \le i \le n$.*

For any smooth representation $(\pi, V_\pi)$ of $GL_n(k)$ we have an action of $\mathcal{H}$ or $\mathcal{H}_K$ on $V_\pi$ as a convolution algebra via

$$\pi(\Phi)v = \int_{GL_n(k)} \Phi(g)\pi(g)v \, dg.$$

Note that since $\pi$ is smooth and $\Phi$ has compact support, this is really a finite sum. In the transition from classical modular forms to automorphic representations and back, this corresponds to the action of the classical Hecke operators on modular forms.

## 1 Unramified representations

Now let $(\pi, V_\pi)$ be an irreducible admissible smooth generic representation of $GL_n(k)$ which is unramified. Then it is known that

$$\pi = Ind_{B(k)}^{GL_n(k)}(\mu_1 \otimes \cdots \otimes \mu_n)$$

is a full induced representation from the Borel subgroup $B(k)$ of unramified characters $\mu_i$ of $k^\times$. Here unramified means that each $\mu_i$ is invariant under the maximal compact subgroup $\mathcal{O}^\times \subset k^\times$. Since $k^\times = \coprod \varpi^j \mathcal{O}^\times$, each character $\mu_i$ is completely determined by its value $\mu_i(\varpi) \in \mathbb{C}^\times$. Thus in turn $\pi$ will be completely determined by the $n$ complex numbers

$$\{\mu_1(\varpi), \ldots, \mu_n(\varpi)\}$$

which can be encoded in a diagonal matrix

$$A_\pi = \begin{pmatrix} \mu_1(\varpi) & & \\ & \ddots & \\ & & \mu_n(\varpi) \end{pmatrix} \in GL_n(\mathbb{C}).$$

These parameters, whether viewed as $n$ non-zero complex numbers, the matrix $A_\pi \in GL_n(\mathbb{C})$ or the conjugacy class $[A_\pi] \subset GL_n(\mathbb{C})$ are the *Satake parameters* of the unramified representation $\pi$.

Since $(\pi, V_\pi)$ is unramified, then there is a unique (up to scalar multiples) non-zero $K$-fixed vector $v^\circ \in V_\pi$. If $\Phi \in \mathcal{H}_K$, the spherical Hecke algebra, then $\pi(\Phi)v^\circ$ will again be $K$-fixed. Thus we obtain

$$\pi(\Phi)v^\circ = \Lambda_\pi(\Phi)v^\circ$$

with $\Lambda_\pi : \mathcal{H}_K \to \mathbb{C}$ a character of $\mathcal{H}_K$ as a convolution algebra. Thus $v^\circ$ is our local Hecke eigen-function.

For $\pi = Ind(\mu_1 \otimes \cdots \otimes \mu_n)$ it is easy to compute this character on the generators $\Phi_i$ of $\mathcal{H}_K$. As in the classical case, we will need to know how to decompose the associated double coset into single cosets.

For each $J \in \mathbb{Z}^n$, say $J = (j_1, \ldots, j_n)$, let

$$\varpi^J = \begin{pmatrix} \varpi^{j_1} & & \\ & \ddots & \\ & & \varpi^{j_n} \end{pmatrix} \in GL_n(k).$$

So if we set $\eta_i = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ with the first $i$ entries of 1 and the others 0, then $\Phi_i$ is the characteristic function of $K\varpi^{\eta_i}K$. To decompose this double coset into single ones, let us set

$$I_i = \left\{ \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{Z}^n \mid \epsilon_j \in \{0, 1\}, \sum \epsilon_j = i \right\}$$

and for each $\epsilon \in I_i$ let

$$N(\mathcal{O}, \epsilon) = N(\mathcal{O}) \cap \varpi^\epsilon K \varpi^{-\epsilon}.$$

**Lemma 7.1**

$$K\varpi^{\eta_i}K = \coprod_{\epsilon\in I_i}\coprod_{n\in N(\mathcal{O})/N(\mathcal{O},\epsilon)} n\varpi^{\epsilon}K.$$

Now let $f^\circ$ be the $K$-fixed vector in $Ind(\mu_1\otimes\cdots\otimes\mu_n)$ normalized so that $f^\circ(e)=1$. Then we have

$$(\pi(\Phi_i)f^\circ)(e) = \Lambda(\Phi_i)f^\circ(e) = \Lambda_\pi(\Phi_i).$$

On the other hand, we can do the explicit computation in the induced model. By definition

$$f^\circ(nak) = \delta_{B_n}^{1/2}(a)\prod_{i=1}^n \mu_i(a_i)f^\circ(e) = \delta_{B_n}^{1/2}(a)\prod_{i=1}^n \mu_i(a_i)$$

for $n\in N_n(k)$, $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix} \in A_n(k)$, and $k\in K_n$. Then we can compute

$$\begin{aligned}(\pi(\Phi_i)f^\circ)(e) &= \int_{GL_n(k)} \Phi_i(g)f^\circ(g)\,dg = \int_{K\varpi^{\eta_i}K} f^\circ(g)\,dg\\ &= \sum_{\epsilon\in I_i}\sum_{n\in N(\mathcal{O})/N(\mathcal{O},\epsilon)} f(n\varpi^\epsilon)\\ &= \sum_{\epsilon\in I_i} |N(\mathcal{O})/N(\mathcal{O},\epsilon)|\delta_{B_n}^{1/2}(\varpi^\epsilon)\prod_{j=1}^n \mu_j(\varpi)^{\epsilon_j}.\end{aligned}$$

An elementary computation then gives

$$|N(\mathcal{O})/N(\mathcal{O},\epsilon)|\delta_{B_n}^{1/2}(\varpi^\epsilon) = q^{i(n-i)/2}$$

so that

$$\begin{aligned}(\pi(\Phi_i)f^\circ)(e) &= q^{i(n-i)/2}\sum_{\epsilon\in I_i}\prod_{j=1}^n \mu_j(\varpi)^{\epsilon_j}\\ &= q^{i(n-i)/2}\sigma_i(\mu_1(\varpi),\ldots,\mu_n(\varpi))\end{aligned}$$

where $\sigma_i$ is the $i^{th}$ elementary symmetric polynomial in the $\mu_j(\varpi)$.

Comparing our two expressions for $(\pi(\Phi_i)f^\circ)(e)$ we obtain

**Proposition 7.2** *For* $\pi = Ind(\mu_1\otimes\cdots\otimes\mu_n)$ *unramified*

$$\Lambda_\pi(\Phi_i) = q^{i(n-i)/2}\sigma_i(\mu_1(\varpi),\ldots,\mu_n(\varpi)).$$

This computes the Hecke eigen-values in terms of the Satake parameters of $\pi$.

## 2 Unramified Whittaker functions

The analogue of the classical recursion relation for the Hecke operators $T_{p^r}$ can now be employed to compute a *formula* for the unramified Whittaker function $W^\circ$ that occurs in our integrals. For $GL_n$ this was first done by Shintani, who we follow.

Take $\psi : k \to \mathbb{C}$ our additive character to also be unramified and non-trivial, so $\psi(\mathcal{O}) = 1$ but $\psi(\varpi^{-1}) \neq 1$. Let $W^\circ \in \mathcal{W}(\pi, \psi)$ be the $K$-fixed Whittaker function in $\mathcal{W}(\pi, \psi)$. By the Iwasawa decomposition, any $g \in GL_n(k)$ can be written

$$g = nak \in NAK \quad \text{with} \quad a = \varpi^J \in A$$

for some $J \in \mathbb{Z}^n$. Then

$$W^\circ(g) = W^\circ(n\varpi^J k) = \psi(n) W^\circ(\varpi^J).$$

So it suffices to compute the values $W^\circ(\varpi^J)$. The same calculation that gave the "rapid decrease" of $W$ on $A$ in the $GL_2(k)$ case now gives

$$W^\circ(\varpi^J) = 0 \quad \text{unless} \quad j_1 \geq j_2 \geq \cdots \geq j_n.$$

We next do an explicit calculation of the action of each $\Phi_i \in \mathcal{H}_K$ in the Whittaker model. We still have that

$$(\pi(\Phi_i)W^\circ)(\varpi^J) = \Lambda_\pi(\Phi_i) W^\circ(\varpi^J)$$

for all $J$, with an explicit formula for $\Lambda(\Phi_i)$. Computing in the Whittaker model we have

$$\begin{aligned}(\pi(\Phi_i)W^\circ)(\varpi^J) &= \int_{K\varpi^{\eta_i}K} W^\circ(\varpi^J g)\ dg \\ &= \sum_{\epsilon \in I_i} \sum_{n \in N(\mathcal{O})/N(\mathcal{O},\epsilon)} W^\circ(\varpi^J n \varpi^\epsilon) \\ &= \sum_{\epsilon \in I_i} \sum_{n \in N(\mathcal{O})/N(\mathcal{O},\epsilon)} \psi(\varpi^J n \varpi^{-J}) W^\circ(\varpi^{J+\epsilon}).\end{aligned}$$

Since $j_1 \geq \cdots \geq j_n$, we have that $\varpi^J n \varpi^{-J} \in N(\mathcal{O})$ so that the value of $\psi$ on this element is 1. Hence

$$\begin{aligned}(\pi(\Phi_i)W^\circ)(\varpi^J) &= \sum_{\epsilon \in I_i} |N(\mathcal{O})/N(\mathcal{O},\epsilon)| W^\circ(\varpi^{J+\epsilon}) \\ &= \sum_{\epsilon \in I_i} \delta_{B_n}^{-1/2}(\varpi^\epsilon) q^{i(n-i)/2} W^\circ(\varpi^{J+\epsilon}).\end{aligned}$$

If we then combine our two expressions for $(\pi(\Phi_i)W^\circ)(\varpi^J)$ we obtain our recursion.

**Proposition 7.3** *For the unramified Whittaker function in $\mathcal{W}(\pi, \psi)$ we have the recursion*

$$\Lambda_\pi(\Phi_i) W^\circ(\varpi^J) = q^{i(n-i)/2} \sum_{\epsilon \in I_i} \delta_{B_n}^{-1/2}(\varpi^\epsilon) W^\circ(\varpi^{J+\epsilon}).$$

The solution to this recursion is quite interesting. It involves the characters of finite dimensional representations of $GL_n(\mathbb{C})$. The $n$-tuples $J = (j_1, \ldots, j_n)$ with $j_1 \geq \cdots \geq j_n$ are the possible highest weights for the finite dimensional representations of $GL_n(\mathbb{C})$. Let $\rho_J$ denote the finite dimensional representation of highest weight $J$ and let $\chi_J = Tr(\rho_J)$ be its character. Then many things are known about these characters, for example, from the formula for the decomposition of the tensor product of two finite dimensional representations we also obtain a recursion

$$\chi_{\eta_i} \chi_J = \sum_{\epsilon \in I_i} \chi_{J+\epsilon}$$

similar to the recursion for $W^\circ(\varpi^J)$. Since the $\chi_J$ are class functions on $GL_n(\mathbb{C})$, it makes sense to evaluate them on our Satake class $A_\pi$ for $\pi$. For example, since $\rho_{\eta_i}$ is the $i^{th}$ exterior power of the standard representation of $GL_n$ we find that

$$\chi_{\eta_i}(A_\pi) = \sigma_i(\mu_1(\varpi), \dots, \mu_n(\varpi)) = q^{-i(n-i)/2}\Lambda_\pi(\Phi_i).$$

Utilizing these facts from finite dimensional representation theory it is then a simple matter to solve the recursion for the $W^\circ(\varpi^J)$ in terms of the $\chi_J(A_\pi)$ and obtain Shintani's formula.

**Proposition 7.4** $W^\circ(\varpi^J) = \delta_{B_n}^{1/2}(\varpi^J)\chi_J(A_\pi)$.

## 3 Calculating the integral

We now return to our local integral. We consider the case $m < n$ and $\pi$, $\pi'$, and $\psi$ all unramified. Let $W^\circ \in \mathcal{W}(\pi, \psi)$ and $W'^\circ \in \mathcal{W}(\pi', \psi^{-1})$ be the normalized $K$-fixed Whittaker functions computed above. We have

$$\Psi(s, W^\circ, W'^\circ) = \int_{N_m(k)\backslash GL_m(k)} W^\circ \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'^\circ(h)|\det(h)|^{s-\frac{n-m}{2}}\, dh.$$

Use the Iwasawa decomposition to write $GL_m = N_m A_m K_m$ so that $h = n\varpi^J k$ and $dh = dn\ \delta_{B_m}^{-1}(\varpi^J)\ dk$. Then

$$\begin{aligned}\Psi(s,&W^\circ, W'^\circ) \\ &= \sum_{J\in\mathbb{Z}^m} W^\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} W'^\circ(\varpi^J)|\det(\varpi^J)|^{s-\frac{n-m}{2}}\beta_{B_m}^{-1}(\varpi^J).\end{aligned}$$

Now $W'^\circ(\varpi^J) = 0$ unless $j_1 \geq \cdots \geq j_m$ and $W^\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} = 0$ unless $j_1 \geq \cdots \geq j_m \geq 0$. Moreover, $|\det(\varpi^J)| = q^{-|J|}$ where $|J| = j_1 + \cdots + j_m$. So our integral becomes

$$\begin{aligned}\Psi(s,&W^\circ, W'^\circ) \\ &= \sum_{j_1\geq\cdots\geq j_m\geq 0} W^\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} W'^\circ(\varpi^J) q^{-|J|(s-\frac{n-m}{2})}\beta_{B_m}^{-1}(\varpi^J).\end{aligned}$$

We next insert the formula from Proposition 7.4 and use the elementary fact that

$$\delta_{B_n}^{1/2}\begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} \delta_{B_m}^{-1/2}(\varpi^J) = q^{-|J|\frac{n-m}{2}}$$

to obtain

$$\Psi(s, W^\circ, W'^\circ) = \sum_{j_1\geq\cdots\geq j_m\geq 0} \chi_{(J,0)}(A_\pi)\chi_J(A_{\pi'})q^{-|J|s}$$

where $(J, 0) = (j_1, \dots, j_m, 0, \dots, 0)$ represents $J$ filled out to be a vector in $\mathbb{Z}^n$.

We next use some fairly standard facts from the finite dimensional representation theory of $GL_n(\mathbb{C})$, namely

$$\chi_{(J,0)}(D_n)\chi_J(D_m) = Tr(\rho_{(J,0)}(D_n) \otimes \rho_J(D_m)),$$

$$\sum_{\substack{j_1 \geq \cdots \geq j_m \geq 0 \\ |J|=r}} Tr(\rho_{_{(J,0)}}(D_n) \otimes \rho_{_J}(D_m)) = Tr(S^r(D_n \otimes D_m)),$$

and

$$\sum_{r=0}^{\infty} Tr(S^r(D))X^r = \det(1 - XD)^{-1},$$

where $S^r(D)$ is the $r^{th}$ symmetric power of $D$. Applying these with $D_n = A_\pi$ and $D_m = A_{\pi'}$ finishes our calculation of the local integral.

**Proposition 7.5** *If $\pi$, $\pi'$, and $\psi$ are all unramified, then*

$$\Psi(s, W^\circ, W'^\circ) = \det(I - q^{-s}A_\pi \otimes A_{\pi'})^{-1} = \prod_{i,j}(1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s})^{-1}.$$

Since $L(s, \pi \times \pi')$ is the minimal inverse polynomial in $q^{-s}$ killing all poles of the family of local integrals this implies

$$\det(I - q^{-s}A_\pi \otimes A_{\pi'}) | L(s, \pi \times \pi')^{-1}.$$

Then comparing the poles of this factor with the potential poles coming from the asymptotics of $W^\circ$ and $W'^\circ$ as the simple roots go to zero from Lecture 6 gives us our result.

**Theorem 7.1** *If $\pi$, $\pi'$, and $\psi$ are all unramified, then*

$$L(s, \pi \times \pi') = \det(I - q^{-s}A_\pi \otimes A_{\pi'})^{-1} = \Psi(s, W^\circ, W'^\circ).$$

Note that the degree of this Euler factor is $mn$. Moreover,

$$L(s, \pi) = \det(I - q^{-s}A_\pi)^{-1} = \prod(1 - \mu_i(\varpi)q^{-s})^{-1}$$

is an Euler factor of degree $n$. The same result holds for $GL_n \times GL_n$. One then takes for the Schwartz function $\Phi \in \mathcal{S}(k^n)$ the characteristic function $\Phi^\circ$ of $\mathcal{O}^n \subset k^n$.

Since the factor $\varepsilon(s, \pi \times \pi', \psi)$ satisfies the local functional equation

$$\frac{\widetilde{\Psi}(1 - s, R(w_{n,m})\widetilde{W}^\circ, \widetilde{W}'^\circ)}{L(1 - s, \widetilde{\pi} \times \widetilde{\pi}')} = \omega_{\pi'}(-1)^{n-1}\varepsilon(s, \pi \times \pi', \psi)\frac{\Psi(s, W^\circ, W'^\circ)}{L(s, \pi \times \pi')}$$

we can conclude the following corollary.

**Corollary 7.1.1** *If $\pi$, $\pi'$, and $\psi$ are all unramified, then*

$$\varepsilon(s, \pi \times \pi', \psi) \equiv 1.$$

*In particular, taking $\pi'$ to be the trivial character of $GL_1$ we see that if $\pi$ is unramified then its conductor $f(\pi) = 0$.*

Finally, as a second corollary we obtain the Jacquet-Shalika bounds on the Satake parameters.

**Corollary 7.1.2** *Suppose that $\pi$ is a irreducible unitary generic unramified representation of $GL_n(k)$, $\pi \simeq Ind(\mu_1 \otimes \cdots \otimes \mu_n)$. Then the Satake parameters $\mu_i(\varpi)$ satisfy*

$$q^{-1/2} < |\mu_i(\varpi)| < q^{1/2}.$$

To see this, we apply the $GL_n \times GL_n$ unramified calculation to $\pi$ and $\pi' = \overline{\pi}$, the complex conjugate representation. Then $A_{\pi'} = A_{\overline{\pi}} = \overline{A_\pi}$ and

$$\det(I - q^{-s} A_\pi \otimes \overline{A_\pi}) \Psi(s, W^\circ, W'^\circ, \Phi^\circ) = 1.$$

The local integral is absolutely convergent for $Re(s) \geq 1$ since $\pi$ is unitary. Then

$$\det(I - q^{-s} A_\pi \otimes \overline{A_\pi}) \neq 0 \quad \text{for} \quad Re(s) \geq 1.$$

This determinant has as a factor $(1 - |\mu_i(\varpi)|^2 q^{-s})$, so this also cannot vanish for $Re(s) \geq 1$. Hence

$$|\mu_i(\varpi)| < q^{1/2}.$$

Applying the same argument to $\widetilde{\pi}$ gives

$$|\mu_i(\varpi)|^{-1} < q^{1/2}$$

since $A_{\widetilde{\pi}} = A_\pi^{-1}$. Thus we have the result.

## 4 References

[1] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations*, I & II. Amer. J. Math. **103** (1981), 499–588 & 777–815.

[2] T. Shintani, *On an explicit formula for class-*1 *"Whittaker functions" on* $GL_n$ *over* $\mathfrak{P}$*-adic fields.* Proc. Japan Acad. **52** (1976), 180–182.

LECTURE 8

# Local $L$-functions: the Archimedean Case

When $k = \mathbb{R}$ or $\mathbb{C}$ we still have our family of local integrals

$$\{\Psi(s, W, W')\} \quad \text{or} \quad \{\Psi_j(s, W, W')\} \quad \text{or} \quad \{\Psi(s, W, W', \Phi)\}$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1})$, and $\Phi \in \mathcal{S}(k^n)$, now for $\pi$ and $\pi'$ irreducible admissible generic representations of $GL_n(k)$ or $GL_m(k)$ which are smooth and of moderate growth. In the current state of affairs the local $L$-functions $L(s, \pi \times \pi')$ are not defined intrinsically through the integrals, but rather extrinsically through the arithmetic Langlands classification and then related to the integrals.

## 1 The arithmetic Langlands classification

Both $k = \mathbb{R}$ and $k = \mathbb{C}$ have attached to them Weil groups $W_k$ which play a role in their local class field theory similar to that of the richer $Gal(\overline{k}/k)$ for non-archimedean $k$.

When $k = \mathbb{C}$, $W_{\mathbb{C}} = \mathbb{C}^\times$ is simply the multiplicative group of $\mathbb{C}$. The only irreducible representations of $W_{\mathbb{C}}$ are thus characters.

When $k = \mathbb{R}$ then $W_{\mathbb{R}}$ can be defined as $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ where $jzj^{-1} = \overline{z}$ and $j^2 = -1 \in \mathbb{C}^\times$. This is an extension of $Gal(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times = W_{\mathbb{C}}$.

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow Gal(\mathbb{C}/\mathbb{R}) \longrightarrow 1$$

Now $W_{\mathbb{R}}$ has both one and two dimensional irreducible representations. Note that $W_{\mathbb{R}}^{ab} \simeq \mathbb{R}^\times$.

In rough terms, the arithmetic Langlands classification says there are natural bijections between

$$\mathcal{A}_n(k) = \{ \text{ irreducible admissible } \mathcal{H} - \text{modules} \quad \text{for} \quad GL_n(k)\}$$

and

$$\mathcal{G}_n(k) = \{n - \text{dimensional, semisimple representations of } W_k\}.$$

On the other hand, if

$$\mathcal{A}_n^\infty(k) = \{ \text{ irreducible admissible smooth moderate growth representations of } GL_n(k)\}$$

then the work of Casselman and Wallach gives a bijection between $\mathcal{A}_n(k)$ and $\mathcal{A}_n^\infty(k)$. Combining these, we can view the arithmetic Langlands classification as giving a natural bijection

$$\begin{array}{ccccc} \mathcal{A}_n^\infty(k) & \stackrel{\sim}{\longleftrightarrow} & \mathcal{A}_n(k) & \stackrel{\sim}{\longleftrightarrow} & \mathcal{G}_n(k) \\ \pi & & \longrightarrow & & \tau = \tau(\pi). \\ \pi = \pi(\tau) & & \longleftarrow & & \tau \end{array}$$

For example:

- If $\dim(\tau) = 1$, then $\pi(\tau)$ is a character of $GL_1(k)$.

- If $k = \mathbb{R}$ and $\tau$ is irreducible, unitary, and $\dim(\tau) = 2$, then $\pi(\tau)$ is a unitary discrete series representation of $GL_2(\mathbb{R})$.

- If $\tau = \bigoplus_{i=1}^{r} \tau_i$ with each $\tau_i$ irreducible, then $\pi(\tau)$ is the Langlands quotient of $Ind_Q^{GL_n}(\pi(\tau_1) \otimes \cdots \otimes \pi(\tau_r))$.

- If in addition $\pi$ is generic, then

$$\pi = \pi(\tau) = Ind_Q^{GL_n}(\pi(\tau_1) \otimes \cdots \otimes \pi(\tau_r))$$

is a full irreducible induced representation from characters of $GL_1(k)$ and possible discrete series representations of $GL_2(k)$ if $k = \mathbb{R}$. (This result is due to Vogan and is not part of the classification per se.)

## 2 The $L$-functions

Set

$$\Gamma_k(s) = \begin{cases} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) & k = \mathbb{R} \\ 2(2\pi)^{-s}\Gamma(s) & k = \mathbb{C} \end{cases}.$$

Then Weil attached to each semi-simple representation $\tau$ of $W_k$ an $L$-function: $L(s, \tau)$. For example:

- If $\tau$ is an unramified character of $W_k^{ab} = k^\times$, say $\tau(x) = |x|_k^r$, then $L(s, \tau) = \Gamma_k(s + r)$.

- If $\dim(\tau) = 2$ and $\pi(\tau)$ is the holomorphic discrete series of weight $k$ for $GL_2(\mathbb{R})$, then $L(s, \tau) = \Gamma_{\mathbb{C}}\left(s + \frac{k-1}{2}\right)$.

- If $\tau = \bigoplus_{i=1}^{r} \tau_i$ with each $\tau_i$ irreducible, then $L(s, \tau) = \prod_{i=1}^{r} L(s, \tau_i)$.

He also attached local $\varepsilon$-factors. For example, if $\tau$ is the character $\tau(x) = x^{-N}|x|_k^t$ and $\psi$ is the standard additive character of $k$, then $\varepsilon(s, \tau, \psi) = i^N$.

There are natural "twisted" $L$-functions and $\varepsilon$-factors in this context, for if $\tau$ is an $n$-dimensional representation of $W_k$ and $\tau'$ in a $m$-dimensional representation of $W_k$ then the tensor product $\tau \otimes \tau'$ is an $mn$-dimensional representation and we have thus defined $L(s, \tau \otimes \tau')$ and $\varepsilon(s, \tau \otimes \tau', \psi)$ as well.

Now return to our representations $\pi$ of $GL_n(k)$ and $\pi'$ of $GL_m(k)$. Suppose that under the arithmetic Langlands classification we have $\pi = \pi(\tau)$ and $\pi' = \pi(\tau')$ with $\tau$ an $n$-dimensional and $\tau'$ an $m-$dimensional representation of $W_k$. Then we **define** the $L$-function for $\pi$ and $\pi'$ through the classification:

$$L(s, \pi \times \pi') = L(s, \tau \otimes \tau')$$
$$\varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \tau \otimes \tau', \psi)$$

and we set

$$\gamma(s, \pi \times \pi', \psi) = \frac{\varepsilon(s, \pi \times \pi', \psi) L(1-s, \widetilde{\pi} \times \widetilde{\pi}')}{L(s, \pi \times \pi')} = \frac{\varepsilon(s, \tau \otimes \tau', \psi) L(1-s, \widetilde{\tau} \otimes \widetilde{\tau}')}{L(s, \tau \otimes \tau')}.$$

Note that $L(s, \pi \times \pi')$ is always an archimedean Euler factor of degree $nm$.

## 3 The integrals ($m < n$)

We now have to prove that this definition of the $L$-function behaves well with respect to our integrals. To analyze the integrals, we begin again with the properties of the Whittaker functions.

**Proposition 8.1** *Let $\pi$ be an irreducible admissible generic representation of $GL_n(k)$ which is smooth of moderate growth. Then there is a finite set of $A$-finite functions on $A$, say $X(\pi) = \{\chi_i\}$, depending only on $\pi$, such that for every $W \in \mathcal{W}(\pi, \psi)$ there exist Schwartz functions $\phi_i \in \mathcal{S}(k^{n-1} \times K)$ such that for $a \in A$ with $a_n = 1$ and $k \in K$ we have*

$$W\left(\begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & a_{n-1} & \\ & & & 1 \end{pmatrix} k\right) = \sum_{X(\pi)} \chi_i(a) \phi_i(\alpha_1(a), \ldots, \alpha_{n-1}(a); k).$$

As in the non-archimedean case, the $A$-finite functions in $X(\pi)$ are related to the archimedean Jacquet module of $\pi$ and then through the classification to the associated representation $\tau$ of $W_k$. This then gives the same convergence estimates as before.

**Proposition 8.2** *Each local integral $\Psi_j(s, W, W')$ converges absolutely for $Re(s) \gg 0$, and if $\pi$ and $\pi'$ are both unitary they converge absolutely for $Re(s) \geq 1$.*

The non-archimedean statements on rationality and "bounded denominators" are replaces by the following analysis.

Let $\mathcal{M}(\pi \times \pi') = \mathcal{M}(\tau \otimes \tau')$ be the space of all meromorphic functions $\phi(s)$ satisfying:

• If $P(s) \in \mathbb{C}[s]$ is a polynomial such that $P(s)L(s, \pi \times \pi')$ is holomorphic in the vertical strip $S[a, b] = \{s \mid a \leq Re(s) \leq b\}$, then $P(s)\phi(s)$ is holomorphic and bounded in $S[a, b]$.

As an exercise, one can show that $\phi \in \mathcal{M}(\pi \times \pi')$ implies that the ratio $\dfrac{\phi(s)}{L(s, \pi \times \pi')}$ is entire.

**Theorem 8.1** *The integrals $\Psi_j(s, W, W')$ extend to meromorphic functions of $s$ and as such $\Psi_j(s, W, W') \in \mathcal{M}(\pi \times \pi')$. In particular, the ratios $e_j(s, W, W') = \dfrac{\Psi_j(s, W, W')}{L(s, \pi \times \pi')}$ are entire.*

This is more than just "bounded denominators" since it specifies $L(s, \pi \times \pi')$ as a common denominator.

The same formal manipulations as in the non-archimedean show that if we set

$$\mathcal{I}_j(\pi \times \pi') = \langle \Psi_j(s, W, W') \mid W \in \mathcal{W}(\pi, \psi), W' \in \mathcal{W}(\pi', \psi^{-1}) \rangle$$

then $\mathcal{I}_j(\pi \times \pi') = \mathcal{I}_{j+1}(\pi \times \pi')$ and hence $\mathcal{I}(\pi \times \pi') = \mathcal{I}_j(\pi \times \pi')$ is independent of $j$. Our theorem then becomes

$$\mathcal{I}(\pi \times \pi') \subset \mathcal{M}(\pi \times \pi').$$

There is also a local functional equation, but unlike the non-archimedean case, the "factor of proportionality" $\gamma(s, \pi \times \pi', \psi)$ is specified a priori.

**Theorem 8.2** *We have the local functional equation*

$$\widetilde{\Psi}(1-s, R(w_{n,m})\widetilde{W}, \widetilde{W}') = \omega_{\pi'}(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi(s, W, W')$$

*with $\gamma(s, \pi \times \pi', \psi) = \gamma(s, \tau \otimes \tau', \psi)$.*

The proofs of Theorems 8.1 and 8.2 are due to Jacquet and Shalika. Their strategy is roughly as follows:

(i) Very interestingly, they essentially show that Theorem 8.2 (the local functional equation) *implies* Theorem 8.1 (that the $L$-functions is essentially the correct denominator). This takes place in the space $\mathcal{M}(\pi \times \pi')$.

(ii) If $m = 1$, so $\pi'$ is a character, they reduce Theorem 8.2 to previous results of Godement and Jacquet on standard $L$-functions for $GL_n$, which in turn reduced to the cases of $GL_2 \times GL_1$ and $GL_1 \times GL_1$ in that context.

(iii) If $m = 2$ and $\pi'$ is a discrete series representation of $GL_2(\mathbb{R})$, then they embed $\pi' \subset Ind(\mu_1 \otimes \mu_2)$ and then reduce to (ii).

(iv) If $m > 2$ and $\pi' = Ind(\pi'_1 \otimes \cdots \otimes \pi'_r)$ with each $\pi'_i$ either a character or discrete series representation then they use a "multiplicativity" argument to again reduce to (ii) or (iii).

## 4 Is the $L$-factor correct?

We know that $\mathcal{I}(\pi \times \pi') \subset \mathcal{M}(\pi \times \pi')$, so that $L(s, \pi \times \pi') = L(s, \tau \otimes \tau')$ contains all poles of our local integrals. We are left with the following two related questions.

1. Is $L(s,\pi\times\pi')$ the minimal such factor?

2. Can we write
$$L(s,\pi\times\pi')=\sum_{i=1}^{r}\Psi(s,W_i,W_i')$$
as a finite linear combination of local integrals?

To investigate these questions, Jacquet and Shalika had to first enlarge the family of local integrals. If $\Lambda$ and $\Lambda'$ are continuous Whittaker functionals on $V_\pi$ and $V_{\pi'}$ then their tensor product $\hat{\Lambda}=\Lambda\otimes\Lambda'$ is a continuous linear functional on the algebraic tensor product $V_\pi\otimes V_{\pi'}$ which extends continuously to the topological tensor product $V_{\pi\hat{\otimes}\pi'}=V_\pi\hat{\otimes}V_{\pi'}$. (Note that this completion is in fact the Casselman-Wallach canonical completion of the algebraic tensor product. So to remain categorical, this is natural.) Then for $\xi\in V_\pi\hat{\otimes}V_{\pi'}$ we can define
$$W_\xi(g,h)=\hat{\Lambda}(\pi(g)\otimes\pi'(h)\xi),$$
so that $W_\xi\in\mathcal{W}(\pi\hat{\otimes}\pi')=\mathcal{W}(\pi,\psi)\hat{\otimes}\mathcal{W}(\pi',\psi^{-1})$, and then
$$\Psi(s,W)=\int_{N_m(k)\backslash GL_m(k)}W_\xi\left(\begin{pmatrix}h&\\&I_{n-m}\end{pmatrix},h\right)|\det(h)|^{s-\frac{n-m}{2}}\,dh.$$

Essentially the same arguments as before give Theorems 8.1 and 8.2 for these extended integrals. If we set
$$\mathcal{I}(\pi\hat{\otimes}\pi')=\langle\Psi(s,W)\mid W\in\mathcal{W}(\pi\hat{\otimes}\pi')\rangle$$
then again we have $\mathcal{I}(\pi\hat{\otimes}\pi')\subset\mathcal{M}(\pi\times\pi')$. But now they are able to show that in fact these spaces are equal.

**Theorem 8.3** $\mathcal{I}(\pi\hat{\otimes}\pi')=\mathcal{M}(\pi\times\pi')$.

So $L(s,\pi\times\pi')$ is the correct denominator for the extended family $\mathcal{I}(\pi\hat{\otimes}\pi')$. This partially answers our first question. We also obtain a partial answer to our second question.

**Corollary 8.3.1** *There exists $W\in\mathcal{I}(\pi\hat{\otimes}\pi')$ such that $\Psi(s,W)=L(s,\pi\times\pi')$.*

In order to investigate our questions for our original family, with Piatetski-Shapiro we showed the following continuity result.

**Proposition 8.3** *The functional*
$$W\mapsto e(s,W)=\frac{\Psi(s,W)}{L(s,\pi\times\pi')}$$
*is continuous on $\mathcal{W}(\pi\hat{\otimes}\pi')$, uniformly for $s$ in compact subsets.*

Since the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ is dense in $\mathcal{W}(\pi\hat{\otimes}\pi')$ and by the above corollary there exists $W\in\mathcal{W}(\pi\hat{\otimes}\pi')$ with $e(s,W)\equiv 1$ we then obtain the following result.

**Corollary 8.3.2** *For each $s_0 \in \mathbb{C}$ there exist Whittaker functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$ such that*

$$e(s_0, W, W') = \frac{\Psi(s, W, W')}{L(s, \pi \times \pi')} \neq 0.$$

*Moreover, one can take $W$ and $W'$ to be $K$-finite Whittaker functions.*

So $L(s, \pi \times \pi')$ is precisely the archimedean Euler factor of degree $nm$ determined by the poles of original family of integrals $\mathcal{I}(\pi \times \pi')$. This finally answers question 1.

As for question 2, the answer is more ambiguous. There are definitive results only in the cases of $m = n$ and $m = n - 1$. In the case where $\pi$ and $\pi'$ are both unramified, Stade has done the archimedean unramified calculation.

**Theorem 8.4** *If $n = m$ or $n = m - 1$ and both $\pi$ and $\pi'$ are unramified then*

$$L(s, \pi \times \pi') = \begin{cases} \Psi(s, W^\circ, W'^\circ, \Phi^\circ) & m = n \\ \Psi(s, W^\circ, W'^\circ) & m = n - 1 \end{cases}$$

*where $W^\circ$, $W'^\circ$, and $\Phi^\circ$ are all normalized and unramified.*

This has been generalized by Jacquet and Shalika, utilizing the last corollary.

**Theorem 8.5** *If $m = n$ or $m = n - 1$ then there are finite collections of $K$-finite Whittaker functions $W_i \in \mathcal{W}(\pi, \psi)$ and $W_i' \in \mathcal{W}(\pi', \psi^{-1})$ and possibly $\Phi_i \in \mathcal{S}(k^n)$ such that*

$$L(s, \pi \times \pi') = \begin{cases} \sum_i \Psi(s, W_i, W_i', \Phi_i) & m = n \\ \\ \sum_i \Psi(s, W_i, W_i') & m = n - 1 \end{cases}.$$

It is somewhat widely believed that this last result will not extend to $m \leq n-2$, even if one relaxes the $K$-finiteness condition.

## 5 References

[1] J.W. Cogdell and I.I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions.* Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002) (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.), Johns Hopkins University Press, Baltimore, to appear.

[2] R. Godement and H. Jacquet, Zeta Functions of Simple Algebras, Springer Lecture Notes in Mathematics, No.260, Springer-Verlag, Berlin, 1972.

[3] H. Jacquet and J. Shalika, *Rankin-Selberg convolutions: Archimedean theory.* Festschrift in Honor of I.I. Piatetski-Shapiro, Part I, Weizmann Science Press, Jerusalem, 1990, 125–207.

[4] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*. Representation Theory and Harmonic Analysis on Semisimple Lie Groups, AMS Mathematical Surveys and Monographs, No.31, 1989, 101–170.

[5] E. Stade, *Mellin transforms of $GL(n,\mathbb{R})$ Whittaker functions*. Amer. J. Math. **123** (2001), 121–161.

[6] E. Stade, *Archimedean $L$-factors on $GL(n)\times GL(n)$ and generalized Barnes integrals*. Israel J. Math. **127** (2002), 201–219.

[7] J. Tate, *Number theoretic background*. Proc. Sympos. Pure Math. **33**, part 2, (1979), 3–26.

LECTURE 9

# Global $L$-functions

We return now to the global setting. So once again $k$ is a number field and $\mathbb{A}$ its ring of adeles. Let $\Sigma$ denote the set of all places of $k$. Take $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$ a non-trivial continuous additive character.

Let $(\pi, V_\pi)$ be a unitary smooth cuspidal representation of $GL_n(\mathbb{A})$, which then decomposes as $\pi \simeq \otimes'\pi_v$. Similarly, $(\pi', V_{\pi'})$ will be a unitary smooth cuspidal representation of $GL_m(\mathbb{A})$ with $\pi' \simeq \otimes'\pi'_v$. We will mainly concentrate on the case of $m < n$. The case of $m = n$ can then be worked out as an exercise.

For each place $v \in \Sigma$ we have defined local $L$- and $\varepsilon$-factors

$$L(s, \pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

We then define the global $L$-function and $\varepsilon$-factor as Euler products.

**Definition 9.1** *The global $L$-function and $\varepsilon$-factors for $\pi$ and $\pi'$ are*

$$L(s, \pi \times \pi') = \prod_{v\in\Sigma} L(s, \pi_v \times \pi'_v)$$

*and*

$$\varepsilon(s, \pi \times \pi') = \prod_{v\in\Sigma} \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

Implicit in this definition is the convergence of the products in a half plane $Re(s) >> 0$ and the independence of the $\varepsilon$-factor from the choice of $\psi$. We will address this below. Then we will turn to showing these $L$-functions are *nice*. Our scheme will be to relate these Euler products to our global integrals and deduce the global properties of the $L$-functions from those of our global integrals.

Throughout, we will take $S \subset \Sigma$ to be a finite set of places, containing the archimedean places, such that for all $v \notin S$ we have that $\pi_v$, $\pi'_v$, and $\psi_v$ are all unramified and $\psi_v$ normalized. The set $S$ can vary, but it should always have these properties.

## 1 Convergence

Choose cusp forms $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ such that under the decomposition $V_\pi \simeq \otimes' V_{\pi_v}$ we have $\varphi \simeq \otimes\xi_v$ and similarly $\varphi' \simeq \otimes\xi'_v$. Choose $S$ as above such that for all $v \notin S$, $\xi_v = \xi_v^\circ$ is the $K_v$-fixed vector in $V_{\pi_v}$ and similarly $\xi'_v = \xi'^\circ_v$. Then we know from Lecture 5 that

$$I(s,\varphi,\varphi') = \Psi(s, W_\varphi, W'_{\varphi'}) = \prod_{v\in\Sigma} \Psi(s, W_{\xi_v}, W'_{\xi'_v})$$

and this converges absolutely for $Re(s) > 1$. By our unramified calculation of Lecture 7 we know that for $v \notin S$ we have

$$\Psi(s, W_{\xi_v^\circ}, W'_{\xi'^\circ_v}) = L(s, \pi_v \times \pi'_v).$$

Hence

$$I(s,\varphi,\varphi') = \left(\prod_{v\in S} \Psi(s, W_{\xi_v}, W'_{\xi'_v})\right) L^S(s, \pi\times\pi')$$

where $L^S(s,\pi\times\pi')$ is the partial $L$-function

$$L^S(s,\pi\times\pi') = \prod_{v\notin S} L(s, \pi_v \times \pi'_v).$$

Thus the Euler product for $L^S(s,\pi\times\pi')$ converges for $Re(s) >> 0$ and hence

- $L(s,\pi\times\pi')$ converges for $Re(s) >> 0$.

Thus our global $L$-function is well defined.

We could have also deduced the convergence of the infinite product from the Jacquet-Shalika bounds on the Satake parameters for unramified representations. As was pointed out, this would give convergence for $Re(s) > \frac{3}{2}$. In fact, with a bit more work than I have done here, Jacquet and Shalika show absolute convergence (and hence non-vanishing) for $Re(s) > 1$.

As for the $\varepsilon$-factor, again from our unramified calculation of Lecture 7 we know that $\varepsilon(s, \pi_v \times \pi'_v, \psi_v) \equiv 1$ for $v \notin S$. So

$$\varepsilon(s,\pi\times\pi') = \prod_{v\in S} \varepsilon(s, \pi_v\times\pi'_v, \psi_v)$$

is only a finite product. From the shape of the local $\varepsilon$-factors from Lectures 6 and 8, we know that it has the form

$$\varepsilon(s,\pi\times\pi') = WN^{\frac{1}{2}-s}$$

with $N$ a positive integer.

The independence of $\varepsilon(s,\pi\times\pi')$ from the choice of $\psi$ can be seen either by investigating how the local $\varepsilon$-factors vary as we vary $\psi$, which can be done through the local integrals, or as a consequence of the global functional equation below.

## 2 Meromorphic continuation

We continuation analyzing the relation between $L(s,\pi\times\pi')$ and our global integrals from above. We have

$$\begin{aligned} I(s,\varphi,\varphi') &= \left(\prod_{v\in S}\Psi(s,W_{\xi_v},W'_{\xi'_v})\right)L^S(s,\pi\times\pi') \\ &= \left(\prod_{v\in S}\frac{\Psi(s,W_{\xi_v},W'_{\xi'_v})}{L(s,\pi_v\times\pi'_v)}\right)L(s,\pi\times\pi') \\ &= \left(\prod_{v\in S}e(s,W_{\xi_v},W'_{\xi'_v})\right)L(s,\pi\times\pi') \end{aligned}$$

From our analysis of the global integrals, we know that $I(s,\varphi,\varphi')$ is entire (or $I(s,\varphi,\varphi',\Phi)$ is meromorphic if $m=n$). For each $v\in S$ we have seen that the local ratios $e(s,W_{\xi_v},W'_{\xi'_v})$ are entire. Since $S$ is a finite set, we can conclude

- $L(s,\pi\times\pi')$ extends to a meromorphic function of $s$.

## 3 Poles of $L$-functions

In our analysis of the local $L$-functions in Lectures 6 and 8 we have shown not only that the local ratios $e(s,W_{\xi_v},W'_{\xi'_v})$ are entire, but in fact that for every $s_0\in\mathbb{C}$ there is a choice of local Whittaker functions $W_v$ and $W'_v$ such that the ratio $e(s_0,W_v,W'_v)\neq 0$. So as we vary $W_v\in\mathcal{W}(\pi_v,\psi_v)$ and $W'_v\in\mathcal{W}(\pi'_v,\psi_v^{-1})$ we obtain that the poles of the global $L$-function $L(s,\pi\times\pi')$ are *precisely* those that occur for the families of global integrals

$$\{I(s,\varphi,\varphi')\}\quad\text{or}\quad\{I(s,\varphi,\varphi'\Phi)\}.$$

Hence

- If $m<n$ then $L(s,\pi\times\pi')$ is entire.

- If $m=n$ then $L(s,\pi\times\pi')$ has simple poles precisely at those $s=i\sigma$ and $s=1+i\sigma$ with $\sigma\in\mathbb{R}$ such that $\widetilde{\pi}\simeq\pi'\otimes|\det|^{i\sigma}$.

In particular,

- $L(s,\pi\times\widetilde{\pi})$ has simple poles at $s=0,1$

- $L(s,\pi\times\widetilde{\pi}')$ has a pole at $s=1$ iff $\pi\simeq\pi'$.

## 4 The global functional equation

We know that our global integrals satisfy a functional equation

$$I(s,\varphi,\varphi')=\widetilde{I}(1-s,\widetilde{\varphi},\widetilde{\varphi}').$$

Furthermore, as above, we have decompositions

$$I(s,\varphi,\varphi') = \left(\prod_{v\in S} e(s,W_{\xi_v},W'_{\xi'_v})\right) L(s,\pi\times\pi')$$

and

$$\widetilde{I}(1-s,\widetilde{\varphi},\widetilde{\varphi}') = \left(\prod_{v\in S} \widetilde{e}(1-s,R(w_{n,m})\widetilde{W}_{\xi_v},\widetilde{W}'_{\xi'_v})\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

By the local functional equations, for each $v\in S$ we have

$$\widetilde{e}(1-s,R(w_{n,m})\widetilde{W}_{\xi_v},\widetilde{W}'_{\xi'_v}) = \omega_{\pi'_v}(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)e(s,W_{\xi_v},W'_{\xi'_v}).$$

We now take the product of both sides over those $v\in S$. Note that since everything is unramified for $v\notin S$, we have

$$\prod_{v\in S}\omega_{\pi'_v}(-1)^{n-1} = \prod_{v\in\Sigma}\omega_{\pi'_v}(-1)^{n-1} = \omega_{\pi'}(-1) = 1$$

and as we have seen above

$$\prod_{v\in S}\varepsilon(s,\pi_v\times\pi'_v,\psi_v) = \prod_{v\in\Sigma}\varepsilon(s,\pi_v\times\pi'_v,\psi_v) = \varepsilon(s,\pi\times\pi').$$

Thus when we take this product we find

$$\prod_{v\in S}\widetilde{e}(1-s,R(w_{n,m})\widetilde{W}_{\xi_v},\widetilde{W}'_{\xi'_v}) = \varepsilon(s,\pi\times\pi')\prod_{v\in S} e(s,W_{\xi_v},W'_{\xi'_v})$$

so that

$$\widetilde{I}(1-s,\widetilde{\varphi},\widetilde{\varphi}') = \left(\prod_{v\in S} e(s,W_{\xi_v},W'_{\xi'_v})\right)\varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

If we combine this with our functional equation for the global integrals, we find our global functional equation

- $L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}')$.

Note that this equality implies that $\varepsilon(s,\pi\times\pi')$ is independent of $\psi$.

## 5 Boundedness in vertical strips

This is not as simple as it should be. Here is the paradigm. We include the case of $m=n$.

For $v\notin S$ we have

$$L(s,\pi_v\times\pi'_v) = \begin{cases}\Psi(s,W_v^\circ,W_v'^\circ,\Phi_v^\circ) & m=n\\ \Psi(s,W_v^\circ,W_v'^\circ) & m<n\end{cases}.$$

For non-archimedean $v \in S$ there are finite collections $\{W_{v,i}\}$, $\{W'_{v,i}\}$, and $\{\Phi_{v,i}\}$ if necessary such that

$$L(s,\pi_v \times \pi'_v) = \begin{cases} \sum_i \Psi(s, W_{v,i}, W'_{v,i}, \Phi_{v,i}) & m=n \\ \sum_i \Psi(s, W_{v,i}, W'_{v,i}) & m<n \end{cases}.$$

For archimedean places $v$ only if $m=n$ or $m=n-1$ do we know that there are finite families of either smooth or even $K_v$-finite Whittaker functions $\{W_{v,i}\}$ and $\{W'_{v,i}\}$, and if necessary Schwartz functions $\{\Phi_{v,i}\}$ such that

$$L(s,\pi_v \times \pi'_v) = \begin{cases} \sum_i \Psi(s, W_{v,i}, W'_{v,iy}, \Phi_{v,i}) & m=n \\ \sum_i \Psi(s, W_{v,i}, W'_{v,i}) & m=n-1 \end{cases}.$$

Hence if $m=n$ or $m=n-1$ the there are finite collections of cusp forms $\{\varphi_i\} \subset V_\pi$ and $\{\varphi'_i\} \subset V_{\pi'}$ and if necessary Schwartz functions $\{\Phi_i\} \subset \mathcal{S}(\mathbb{A}^n)$ such that

$$L(s,\pi \times \pi') = \begin{cases} \sum_i I(s, \varphi_i, \varphi'_i, \Phi_i) & m=n \\ \sum_i I(s, \varphi_i, \varphi'_i) & m=n-1. \end{cases}$$

Now boundedness in vertical strips of the $L$-function $L(s,\pi\times\pi')$ follows from that of the global integrals.

If $m < n-1$ then at the archimedean places we must pass to the topological product $V_{\pi_v} \hat{\otimes} V_{\pi'_v}$ in order to obtain $L(s,\pi_v \times \pi'_v)$, that is,

$$L(s,\pi_v \times \pi'_v) = \Psi(s,W) \quad \text{for} \quad W \in \mathcal{W}(\pi_v \hat{\otimes} \pi'_v, \psi_v).$$

To make our paradigm work we should re-develop the analysis of our global integrals for cusp forms $\varphi(g,h) \in V_\pi \hat{\otimes} V_{\pi'}$, which is a smooth cuspidal representation of the product $GL_n(\mathbb{A}) \times GL_m(\mathbb{A})$. Then we would obtain an equality

$$L(s,\pi\times\pi') = I(s,\varphi) \quad \text{with} \quad \varphi \in V_\pi \hat{\otimes} V_{\pi'}$$

and would then have boundedness in vertical strips as before. There seems to be no obstruction to carrying this out and we hope to soon write up the details. This then gives boundedness in vertical strips in general.

If this makes you nervous, Gelbart and Shahidi have proven boundedness in vertical strips for a wide class of $L$-functions, including ours, via the Langlands-Shahidi method of analyzing $L$-functions through the Fourier coefficients of Eisenstein series.

So, no matter how you cut it,

- $L(s,\pi\times\pi')$ is bounded in vertical strips of finite width.

## 6 Summary

If we combine these results, we obtain a statement of the basic analytic properties of out $L$-functions.

**Theorem 9.1** *If $\pi$ is a unitary cuspidal representation of $GL_n(\mathbb{A})$ and $\pi'$ is a unitary cuspidal representation of $GL_m(\mathbb{A})$ with $m < n$ then $L(s, \pi \times \pi')$ is* **nice**, *i.e.,*

(i) *$L(s, \pi \times \pi')$ converges for $Re(s) >> 0$ and extends to an entire function of $s$;*
(ii) *this extension is bounded in vertical strips of finite width;*
(iii) *it satisfies the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi')L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

In the case of $m = n$ we have a similar result.

**Theorem 9.2** *If $\pi$ and $\pi'$ are two unitary cuspidal representation of $GL_n(\mathbb{A})$ then*

(i) *$L(s, \pi \times \pi')$ converges for $Re(s) >> 0$ and extends to a meromorphic function of $s$ with simple poles at those $s = i\sigma$ and $s = 1 + i\sigma$ such that $\widetilde{\pi} \simeq \pi' \otimes |\det|^{i\sigma}$; if there are no such $i\sigma$ then $L(s, \pi \times \pi')$ is entire;*
(ii) *this extension is bounded in vertical strips of finite width (away from its poles);*
(iii) *it satisfies the functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi')L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

## 7 Strong Multiplicity One revisited

We will now present the analytic proof of the Strong Multiplicity One Theorem due to Jacquet and Shalika. It is based on the analytic properties of $L$-functions. First, recall the statement.

**Theorem 9.3 (Strong Multiplicity One)** *Let $(\pi_1, V_{\pi_1})$ and $(\pi_2, V_{\pi_2})$ be two cuspidal representations of $GL_n(\mathbb{A})$. Decompose them as $\pi_1 \simeq \otimes'\pi_{1,v}$ and $\pi_1 \simeq \otimes'\pi_{2,v}$. Suppose that there is a finite set of places $S$ such that $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$. Then $(\pi_1, V_{\pi_1}) = (\pi_2, V_{\pi_2})$.*

Without loss of generality we may assume $\pi_1$ and $\pi_2$ are unitary. We know from Section 3 of this Lecture or Theorem 9.2 that $L(s, \pi_1 \times \widetilde{\pi}_2)$ has a pole at $s = 1$ iff $\pi_1 = \pi_2$.

Let us write

$$L(s, \pi_1 \times \widetilde{\pi}_2) = \left( \prod_{v \in S} L(s, \pi_{1,v} \times \widetilde{\pi}_{2,v}) \right) L^S(s, \pi_1 \times \widetilde{\pi}_2).$$

The local $L$-functions for $v \in S$ are all of the form

$$L(s, \pi_{1,v} \times \widetilde{\pi}_{2,v}) = \begin{cases} P_v(q_v^{-s})^{-1} & v < \infty \\ \prod \Gamma_v(s + *) & v | \infty \end{cases}.$$

So in either case they are never zero. We also know from Lectures 6 and 8 that the local integrals are absolutely convergent for $Re(s) \geq 1$. So the local $L$-factors can have no poles in this region either. Hence the finite product

$$\prod_{v \in S} L(s, \pi_{1,v} \times \widetilde{\pi}_{2,v})$$

has no zeros or poles in $Re(s) \geq 1$. Thus $L(s, \pi_1 \times \widetilde{\pi}_2)$ has a pole at $s = 1$ iff $L^S(s, \pi_1 \times \widetilde{\pi}_2)$ does.

Since $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$ we have

$$L^S(s, \pi_1 \times \widetilde{\pi}_2) = L^S(s, \pi_1 \times \widetilde{\pi}_1).$$

By the same argument as above, $L^S(s, \pi_1 \times \widetilde{\pi}_1)$ will have a pole at $s = 1$ since the full $L$-function $L(s, \pi_1 \times \widetilde{\pi}_1)$ does, again by Theorem 9.2.

Thus $L(s, \pi_1 \times \widetilde{\pi}_2)$ does indeed have a pole at $s = 1$ and so $\pi_1 \simeq \pi_2$. Then Multiplicity One for $GL_n$ gives that in fact $(\pi_1, V_{\pi_1}) = (\pi_2, V_{\pi_2})$.

## 8 Generalized Strong Multiplicity One

Jacquet and Shalika were able to push this technique further to obtain a version of the Strong Multiplicity One Theorem for non-cuspidal representations. To state it, we must first recall a theorem of Langlands.

If $\pi$ is any irreducible automorphic representation of $GL_n(\mathbb{A})$ then there exists a partition $n = n_1 + \cdots + n_r$ of $n$ and cuspidal representations $\tau_i$ of $GL_{n_i}(\mathbb{A})$ such that $\pi$ is a constituent of the induced representation

$$\Xi = \mathrm{Ind}_{Q(\mathbb{A})}^{GL_n(\mathbb{A})}(\tau_1 \otimes \cdots \otimes \tau_r).$$

Langlands worked in the context of $K$-finite automorphic representations, but the result is valid for smooth automorphic representations as well. It is a consequence of the theory of Eisenstein series. Similarly, if $\pi'$ is another automorphic representation of $GL_n(\mathbb{A})$ then $\pi'$ will be a constituent of a similarly induced representation

$$\Xi' = \mathrm{Ind}_{Q'(\mathbb{A})}^{GL_n(\mathbb{A})}(\tau'_1 \otimes \cdots \otimes \tau'_{r'})$$

associated to a second partition $n = n'_1 + \cdots + n'_{r'}$.

**Theorem 9.4 (Generalized Strong Multiplicity One)** *Let $\pi$ and $\pi'$ be two automorphic representations of $GL_n(\mathbb{A})$ as above. Suppose that there is a finite set of places $S$ such that $\pi_v \simeq \pi'_v$ for all $v \notin S$. Then $r = r'$ and there is a permutation $\sigma$ of $\{1, \ldots, r\}$ such that $n_i = n'_{\sigma(i)}$ and $\tau_i \simeq \tau_{\sigma(i)}$.*

Thus the knowledge of the local components of $\pi$ at almost all places completely determines the "cuspidal support" of $\pi$. In particular, the "cuspidal support" of $\pi$ is well defined. As a consequence of this result Jacquet and Shalika showed the existence of the category of *isobaric representations* for $GL_n(\mathbb{A})$.

## 9 References

[1] J.W. Cogdell and I.I. Piatetski-Shapiro, *Remarks on Rankin-Selberg convolutions*. Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002) (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.), Johns Hopkins University Press, Baltimore, to appear.

[2] S. Gelbart and F. Shahidi, *Boundedness of automorphic L-functions in vertical strips*. J. Amer. Math. Soc. **14** (2001), 79–107.

[3] H. Jacquet and J. Shalika, *On Euler products and the classification of automorphic representations*, I & II. Amer. J. Math. **103** (1981), 499–588 & 777–815.

[4] R.P. Langlands, *On the notion of an automorphic representation*. Proc. Sympos. Pure Math. **33**, part 1, (1979), 203–207.

LECTURE 10

# Converse Theorems

Once again, we take $k$ to be a global field, which we have taken to be a number field – but that is irrelevant. Then $\mathbb{A}$ is its ring of adeles and we take $\psi : k\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ a non-trivial continuous additive character.

## 1 Converse Theorems for $GL_n$

For automorphic representations of $GL_n(\mathbb{A})$ the "Converse Theorem", i.e., the converse to the theory of global $L$-functions developed in the last lecture, has a slightly different flavor from the classical ones. It addresses the following question.

Let us take $\pi \simeq \otimes' \pi_v$ to be an arbitrary (i.e., not necessarily automorphic) irreducible admissible smooth representation of $GL_n(\mathbb{A})$.

**Question:** *How can we tell if the local pieces $\pi_v$ of $\pi$ are "coherent enough" that we have an embedding*

$$V_\pi \hookrightarrow \mathcal{A}_0^\infty(GL_n(k)\backslash GL_n(\mathbb{A}))?$$

Our Converse Theorems gives an analytic answer to this question in terms of $L$-functions. From our local theory of $L$-functions, to each local component $\pi_v$ we have attached a local $L$-factor $L(s, \pi_v)$ and a local $\varepsilon$-factor $\varepsilon(s, \pi_v, \psi_v)$. Thus we can (at least formally) form the Euler products

$$L(s, \pi) = \prod_v L(s, \pi_v) \quad \text{and} \quad \varepsilon(s, \pi, \psi) = \prod_v \varepsilon(s, \pi_v, \psi_v).$$

Then $L(s, \pi)$ is a formal Euler product of degree $n$ and our question can be rephrased as:

**Question:** *Is the Dirichlet series defined by this formal Euler product modular?*

This is closer to the classical Converse Theorems.

To begin we must make some mild coherence and modularity assumptions, namely that

(i) the Euler product for $L(s, \pi)$ is absolutely convergent in some right half plane $Re(s) >> 0$;

(ii) the central character $\omega_\pi$ of $\pi$ is an automorphic form on $GL_1(\mathbb{A})$, that is, an idele class character of $k^\times \backslash \mathbb{A}^\times$.

Note that one can show that (ii) implies that $\varepsilon(s,\pi,\psi) = \varepsilon(s,\pi)$ is independent of $\psi$.

Under these conditions, if $\pi' \simeq \otimes' \pi'_v$ is any *cuspidal* (hence automorphic) representation of $GL_m(\mathbb{A})$ with $1 \le m \le n-1$ then we can similarly form

$$L(s,\pi\times\pi') = \prod_v L(s,\pi_v\times\pi'_v) \text{ and } \varepsilon(s,\pi\times\pi',\psi) = \prod_v \varepsilon(s,\pi_v\times\pi'_v,\psi_v)$$

and still have that

• both the Euler products for $L(s,\pi\times\pi')$ and $L(s,\widetilde{\pi}\times\widetilde{\pi}')$ converge absolutely for $Re(s) >> 0$; and that

• $\varepsilon(s,\pi\times\pi',\psi) = \varepsilon(s,\pi\times\pi')$ is independent of $\psi$.

We say that $L(s,\pi\times\pi')$ is *nice* if it behaves as it would if $\pi$ were cuspidal, i.e.,

(i) $L(s,\pi\times\pi')$ and $L(s,\widetilde{\pi}\times\widetilde{\pi}')$ extend to entire functions of $s$;
(ii) these extensions are bounded in vertical strips;
(iii) they satisfy the functional equation

$$L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Our Converse Theorems, like Weil's, will involve these twists. To that end, for any $m$ with $1 \le m \le n-1$ let us set

$$\mathcal{T}(m) = \coprod_{d=1}^{m} \{\pi' \text{ cuspidal}, \ V_{\pi'} \subset \mathcal{A}_0^\infty(GL_d(k)\backslash GL_d(\mathbb{A}))\}$$

and for any finite set $S$ of finite places we set

$$\mathcal{T}^S(m) = \{\pi' \in \mathcal{T}(m) \mid \pi'_v \text{ is unramified for all } \ v \in S\}.$$

The basic Converse Theorem, the analogue of those of Hecke and Weil, is the following result.

**Theorem 10.1** *Let $\pi$ be as above, an irreducible admissible smooth representation of $GL_n(\mathbb{A})$ having automorphic central character and such that $L(s,\pi)$ converges for $Re(s) >> 0$. Let $S$ be a finite set of finite places. Suppose that $L(s,\pi\times\pi')$ is nice for all $\pi' \in \mathcal{T}^S(n-1)$. Then*

(i) *if $S = \emptyset$ then $\pi$ is automorphic and cuspidal;*
(ii) *if $S \neq \emptyset$ then $\pi$ is quasi-automorphic in the sense that there exists an automorphic representation $\pi_1$ such that $\pi_{1,v} \simeq \pi_v$ for all $v \notin S$.*

A stronger result, but somewhat harder to prove, is the following.

**Theorem 10.2** *Let $n \ge 3$ and let $\pi$ be as above, an irreducible admissible smooth representation of $GL_n(\mathbb{A})$ having automorphic central character and such that $L(s,\pi)$ converges for $Re(s) >> 0$. Let $S$ be a finite set of finite places. Suppose that $L(s,\pi\times\pi')$ is nice for all $\pi' \in \mathcal{T}^S(n-2)$. Then*

(i) *if* $S = \emptyset$ *then* $\pi$ *is automorphic and cuspidal;*
(ii) *if* $S \neq \emptyset$ *then* $\pi$ *is quasi-automorphic in the sense that there exists an automorphic representation* $\pi_1$ *such that* $\pi_{1,v} \simeq \pi_v$ *for all* $v \notin S$.

I would like to sketch the proof of Theorem 10.1. For simplicity let us assume that in addition $(\pi, V_\pi)$ is generic. (We have discussed how to get around this in practice.)

## 2 Inverting the integral representation

Take $\pi \simeq \otimes' \pi_v$ as in the statement of Theorem 10.1. For this section we assume that $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-1)$ and see what this leads to when we invert our integral representation.

We first need to produce some functions on $GL_n(\mathbb{A})$. Since we have assumed $(\pi, V_\pi)$ is generic we can do this via the Whittaker model. If $\xi \in V_\pi$ is such that under the decomposition $V_\pi \simeq \otimes' V_{\pi_v}$ we have $\xi \simeq \otimes \xi_v$ then to each $\xi_v$ we have associated a Whittaker function $W_{\xi_v} \in \mathcal{W}(\pi_v, \psi_v)$ and hence

$$W_\xi(g) = \prod_v W_{\xi_v}(g_v) \in \mathcal{W}(\pi, \psi)$$

is a smooth function on $N_n(k)\backslash GL_n(\mathbb{A})$.

We could try to embed $V_\pi$ into $\mathcal{A}_0^\infty$ by averaging $W_\xi$ over $GL_n(k)$, but this would not converge. However the standard estimates on Whittaker functions do let us average over the rational points of the mirabolic

$$P = \mathrm{Stab}_{GL_n}((0, \ldots, 0, 1)) = \left\{ p = \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \right\}.$$

So we form

$$U_\xi(g) = \sum_{p \in N(k)\backslash P(k)} W_\xi(pg) = \sum_{\gamma \in N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left( \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

**Proposition 10.1** $U_\xi(g)$ *converges absolutely and uniformly for* $g$ *in compact subsets, is left invariant under* $P(k)$*, and its restriction to* $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$ *is rapidly deceasing (modulo the center).*

Note that if $\xi = \varphi$ was indeed a cusp form, this would be its Fourier expansion.

We can make a similar construction for any mirabolic subgroup and to utilize our functional equation we will need to do this. To this end, let $Q$ be the opposite mirabolic

$$Q = \mathrm{Stab}_{GL_n}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \left\{ q = \begin{pmatrix} * & \cdots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & 0 \\ * & \cdots & * & 1 \end{pmatrix} \right\}$$

and let $\alpha = \begin{pmatrix} & 1 \\ I_{n-1} & \end{pmatrix}$, a permutation matrix. Then set

$$V_\xi(g) = \sum_{q \in N'(k) \backslash Q(k)} W_\xi(\alpha q g) \quad \text{where} \quad N' = \alpha^{-1} N \alpha.$$

This is again absolutely convergent, uniformly on compact subsets, left invariant under $Q(k)$, and rapidly decreasing upon restriction to $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$ (mod center, of course).

Since $P(k)$ and $Q(k)$ together generate $GL_n(k)$, it suffices to show that $U_\xi(g) = V_\xi(g)$, for then

$$\xi \mapsto U_\xi(g) \quad \text{embeds} \quad V_\pi \hookrightarrow \mathcal{A}_0^\infty.$$

We will obtain this equality from the analytic properties of $L(s, \pi \times \pi')$.

Let $V_{\pi'} \subset \mathcal{A}^\infty(GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A}))$ be *any* irreducible subspace of the space of smooth automorphic forms on $GL_{n-1}$. For example $\pi'$ could be cuspidal. We call such $\pi'$ *proper* automorphic representations. They consist of spaces of automorphic forms.

If $\varphi' \in V_{\pi'}$ then we can form

$$I(s, U_\xi, \varphi') = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})} U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh$$

and show that this converges for $Re(s) >> 0$. This will then factor in the usual manner into

$$I(s, U_\xi, \varphi') = \prod_v \Psi(s, W_{\xi_v}, W'_{\varphi'_v}).$$

Suppose first that $\pi'$ is cuspidal. Let $T$ be the finite set of places, containing the archimedean ones, such that $\pi_v$, $\pi'_v$, and $\psi_v$ are all unramified for $v \notin T$. Then as before

$$\begin{aligned} I(s, U_\xi, \varphi') &= \left( \prod_{v \in T} \Psi(s, W_{\xi_v}, W'_{\varphi'_v}) \right) L^T(s, \pi \times \pi') \\ &= \left( \prod_{v \in T} e(s, W_{\xi_v}, W'_{\varphi'_v}) \right) L(s, \pi \times \pi'). \end{aligned}$$

From our local theory we know that the factors $e_v(s)$ are entire and by assumption $L(s, \pi \times \pi')$ is entire. Hence $I(s, U_\xi, \varphi')$ extends to an entire function of $s$.

If $\pi'$ is not cuspidal, then by Langlands' Theorem given last lecture we know that $\pi'$ is a constituent, and in fact a sub-representation, of an induced representation $\Xi = Ind(\tau_1 \otimes \cdots \otimes \tau_r)$ with each $\tau_i$ a cuspidal representation of some $GL_{n_i}$ with $n_i < n-1$. So each $L(s, \pi \times \tau_i)$ is nice and we can use these to reach the same conclusion, namely that $I(s, U_\xi, \varphi')$ is entire for any $\varphi' \in V_{\pi'}$ for any proper $\pi'$.

Similarly if we form

$$I(s, V_\xi, \varphi') = \int_{GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})} V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det h|^{s-\frac{1}{2}} \, dh$$

then this will converge for $Re(s) << 0$, unfolds to

$$I(s,V_\xi,\varphi') = \left(\prod_{v\in T} \widetilde{e}(1-s,R(w_{n,n-1}\widetilde{W}_{\xi_v},\widetilde{W}'_{\varphi'_v})\right) L(1-s,\widetilde{\pi}\times\widetilde{\pi}'),$$

and continues to an entire function of $s$.

If we now apply the assumed global functional equation for either $L(s,\pi\times\pi')$ or the $L(s,\pi\times\tau_i)$ and the local functional equations for $v\in T$ we may conclude that

$$I(s,U_\xi,\varphi') = I(s,V_\xi,\varphi') \quad \text{for all} \quad \varphi'\in V_{\pi'}\subset\mathcal{A}^\infty(GL_{n-1}).$$

Then an application of the Phragmen–Lindelöf principle implies that these functions are bounded in vertical strips of finite width.

Thus we have

$$\int U_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix}\varphi'(h)|\det h|^{s-\frac{1}{2}}\,dh = \int V_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix}\varphi'(h)|\det h|^{s-\frac{1}{2}}\,dh$$

in the sense of analytic continuation; the integration is over $GL_{n-1}(k)\backslash GL_{n-1}(\mathbb{A})$. Using the boundedness in vertical strips, we can apply Jacquet-Langlands' version of Mellin inversion to obtain

$$\int U_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix}\varphi'(h)\,dh = \int V_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix}\varphi'(h)\,dh$$

now with the integration over $SL_{n-1}(k)\backslash SL_{n-1}(\mathbb{A})$. Then using the weak form of Langlands spectral theory for $SL_{n-1}(k)\backslash SL_{n-1}(\mathbb{A})$ we can conclude that the functions $\varphi'$ are "complete" and that

$$U_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix} = V_\xi\begin{pmatrix} h & \\ & 1\end{pmatrix} \quad \text{for} \quad h\in SL_{n-1}(\mathbb{A}),\ \xi\in V_\pi$$

and in particular

$$U_\xi(I_n) = V_\xi(I_n) \quad \text{for all} \quad \xi\in V_\pi.$$

## 3 Proof of Theorem 10.1 (i)

To conclude the proof of part (i) of Theorem 10.1 we just note that since we have

$$U_\xi(I_n) = V_\xi(I_n) \quad \text{for all} \quad \xi\in V_\pi$$

then for any $g\in GL_n(\mathbb{A})$ we have

$$U_\xi(g) = U_{\pi(g)\xi}(I_n) = V_{\pi(g)\xi}(I_n) = V_\xi(g).$$

So the map $\xi\mapsto U_\xi$ maps $V_\pi\to\mathcal{A}^\infty$. Since $U_\xi$ is given by a Fourier expansion

$$U_\xi(g) = \sum_{N_{n-1}(k)\backslash GL_{n-1}(k)} W_\xi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)$$

we can compute a non-zero Fourier coefficient to conclude that $U_\xi\not\equiv 0$, and hence the map is injective, and explicitly show that all unipotent periods are zero, and hence that $U_\xi$ is in fact cuspidal. Thus we have $V_\pi\hookrightarrow\mathcal{A}_0^\infty$ as desired.

## 4 Proof of Theorem 10.1 (ii)

In part (ii) of Theorem 10.1 we can no longer directly apply the inversion of the integral representation since we can no longer control $I(s, U_\xi, \varphi')$ for $\varphi' \in V_{\pi'}$ for every proper automorphic representation $\pi'$, rather only for those which are unramified for $v \in S$. Our first idea to get around this is to place local conditions on our vector $\xi$ at $v \in S$ to ensure that this is all you need. For $v \in S$, let $\xi_v^\circ \in V_{\pi_v}$ be the "new vector", that is, the essentially unique vector fixed by $K_1(\mathfrak{p}^{f(\pi_v)})$ where $f(\pi_v)$ is the conductor of $\pi_v$ as in Lecture 6. Note that for any $t$ we have

$$K_1(\mathfrak{p}_v^t) = \left\{ k_v \in GL_n(\mathcal{O}_v) \middle| k_v \equiv \begin{pmatrix} * & \cdots & * & * \\ \vdots & & \vdots & \vdots \\ * & \cdots & * & * \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{\mathfrak{p}_v^t} \right\}$$
$$\supset \left\{ \begin{pmatrix} k_v' & \\ & 1 \end{pmatrix} \,\middle|\, k_v' \in GL_{n-1}(\mathcal{O}_v) \right\}.$$

Set $\xi_S^\circ = \otimes_{v \in S} \xi_v^\circ \in V_{\pi_S}$. This is then fixed by

$$K_1(\mathfrak{n}) = \prod_{v \in S} K_1\left(\mathfrak{p}_v^{(f(\pi_v)}\right) \supset GL_{n-1}(\mathcal{O}_S).$$

For any $\xi^S \in V_{\pi^S} \simeq \otimes'_{v \notin S} V_{\pi_v}$ we can form $\xi = \xi_S^\circ \otimes \xi^S$ and for such restricted $\xi \in V_\pi$ we form $U_\xi$ and $V_\xi$ as before. Note that when we restrict these functions to $GL_{n-1}(\mathbb{A})$ we see that $U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ and $V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ are now unramified for $v \in S$. So when we form $I(s, U_\xi, \varphi')$ and $I(s, V_\xi, \varphi')$ for $\varphi' \in V_{\pi'}$ a proper automorphic representation of $GL_{n-1}(\mathbb{A})$ we find that either

- $I(s, U_\xi, \varphi') = 0 = I(s, V_\xi, \varphi')$ if $\pi'$ is not unramified for $v \in S$, or
- $I(s, U_\xi, \varphi') = I(s, V_\xi, \varphi')$ as before if $\pi'$ is unramified for $v \in S$.

Thus, arguing as before, we may now conclude that we have

$$U_\xi(g) = V_\xi(g) \quad \text{for all} \quad g \in K_1(\mathfrak{n}) G^S$$

where $G^S = \prod_{v \notin S} GL_n(k_v)$.

We now use the weak approximation theorem to get back to $GL_n(\mathbb{A})$. Note that

- $U_\xi(g)$ is left invariant under $P(\mathfrak{n}) = P(k) \cap K_1(\mathfrak{n}) G^S$
- $V_\xi(g)$ is left invariant under $Q(\mathfrak{n}) = Q(k) \cap K_1(\mathfrak{n}) G^S$
- $P(\mathfrak{n})$ and $Q(\mathfrak{n})$ generate $\Gamma(\mathfrak{n}) = GL_n(k) \cap K_1(\mathfrak{n}) G^S$.

Thus as we let $\xi^S$ vary in $V_{\pi^S}$ we obtain that

$$\xi^S \mapsto \xi = \xi_S^\circ \otimes \xi^S \mapsto U_\xi(g) \quad \text{embeds} \quad V_{\pi^S} \hookrightarrow \mathcal{A}^\infty(\Gamma(\mathfrak{n}) \backslash K_1(\mathfrak{n}) G^S).$$

Now weak approximation gives that $GL_n(\mathbb{A}) = GL_n(k)K_1(\mathfrak{n})G^S$ so that

$$\mathcal{A}^\infty(\Gamma(\mathfrak{n})\backslash K_1(\mathfrak{n})G^S) = \mathcal{A}^\infty(GL_n(k)\backslash GL_n(\mathbb{A})).$$

Then $\pi^S$ determines a sub-representation of the space of automorphic forms on $GL_n(\mathbb{A})$ and for our $\pi_1$ we may take any irreducible constituent of this. Fortunately we still retain that $\pi_{1,v} \simeq \pi_v$ for $v \notin S$. This is the $\pi_1$ claimed in the Theorem.

## 5 Theorem 2 and beyond

What can we expect if we only assume that $L(s, \pi \times \pi')$ is nice for all $\pi'$ in say $\mathcal{T}(m)$ or $\mathcal{T}^S(m)$?

If $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(m)$ then we can proceed as above to invert the integral representation for $GL_n \times GL_m$. We form $U_\xi$ as before, but must use a $V_\xi$ which is adapted to this functional equation. To this end, we let $Q_m$ be the mirabolic subgroup defined as the stabilizer in $GL_n$ of the vector ${}^te_{m+1}$, that is, the column vector all of whose entries are 0 except for the $(m+1)^{st}$ which is 1. We take for our permutation matrix the matrix

$$\alpha_m = \begin{pmatrix} & & 1 \\ I_m & & \\ & I_{n-m-1} & \end{pmatrix}.$$

Then we set

$$V_\xi(g) = \sum_{q \in N' \backslash Q_m} W_\xi(\alpha_m q g) \quad \text{where now} \quad N' = \alpha_m^{-1} N \alpha_m.$$

Then if we invert the $GL_n \times GL_m$ integral representation as before we obtain

$$\mathbb{P}_m U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \mathbb{P}_m V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \quad \text{for} \quad h \in SL_m(\mathbb{A}),\ \xi \in V_\pi$$

or

$$\mathbb{P}_m U_\xi(I_{m+1}) = \mathbb{P}_m V_\xi(I_{m+1}) \quad \text{for} \quad \xi \in V_\pi.$$

If we now set $m = n-2$ as in Theorem 10.2 (i), then this last equation becomes

$$\int U_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \psi^{-1}(u_{n-1})\, du = \int V_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \psi^{-1}(u_{n-1})\, du$$

where the integral is over $k^{n-1}\backslash\mathbb{A}^{n-1}$. We can rewrite this as

$$\int_{k^{n-1}\backslash\mathbb{A}^{n-1}} F_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix} \psi^{-1}(u_{n-1})\, du = 0$$

with $F_\xi(g) = U_\xi(g) - V_\xi(g)$. Then our desired equality $U_\xi(I_n) = V_\xi(I_n)$ becomes $F_\xi(I_n) = 0$.

If we set $f_\xi(u) = F_\xi \begin{pmatrix} I_{n-1} & u \\ & 1 \end{pmatrix}$ then $f_\xi$ is a periodic function on $k^{n-1}\backslash\mathbb{A}^{n-1}$ and we wish to know that $f_\xi(0) = F_\xi(I_n) = 0$ for all $\xi$. Instead, what we have from the above is that a certain Fourier coefficient of $f_\xi$ vanishes. But we also know that $F_\xi(g)$ is left invariant under $P(k) \cap Q_{n-2}(k)$. Using this allows us to show that many more Fourier coefficients of $f_\xi$ vanish. Eventually this analysis leads to the fact that $f_\xi({}^t(0,\ldots,0,u_{n-1}))$ is constant, and moreover this constant is

$$f_\xi({}^t(0,\ldots,0,0)) = f_\xi(0) = F_\xi(I_n).$$

To conclude, we now take any finite place $v_1$ and working in the local Kirillov model at the place $v_1$ we are able to place a local condition on the component $\xi_{v_1}$ which guarantees that this common value is 0. Hence we may conclude $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\pi$ with $\xi_{v_1}$ fixed.

Now we more or less proceed as in the proof of Theorem 10.1 (ii). We use weak approximation to obtain an automorphic representation $\pi_1$ which agrees with $\pi$ except possibly at $v_1$. Then we repeat the argument with a second fixed place $v_2$ to get an automorphic representation $\pi_2$ which agrees with $\pi$ except possibly at $v_2$. Then we use the Generalized Strong Multiplicity One Theorem and what we know about the entirety of the twisted $L$-functions to conclude that $\pi_1 = \pi_2 = \pi$ and $\pi$ is cuspidal. This gives Theorem 10.2 (i).

Theorem 10.2 (ii) is then obtained by combining this method with the proof of Theorem 10.1 (ii). Once can take the place $v_1$ used above to lie in $S$, and then once you have used the weak approximation theorem, you are done.

Note that if $m < n-2$ then the unipotent integration in $\mathbb{P}_m$ is now *non-abelian* and our abelian Fourier expansion method (thus far) breaks down.

## 6 A useful variant

For applications, these theorems are used in the following useful variant form.

**Useful Variant:** *Let $\pi$ be as in Theorems 10.1 and 10.2. Let $\mathcal{T}$ be the twisting set of either theorem. Let $\eta : k^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be an fixed idele class character. Suppose that $L(s,\pi\times\pi')$ is nice for every $\pi' \in \mathcal{T}\otimes\eta$. Then we have the same conclusions for $\pi$ as in those theorems.*

To see this, note that $L(s,\pi\times\pi')$ is nice for every $\pi' \in \mathcal{T}\otimes\eta$ iff $L(s,(\pi\otimes\eta)\times\pi'_0)$ is nice for every $\pi'_0 \in \mathcal{T}$. Hence $\pi\otimes\eta$ satisfies the conclusions of either Theorem 10.1 or 10.2. But since $\eta$ is automorphic, $\pi$ will as well.

In practice, the set of places $S$ often is taken to be the places where $\pi$ is ramified and $\eta$ is taken to be highly ramified at those place so that stability of $\gamma$ can be used.

## 7 Conjectures

The most widely held belief is the conjecture of Jacquet:

**Conjecture 10.1** *Let $\pi$ be as in Theorem 10.1 or Theorem10.2. Suppose that $L(s,\pi\times\pi')$ is nice for all $\pi' \in \mathcal{T}^S\left(\left[\frac{n}{2}\right]\right)$. Then we have the same conclusions as in those theorems. In particular, if $S$ is empty then $\pi$ should be cuspidal.*

The most interesting and useful conjecture is due to Piatetski-Shapiro:

**Conjecture 10.2** *Let $\pi$ be as in Theorem 10.1 or Theorem10.2. Suppose that $L(s,\pi\otimes\chi)$ is nice for all $\chi \in \mathcal{T}(1)$, that is, for all idele class characters. Then there exists an automorphic representation $\pi_1$ such that $\pi_{1,v} \simeq \pi_v$ at all places where they are both unramified and*

$$L(s,\pi \otimes \chi) = L(s,\pi_1 \otimes \chi) \quad \textit{for all } \chi \in \mathcal{T}(1).$$

*In particular $\pi$ and $\pi_1$ have the same L-function, so that the formal Euler product defining $L(s,\pi)$ is in fact modular.*

One can easily formulate a version of this conjecture for $\mathcal{T}^S(1)$.

## 8 References

[1] J.W. Cogdell, H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to $GL_N$*. Publ, Math. IHES **93** (2001), 5–30.

[2] J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for $GL_n$*. Publ. Math. IHES **79** (1994), 157–214.

[3] J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for $GL_n$*, II. J. reine angew. Math. **507** (1999), 165–188.

[4] H. Jacquet and R.P. Langlands, Automorphic Forms on $GL(2)$, Springer Lecture Notes in Mathematics No.114, Springer Verlag, Berlin, 1970.

[5] H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on* $GL(3)$, I & II. Ann. Math. **109** (1979), 169–258.

LECTURE 11

# Functoriality

In this lecture we would like to give a brief introduction to functoriality and how one uses the Converse Theorem to attack the problem of functoriality from reductive groups $G$ to $GL_n$.

## 1 The Weil-Deligne group

Local functoriality is mediated by admissible maps of the Weil-Deligne group into the Langlands dual group or the $L$-group.

Let $k$ be a local field. We have defined the Weil group $W_k$ when $k = \mathbb{R}$ or $\mathbb{C}$. So we will let $k$ denote a non-achimedean local field (of characteristic 0 as usual). Let $\mathcal{O}$ be the ring of integers of $k$, $\mathfrak{p}$ its unique prime ideal, and $\kappa = \mathcal{O}/\mathfrak{p}$ its residue field. Let $p$ be the characteristic of $\kappa$ and $q = |\kappa|$. Let $\overline{k}$ denote the algebraic closure of $k$.

Reduction mod $\mathfrak{p}$ gives a surjective map $Gal(\overline{k}/k)$ to $Gal(\overline{\kappa}/\kappa)$ and we let $I$ denote its kernel:

$$1 \longrightarrow I \longrightarrow Gal(\overline{k}/k) \longrightarrow Gal(\overline{\kappa}/\kappa) \longrightarrow 1.$$

$I$ is called the inertia group. We know that $Gal(\overline{\kappa}/\kappa)$ is cyclic and generated by the Frobenius automorphism. Let $\Phi \in Gal(\overline{k}/k)$ be any inverse image of the inverse of Frobenius (a so-called geometric Frobenius).

Since the Galois group $Gal(\overline{k}/k)$ is a pro-finite compact group, to obtain a sufficiently rich class of representations to hopefully classify admissible representation of $GL_n$, we need to relax this topology. So we let $W_k$ denote the subgroup of $Gal(\overline{k}/k)$ generated by $I$ and $\Phi$, but we topologize $W_k$ so that $I$ retains its induced topology from the Galois group, $I$ is open in $W_k$, and multiplication by $\Phi$ is a homeomorphism. $W_k$ with this topology is the *Weil group of* $k$. (It carries the structure of a group scheme over $\mathbb{Q}$.) $W_k$ has a natural character $|| \, || : W_k \to q^{\mathbb{Z}} \subset \mathbb{Q}^\times$ given by $||w|| = 1$ for $w \in I$ and $||\Phi|| = q^{-1}$.

The topology on $W_k$, being essentially pro-finite on $I$, is still too restrictive to have a sufficiently interesting theory of complex representations. However it has many interesting $\overline{\mathbb{Q}}_\ell$-representations and these are the ones that arise in arithmetic geometry. In order to free the representation theory from incompatible topologies, Deligne introduced the *Weil-Deligne group* $W_k'$. Following Deligne and Tate we take $W_k'$ to be the semi-direct product $W_k \ltimes \mathbb{G}_a$ of the Weil group with the additive group where $W_k$ acts on $\mathbb{G}_a$ by $wxw^{-1} = ||w||x$.

What is important about $W_k'$ is not so much its structure but its representation theory. A representation $\rho'$ of $W_k'$ is a pair $\rho' = (\rho, N)$ consisting of

(i) an $n$-dimensional vector space $V$ and a group homomorphism $\rho : W_k \to GL(V)$ whose kernel contains an open subgroup of $I$ (so it is continuous with respect to the discrete topology on $V$);
(ii) a nilpotent endomorphism $N$ of $V$ such that $\rho(w)N\rho(w)^{-1} = ||w||N$.

The representation $\rho'$ is called semi-simple if $\rho$ is. This category of representations is independent of the (characteristic 0) coefficient field.

[Often one sees $W_k' = W_k \times SL_2$. This can be made consistent in terms of the representation theory via the Jacobson-Morozov Theorem. However it is the nilpotent endomorphism $N$ that arises naturally as a monodromy operator in the theory of $\ell$-adic Galois representations (Grothendieck) so I have chosen to retain this formulation.]

When $k$ is $\mathbb{R}$ or $\mathbb{C}$, we simply take $W_k' = W_k$.

## 2 The dual group

Now let $k$ be either local or global and let $G$ be a connected reductive algebraic group over $k$. For simplicity we will take $G$ split, so things behave as if $k$ were algebraically closed.

Recall from Kim's lectures that over an algebraically closed field $G$ is determined by its root data. If $T$ is a maximal split torus in $G$ then the root data for $G$ is $\Psi = (X^*(T), \Phi, X_*(T), \Phi^\vee)$ where:

$X^*(T)$ is the set of rational characters of $T$
$\Phi \subset X^*(T)$ is the root system $\Phi(G, T)$
$X_*(T)$ is the set of rational co-characters of $T$ (one parameter subgroups)
$\Phi^\vee \subset X_*(T)$ is the co-root system.

If we dualize this to obtain $\Psi^\vee = (X_*(T), \Phi^\vee, X^*(T), \Phi)$ then this dual data determines a complex group ${}^LG^\circ = {}^LG$ which is the Langlands dual group or the (connected component of the) $L$-group of $G$.

## 3 The local Langlands conjecture

Let $k$ be a local field and let $G$ be a reductive algebraic group over $k$, assumed split as before. The local Langlands conjecture essentially says that the irreducible admissible representations of $G(k)$ are parameterized by admissible homomorphisms of the Weil-Deligne group $W_k'$ to the $L$-group ${}^LG$. To be more precise, for this lecture let us set $\mathcal{A}(G)$ denote the equivalence classes of irreducible admissible (complex) representations of $G(k)$ and let $\Phi(G)$ denote the set of all admissible homomorphisms $\phi : W_k' \to {}^LG$ (module inner automorphisms). We won't worry about the precise definition of admissible, but just note that for $G = GL_n$ an admissible homomorphism is simply a semi-simple representation as above.

**Local Langlands Conjecture:** *There is a surjective map $\mathcal{A}(G) \longrightarrow \Phi(G)$ with finite fibres which partitions $\mathcal{A}(G)$ into finite sets $\mathcal{A}_\phi(G)$, called $L$-packets, satisfying certain naturality conditions.*

This is known in the following cases which will be of relevance to us. (This list is not exhaustive.)

1. If $k = \mathbb{R}$ or $\mathbb{C}$ this was completely established by Langlands. His naturality conditions were representation theoretic in nature.

2. If $k$ is non-archimedean (recalling that $G$ is split) then one knows how to parameterize the unramified representations of $G(k)$ by unramified admissible homomorphisms. This is the Satake classification.

3. If $k$ is non-archimedean and $G = GL_n$ this is known and due to Harris-Taylor and then Henniart (remember we have taken $k$ of characteristic 0) and in fact the map is a bijection. In these works the naturality conditions were phrased in terms of matching twisted $L$- and $\varepsilon$-factors for the Weil-Deligne representations with those we presented here for $GL_n$.

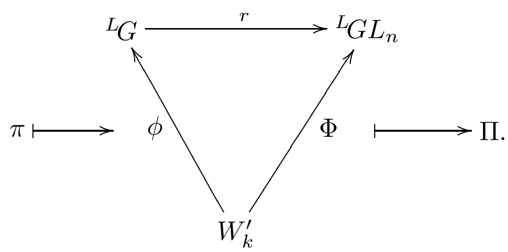
Note that there is at present no similar formulation of a global Langlands conjecture for global fields of characteristic 0. To obtain one, one would need to replace the local Weil-Deligne group by the conjectural Langlands group $\mathcal{L}_k$ that Jim Arthur talked about in the Shimura Variety Workshop. With $\mathcal{L}_k$ in hand it would be relatively easy to formulate a conjecture like the one above.

## 4 Local Functoriality

We still take $k$ to be a local field. Let $G$ be a split reductive algebraic group over $k$. Let $r : {}^LG \to GL_n(\mathbb{C})$ be a complex analytic representation. Since ${}^LGL_n = GL_n(\mathbb{C})$, the map $r$ is an example of what Langlands referred to as an $L$-homomorphism. Langlands' Principle of Functoriality can then be roughly stated as saying:

**Principle of Functoriality:** *Associated to the $L$-homomorphism $r : {}^LG \to {}^LGL_n$ there should be associated a natural lift or transfer of admissible representations from $\mathcal{A}(G)$ to $\mathcal{A}(GL_n)$.*

If we assume the local Langlands conjecture for $G$, this is easy to formulate. We begin with $\pi \in \mathcal{A}(G)$. Associated to $\pi$ we have a parameter $\phi \in \Phi(G)$. Then via the diagram




we obtain a parameter $\Phi \in \Phi(GL_n)$ and hence a representation $\Pi$ of $GL_n(k)$. We refer to $\Pi$ as the local functorial lift of $\pi$. As part of the formalism we obtain

$$L(s,\pi,r) = L(s, r\circ\phi) = L(s,\Phi) = L(s,\Pi)$$

with similar equalities for $\varepsilon$-factors and twisted versions.

## 5 Global functoriality

Now let us take $k$ a global field of characteristic 0, that is, a number field and let $G$ be a connected reductive group defined and split over $k$. As before, we let $r : {}^LG \to GL_n(\mathbb{C}) = {}^LGL_n$ be an $L$-homomorphism. Then there is also a global Principle of Functoriality, namely:

**Principle of Functoriality:** *Associated to the $L$-homomorphism $r : {}^LG \to {}^LGL_n$ there should be associated a natural lift or transfer of automorphic representations of $G(\mathbb{A})$ to automorphic representations of $GL_n(\mathbb{A})$ .*

We can give a precise formulation of this through local Langlands functoriality and a local-global principle. Let $\pi = \otimes'\pi_v$ be an irreducible automorphic representation of $G(\mathbb{A})$. Then there is a finite set $S$ of finite places such that for all $v \notin S$ we have that either $v$ is archimedean or $\pi_v$ is unramified. In either case, we understand the local Langlands conjecture for $\pi_v$ and hence we have a local functorial lift $\Pi_v$ as a representation of $GL_n(k_v)$.

**Definition 11.1** *Let $\pi = \otimes'\pi_v$ be an automorphic representation of $G(\mathbb{A})$. An automorphic representation $\Pi = \otimes'\Pi_v$ of $GL_n(\mathbb{A})$ will be called a* functorial lift or transfer *of $\pi$ if there is a finite set of places $S$ such that $\Pi_v$ is the local Langlands lift of $\pi_v$ for all $v \notin S$.*

Then Langlands' Principle of Functoriality predicts that every automorphic representation $\pi$ of $G(\mathbb{A})$ does indeed have a functorial lift to $GL_n(\mathbb{A})$. Note that $\Pi$ being a functorial lift of $\pi$ entails an equality of partial $L$-functions $L^S(s,\pi,r) = L^S(s,\Pi)$ as well as for $\varepsilon$-factors and twisted versions. (A one point we called this a weak lift. But the terminology of functorial lift (without any prejudicial adjective) is consistent with the recent formulations of functoriality due to Arthur and Langlands himself.)

## 6 Functoriality and the Converse Theorem

It should be clear how to approach the problem of global functoriality via the Converse Theorem. We begin with a cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $G(\mathbb{A})$. There are three basic steps:

1. *Construction of a candidate lift.* If we know the local Langlands conjecture for all $\pi_v$ then we simply take for $\Pi_v$ the local Langlands lift of $\pi_v$. Note that these local lifts will satisfy

$$L(s, \pi_v \times \pi'_v, r \otimes \iota) = L(s, \Pi_v \times \pi'_v)$$
$$\varepsilon(s, \pi_v \times \pi'_v, r \otimes \iota, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

for all irreducible admissible generic representations $\pi'_v$ of $GL_r(k_v)$, where the map $\iota : GL_m(\mathbb{C}) \to GL_m(\mathbb{C})$ is the identity map, viewed as an $L$-homomorphism. Then we take $\Pi = \otimes' \Pi_v$ to be our candidate lift. We then have

$$L(s, \pi \times \pi', r \otimes \iota) = L(s, \Pi \times \pi')$$
$$\varepsilon(s, \pi \times \pi', r \otimes \iota) = \varepsilon(s, \Pi \times \pi')$$

for all cuspidal $\pi'$ of $GL_m(\mathbb{A})$.

In practice, there will be a finite set of places $S$ where we do not know the local Langlands conjecture for $\pi_v$ and we will have to deal with this.

2. *Analytic properties of $L$-functions.* By the equality of $L$- and $\varepsilon$-factors above, to show that $L(s, \Pi \times \pi')$ is nice for $\pi'$ in a suitable twisting set $\mathcal{T}$ it suffices to know this for $L(s, \pi \times \pi', r \otimes \iota)$. But this is what Kim has been lecturing on all semester.

In practice we do not expect $L(s, \pi \times \pi')$ to be entire always, since we do expect some cuspidal representations $\pi$ of $G(\mathbb{A})$ to lift to non-cuspidal representations $\Pi$ of $GL_n(\mathbb{A})$. This will also have to be dealt with.

3. *Apply the Converse Theorem.* Once we know that $L(s, \Pi \times \pi')$ is nice for a suitable twisting set $\mathcal{T}$, then we can apply the appropriate Converse Theorem to conclude that a functorial lift exists.

We have left two problems unresolved: (i) the lack of the local Langlands conjecture at the $v \in S$, and (ii) the fact that some $L(s, \pi \times \pi')$ could have poles. We are able to finesse both of these using an appropriately chosen idele class character $\eta$ and the Useful Variant of our Converse Theorems. We will explain these in the next lecture when we deal with the functoriality for the classical groups.

## 7 References

[1] J. Arthur, *The principle of functoriality.* Bull. AMS **40** (2002), 39–53.

[2] A. Borel, *Automorphic L-functions.* Proc. Sympos. Pure Math. **33**, part 2, (1979), 27–61.

[3] J.W. Cogdell, *Langlands conjectures for $GL_n$*. An Introduction to the Langlands Program (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston, 2003, 229–249.

[4] J.W. Cogdell, *Dual groups and Langlands Functoriality.* An Introduction to the Langlands Program (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston, 2003, 251–268.

[5] J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems, Functoriality, and applications to number theory.* Proc. ICM 2002, Beijing **II** (2002), 119–128.

[6] M. Harris and R. Taylor, The Geometry and Cohomology of Some Simple Shimura Varieties, Annals of Math. Studies No. 151, Princeton University Press, Princeton, 2001.

[7] G. Henniart, *Une preuve simple des conjectures de Langlands pour* GL($n$) *sur un corps p-adique*. Invent. Math. **139** (2000), 439–455.

[8] H. Kim, *Automorphic L-functions*. This volume.

[9] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups.* Representation Theory and Harmonic Analysis on Semisimple Lie Groups, Math. Surveys Monographs, 31, Amer. Math. Soc., Providence, RI, 1989, 101–170

[10] R.P. Langlands, *Where stands functoriality today.* Proc. Sympos. Pure Math. **61** (1997), 457–471.

[11] J. Tate, *Number theoretic background*. Proc. Sympos. Pure Math. **33**, part 2, 3–26.

LECTURE 12

# Functoriality for the Classical Groups

We again take $k$ to be a number field. In this Lecture, this is currently a necessary restriction. We let $\mathbb{A}$ denote its ring of adeles and fix a non-trivial character $\psi$ of $k\backslash\mathbb{A}$.

## 1 The result

We take $G = G_n$ to be a split classical group of rank $n$ defined over $k$. More specifically, we consider the following cases.

(a) $G_n = SO_{2n+1}$ or $SO_{2n}$, the special orthogonal group over $k$ with respect to the symmetric bilinear form represented by

$$\Phi_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \quad \text{with } m = 2n+1,\ 2n.$$

(b) $G_n = Sp_{2n}$ the symplectic group with respect to the alternating form represented by

$$J_{2n} = \begin{pmatrix} & \Phi_n \\ -\Phi_n & \end{pmatrix}.$$

In each case, there is a standard embedding $r : {}^LG \hookrightarrow GL_N(\mathbb{C}) = {}^LGL_N$ for an appropriate $N$ as given in the following table.

| $G_n$ | $r : {}^LG_n \hookrightarrow {}^LGL_N$ | $GL_N$ |
|---|---|---|
| $SO_{2n+1}$ | $Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $SO_{2n}$ | $SO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $Sp_{2n}$ | $SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$ | $GL_{2n+1}$ |

Let $\pi = \otimes'\pi_v$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. [Recall that if $B = TU$ is the standard (upper triangular) Borel subgroup of $G(\mathbb{A})$ and we extend our additive character to one of $U(k)\backslash U(\mathbb{A})$ in the standard way then $\pi$ is globally generic if for $\varphi \in V_\pi$ we have

$$\int_{U(k)\backslash U(\mathbb{A})} \varphi(ug)\psi^{-1}(u)\ du \not\equiv 0.]$$

Our result is then the following.

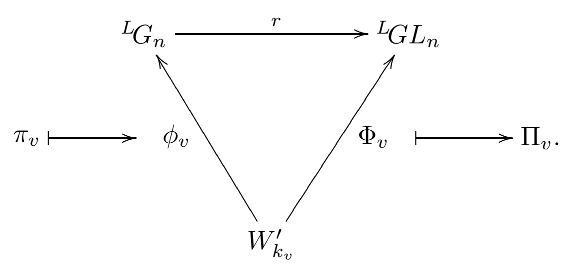
**Theorem 12.1** *Let $\pi$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. Then $\pi$ has a functorial lift $\Pi$ to $GL_N(\mathbb{A})$.*

Our proof will be by the Converse Theorem. We will follow the three steps given above.

Let $S$ be a non-empty set of finite places such that $\pi_v$ is unramified for all finite $v \notin S$.

## 2 Construction of a candidate lift

(i) If $v \notin S$, then either $v|\infty$ or $v < \infty$ and $\pi_v$ is unramified. In either case we have the local Langlands parameterization for $\pi_v$ and hence a local functorial lift $\Pi_v$ as an irreducible admissible representation of $GL_N(k_v)$.




As suggested by the formalism, one can show the following.

**Proposition 12.1** *Let $\Pi_v$ be the local functorial lift of $\pi_v$. Let $\pi'_v$ be an irreducible admissible generic representation of $GL_d(k_v)$ with $1 \le d \le N-1$. Then*

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v)$$
$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

For simplicity, since $r$ is the standard embedding of the $L$-groups, we have dropped it from our notation and written

$$L(s, \pi_v \times \pi'_v) = L(s, \pi_v \times \pi'_v, r \otimes \iota),$$

etc..

(ii) If $v \in S$ then we may not have the local Langlands parameterization of $\pi_v$. We replace this knowledge with the following two local results.

**Proposition 12.2 (Multiplicativity of $\gamma$)** *If $\pi_v$ is an irreducible admissible generic representation of $G_n(k_v)$ and $\pi'_v$ is an irreducible admissible generic representation of $GL_d(k_v)$ of the form*

$$\pi'_v \simeq \mathrm{Ind}_{Q(k_v)}^{GL_d(k_v)}(\pi'_{1,v} \otimes \pi'_{2,v})$$

*then*

$$\gamma(s, \pi_v \times \pi'_v, \psi_v) = \gamma(s, \pi_v \times \pi'_{1,v}, \psi_v)\gamma(s, \pi_v \times \pi'_{2,v}, \psi_v).$$

In this case, one also has a divisibility among the $L$-functions

$$L(s, \pi_v \times \pi'_v)^{-1} \big| \; [L(s, \pi_v \times \pi'_{1,v})L(s, \pi_v \times \pi'_{2,v})]^{-1}.$$

There is a similar multiplicativity in the first variable, that is, when the representation $\pi_v$ of $G_n(k_v)$ is induced.

**Proposition 12.3 (Stability of $\gamma$)** *Let $\pi_{1,v}$ and $\pi_{2,v}$ be two irreducible admissible smooth generic representations of $G_n(k_v)$. Then for every sufficiently highly ramified character $\eta_v$ of $k_v^\times$ we have*

$$\gamma(s, \pi_{1,v} \times \eta_v, \psi_v) = \gamma(s, \pi_{2,v} \times \eta_v, \psi_v).$$

In this situation, one also has that the $L$-functions stabilize

$$L(s, \pi_{1,v} \times \eta_v) = L(s, \pi_{2,v} \times \eta_v) \equiv 1$$

so that the $\varepsilon(s, \pi_{i,v} \times \eta_v, \psi_v)$ stabilize as well.

Recall from Lecture 6 that we had analogous statements for $GL_n(k_v)$. Moreover, as noted there, by using the multiplicativity in the $G_n$-variable one can compute the stable form of the $\gamma$-factor in terms of abelian $\gamma$-factors. Comparing these stable forms for $G_n(k_v)$ with those for $GL_N(k_v)$ one finds:

**Proposition 12.4 (Comparison of stable forms)** *Let $\pi_v$ be an irreducible admissible generic representation of $G_n(k_v)$. Let $\Pi_v$ be an irreducible admissible representation of $GL_N(k_v)$ having trivial central character. Then for every sufficiently ramified character $\eta_v$ of $GL_1(k_v)$ we have*

$$\gamma(s, \pi_v \times \eta_v, \psi_v) = \gamma(s, \Pi_v \times \eta_v, \psi_v).$$

Of course since both $L$-functions stabilize to 1, this gives the equality of the stable $L$- and $\varepsilon$-factors.

So at the places $v \in S$ we can now take as the local component $\Pi_v$ of our candidate lift any irreducible admissible representation of $GL_N(k_v)$ with $\omega_{\Pi_v} \equiv 1$. With this choice of $\Pi_v$ we have the following result.

**Proposition 12.5** *Let $\pi'_v$ be an irreducible admissible generic representation of $GL_d(k_v)$ of the form $\pi'_v = \pi'_{0,v} \otimes \eta_v$ with $\pi'_{0,v}$ unramified and $\eta_v$ chosen as above. Then we have*

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v)$$
$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

To see this on the level of $\gamma$-factors, write $\pi'_{0,v} = \mathrm{Ind}(|\ |_v^{s_1} \otimes \cdots \otimes |\ |_v^{s_d})$. Then $\pi'_v = \mathrm{Ind}(|\ |_v^{s_1}\eta_v \otimes \cdots \otimes |\ |_v^{s_d}\eta_v)$ and we have

$$\begin{aligned}
\gamma(s, \pi_v \times \pi'_v, \psi_v) &= \prod_{i=1}^{d} \gamma(s + s_i, \pi_v \times \eta_v, \psi_v) && \text{(multiplicativity)} \\
&= \prod_{i=1}^{d} \gamma(s + s_i, \Pi_v \times \eta_v, \psi_v) && \text{(stability)} \\
&= \gamma(s, \Pi_v \times \pi'_v, \psi_v) && \text{(multiplicativity)}
\end{aligned}$$

Return to our generic cuspidal representation $\pi = \otimes'\pi_v$ of $G_n(\mathbb{A})$. For each $\pi_v$ we have attached a local representation $\Pi_v$ of $GL_N(k_v)$, which is the local functorial lift for those $v \notin S$. Then $\Pi = \otimes'\Pi_v$ is an irreducible admissible representation of $GL_N(\mathbb{A})$. This is our candidate lift. Combining our local results, we have:

**Proposition 12.6** *Let $\pi$ and $\Pi$ be as above. Then there exists an idele class character $\eta : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ such that for all $\pi' \in \mathcal{T}^S(N-1) \otimes \eta$ we have*

$$\begin{aligned}
L(s, \pi \times \pi') &= L(s, \Pi \times \pi') \\
\varepsilon(s, \pi \times \pi') &= \varepsilon(s, \Pi \times \pi')
\end{aligned}$$

## 3 Analytic properties of $L$-functions

The analytic properties of the $L(s, \pi \times \pi')$ are controlled through the Fourier coefficients of Eisenstein series as in Kim's lectures. We summarize the results from there that we need in the following result.

**Proposition 12.7** *Let $\pi$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. Let $S$ be a non-empty set of finite places and let $\eta : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be an idele class character such that at one place $v_0 \in S$ we have that both $\eta_{v_0}$ and $\eta_{v_0}^2$ are ramified. Then $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S(N-1) \otimes \eta$, that is,*

(i) *$L(s, \pi \times \pi')$ and $L(s, \widetilde{\pi} \times \widetilde{\pi}')$ are entire functions of $s$;*
(ii) *these functions are bounded in vertical strips;*
(iii) *we have the standard functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

Recall that $\eta$ is necessary only to ensure that the $L(s, \pi \times \pi')$ are all entire. This resolves our global problem that the lift $\Pi$ of $\pi$ need not be cuspidal so that $L(s, \pi \times \pi')$ might have poles if some restriction is not placed on the $\pi'$.

It is in the use of the Eisenstein series to control the $L$-functions that $k$ is required to be a number field. In reality this should not matter, but at present this method of controlling the $L$-functions is only worked out in characteristic zero, that is, the number field case.

## 4 Apply the Converse Theorem

Take $\pi = \otimes' \pi_v$ to be our globally generic cuspidal representation of $G_n(\mathbb{A})$. Let $S$ be a non-empty set of finite places such that $\pi_v$ is unramified for all finite places $v \notin S$. Construct the candidate lift $\Pi = \otimes' \Pi_v$ as above.

For an appropriate choice if idele class character $\eta : k^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, chosen to satisfy both our local requirements of Proposition 12.6 and our global requirement of Proposition 12.7, we know that for all $\pi' \in \mathcal{T}^S(N-1) \otimes \eta$ both

$$L(s, \pi \times \pi') = L(s, \Pi \times \pi')$$
$$\varepsilon(s, \pi \times \pi') = \varepsilon(s, \Pi \times \pi')$$

and

$$L(s, \pi \times \pi'), \text{ and hence } L(s, \Pi \times \pi'), \text{ is nice.}$$

Now applying the useful variant of our Converse Theorem there exists an automorphic representation $\Pi'$ of $GL_N(\mathbb{A})$ such that for all $v \notin S$ we have

$$\Pi'_v \simeq \Pi_v = \text{ the local functional lift of } \pi_v.$$

Then $\Pi'$ is our functorial lift of $\pi$.

## 5 References

[1] J.W. Cogdell, H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to* $GL_N$. Publ. Math. IHES **93** (2001), 5–30.

[2] J.W. Cogdell, H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups.* Preprint (2003).

[3] H. Kim, *Automorphic L-Functions.* This volume.

LECTURE 13

# Functoriality for the Classical Groups, II

[This are the notes that accompanied my talk at the Workshop on Automorphic $L$-functions. It has a slight overlap with Lecture 12.]

We let $k$ be a number field, $\mathbb{A}$ its ring of adeles, and $\psi : k\backslash\mathbb{A} \to \mathbb{C}^\times$ a non-trivial additive character.

## 1 Functoriality

We will be interested in global functoriality from the split classical groups to $GL_N$. More precisely, let $G_n$ be a split classical group of rank $n$ defined over $k$ as below:

(a) $G_n = SO_{2n+1}$ or $SO_{2n}$ with respect to the split symmetric form

$$\Phi_m = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \text{ with } m = 2n+1, 2n.$$

(b) $G_n = Sp_{2n}$ with respect to the alternating form

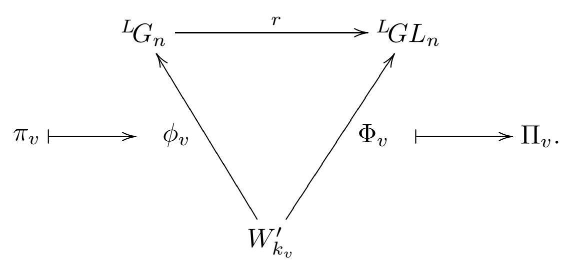
$$J_{2n} = \begin{pmatrix} & \Phi_n \\ -\Phi_n & \end{pmatrix}.$$

For each group there is a standard embedding $r : {}^LG \hookrightarrow GL_N(\mathbb{C}) = {}^LGL_N$:

| $G_n$ | $r : {}^LG_n \hookrightarrow {}^LGL_N$ | $GL_N$ |
|---|---|---|
| $SO_{2n+1}$ | $Sp_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $SO_{2n}$ | $SO_{2n}(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{C})$ | $GL_{2n}$ |
| $Sp_{2n}$ | $SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C})$ | $GL_{2n+1}$ |

By Langlands principle of functoriality there should be an associated functorial lift or transfer of automorphic representations $\pi$ of $G_n(\mathbb{A})$ to automorphic representations $\Pi$ of $GL_N(\mathbb{A})$. Together with Kim, Piatetski-Shapiro and Shahidi, we have recently completed this functoriality for globally generic cuspidal representations $\pi$.

**Theorem 13.1 (Functoriality)** *Let $\pi = \otimes' \pi_v$ be a globally generic representation of $G_n(\mathbb{A})$. Then $\pi$ has a functorial lift to an automorphic representation $\Pi$ of $GL_N(\mathbb{A})$. More precisely, there is a finite set of (finite) places $S$ such that for all $v \notin S$, $\Pi_v$ is the local functorial lift of $\pi_v$ in the sense of the local Langlands parameterization:*



Philosophically, through functoriality one hopes to pull back structural results from $GL_N$ to the classical groups $G_n$. In this lecture I would like to outline some of what we know in these cases.

## 2 Descent

A bit earlier than our proof of functoriality, Ginzburg, Rallis, and Soudry were developing a theory of local and global descent from self dual automorphic representations $\Pi$ of $GL_N(\mathbb{A})$ to cuspidal automorphic representations $\pi$ of the classical groups $G_n(\mathbb{A})$. I would like to give some idea of the descent in the case of $GL_N = GL_{2n}$ to $G_n = SO_{2n+1}$.

Begin with $\Pi$ a self dual cuspidal representation of $GL_N(\mathbb{A})$ having trivial central character. Let $H = SO_{4n}$. Then $H$ has a maximal (Siegel) parabolic subgroup $P \simeq MN$ with Levi subgroup $M \simeq GL_{2n}$. Hence we can form the globally induced representation

$$\Xi(\Pi) = \mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\Pi \otimes |\det|^{s-1/2}).$$

For any function $f \in V_{\Xi(\Pi)}$ we can then form an Eisenstein series $E(s, f, h)$ on $H(\mathbb{A})$ in the usual manner.

$H$ also has a parabolic subgroup $P' = M'N'$ with Levi subgroup $M' \simeq (GL_1)^{n-1} \times SO_{2n+2}$. If one takes an appropriate additive character $\psi'$ of $N'$ then its stabilizer in $M'$ is precisely $SO_{2n+1} = G_n$. Ginzburg, Rallis, and Soudry refer to the corresponding Fourier coefficient

$$E^{\psi'}(s, f, h) = \int_{N'(k)\backslash N'(\mathbb{A})} E(s, f, nh)\psi'(n)\, dn$$

a Gelfand-Graev coefficient. These naturally restrict to automorphic functions of $g \in G_n(\mathbb{A}) \hookrightarrow H(\mathbb{A})$. Hence if $\pi$ is any cuspidal representation of $G_n(\mathbb{A})$ we can consider the Petersson inner products of $\varphi \in V_\pi$ with these coefficients

$$\langle \varphi, E^{\psi'}(s,f)\rangle = \int_{G_n(k)\backslash G_n(\mathbb{A})} \varphi(g) E^{\psi'}(s,f,g)\ dg.$$

This will vanish unless $\pi$ is globally generic and in that case one finds that outside a finite set of places $T$ we have

$$\langle \varphi, E^{\psi'}(s,f)\rangle \sim \frac{L^T(s,\pi\times\Pi)}{L^T(2s,\Pi,\wedge^2)}.$$

The condition for $L^T(s,\pi\times\Pi)$ to have a pole at $s=1$ is that $\Pi$ be a functorial lift of $\pi$. On the other hand, if $L^T(s,\pi\times\Pi)$ is to have a pole at $s=1$ with a non-zero residue, then the above formula gives that $V_\pi$ will have a non-zero $G_n$-invariant pairing with the space of residues

$$\pi_{\psi'}(\Pi) = \langle Res_{s=1}(E^{\psi'}(s,f,g)) \mid f \in V_{\Xi(\Pi)}\rangle.$$

On the other hand, if the Gelfand-Graev coefficients $E^{\psi'}(s,f)$ is to have a pole at $s=1$ then the full Eisenstein series $E(s,f)$ must as well and this happens iff (from the constant term calculation) $L^T(s,\Pi,\wedge^2)$ does.

If we run this analysis backwards, we obtain the descent theorem for self dual cuspidal representations $\Pi$ of $GL_{2n}(\mathbb{A})$ such that $L^T(s,\Pi,\wedge^2)$ has a pole at $s=1$.

**Theorem 13.2 (Descent)** *Let $\Pi$ be a self dual cuspidal representation of $GL_{2n}(\mathbb{A})$ with trivial central character and such that $L^T(s,\Pi,\wedge^2)$ has a pole at $s=1$. Let*

$$\pi_{\psi'}(\Pi) = \langle Res_{s=1}(E^{\psi'}(s,f,g)) \mid f \in V_{\Xi(\Pi)}\rangle.$$

*Then*

(i) $\pi_{\psi'}(\Pi) \neq 0$,
(ii) $\pi_{\psi'}(\Pi)$ *is cuspidal,*
(iii) *each summand of* $\pi_{\psi'}(\Pi)$ *is globally generic,*
(iv) *each summand of* $\pi_{\psi'}(\Pi)$ *functorially lifts to* $\Pi$,
(v) $\pi_{\psi'}(\Pi)$ *is multiplicity one,*
(vi) *if* $\pi$ *is a globally generic cuspidal representation which functorially lifts to* $\Pi$ *then* $\pi$ *has a non-zero invariant pairing with* $\pi_{\psi'}(\Pi)$.

Of course the conjecture is the following.

**Conjecture 13.1** $\pi_{\psi'}(\Pi)$ *is irreducible (and hence cuspidal and the only globally generic representation of* $G_n(\mathbb{A})$ *which functorially lifts to* $\Pi$*).*

One has precisely the same result for the other classical groups with modifications as indicated in the following table.

| $GL_N$ | $H$ | $G_n$ | $L^T(s,\Pi,\beta_{G_n})$ with a pole at $s=1$ |
|---|---|---|---|
| $GL_{2n}$ | $SO_{4n}$ | $SO_{2n+1}$ | $L^T(s,\Pi,\wedge^2)$ |
| $GL_{2n}$ | $SO_{4n+1}$ | $SO_{2n}$ | $L^T(s,\Pi,Sym^2)$ |
| $GL_{2n+1}$ | $\widetilde{Sp}_{4n+2}$ | $Sp_{2n}$ | $L^T(s,\Pi,Sym^2)$ |

As a consequence of this descent theorem, they obtain the following characterization of the image of functoriality.

**Theorem 13.3 (Characterization of the image)** *Let $\pi$ be a globally generic representation of $G_n(\mathbb{A})$. Then any functorial lift of $\pi$ to an automorphic representation $\Pi$ of $GL_N(\mathbb{A})$ has trivial central character and is of the form*

$$\Pi = \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_{\mathrm{d}}) = \Pi_1 \boxplus \cdots \boxplus \Pi_{\mathrm{d}}$$

*with $\Pi_i \not\simeq \Pi_j$ for $i \neq j$ and such that each $\Pi_i$ is a unitary self dual cuspidal representation of $GL_{N_i}(\mathbb{A})$ such that $L^T(s,\Pi_i,\beta_{G_n})$ has a pole at $s=1$. Moreover, any such $\Pi$ is the functorial lift of some $\pi$.*

Note that since these functorial lifts are isobaric, then by the Strong Multiplicity One Theorem for $GL_N$, the functorial image $\Pi$ is completely determined by the $\pi_v$ for $v \notin S$, i.e, those places where we know the local functorial lifts.

## 3 Bounds towards Ramanujan

As a first consequence of functoriality, we obtain bounds towards Ramanujan for globally generic cuspidal representations of $G_n$ by pulling back the known bounds for $GL_N$.

**Theorem 13.4** *Let $\pi \simeq \otimes' \pi_v$ be a globally generic cuspidal representation of $G_n(\mathbb{A})$. Then at the places $v$ where $\pi_v$ is unramified the Satake parameters for $\pi_v$ satisfy*

$$q_v^{-(\frac{1}{2}-\frac{1}{N^2+1})} \leq |\alpha_v| \leq q_v^{\frac{1}{2}-\frac{1}{N^2+1}}.$$

We obtain similar estimates towards temperedness at all places. As far as we know, these are the first Ramanujan type bounds for classical groups.

## 4 The local converse theorem

The results in this section are due to Jiang and Soudry. We now restrict ourselves to the case of $G_n = SO_{2n+1}$. Also, let $v$ be a non-archimedean place of $k$, so $k_v$ is a $p$-adic field. One of the most powerful local results for $SO_{2n+1}(k_v)$ to be pulled back from $GL_{2n}$ is the "local converse theorem" for $GL_{2n}$, first formulated by Jacquet, Piatetski-Shapiro, and Shalika but finally proved by Henniart.

**Theorem 13.5 (Local Converse Theorem)** *Let $\pi_{1,v}$ and $\pi_{2,v}$ be two irreducible admissible generic representations of $SO_{2n+1}(k_v)$. Suppose that*

$$\gamma(s, \pi_{1,v} \times \pi'_v, \psi_v) = \gamma(s, \pi_{2,v} \times \pi'_v, \psi_v)$$

*for all irreducible super-cuspidal $\pi'_v$ of $GL_d(k_v)$ for $1 \leq d \leq 2n-1$. Then $\pi_{1,v} \simeq \pi_{2,v}$.*

To obtain this result, Jiang and Soudry needed to combine global functoriality from $SO_{2n+1}$ to $GL_{2n}$ with local descent from $GL_{2n}$ to $\widetilde{Sp}_{2n}$ and then the local theta correspondence from $\widetilde{Sp}_{2n}$ to $SO_{2n+1}$ to be able to pull Henniart's result back from $GL_{2n}$ to $SO_{2n+1}$.

One immediate consequence of this is that Conjecture 13.1 is true for $G_n = SO_{2n+1}$ and hence the global functoriality from $SO_{2n+1}$ to $GL_{2n}$ for globally generic cuspidal representations is injective. This allows them to pull the Strong Multiplicity One Theorem for $GL_{2n}(\mathbb{A})$ back to globally generic cuspidal representations of $SO_{2n+1}(\mathbb{A})$ to obtain a rigidity theorem.

**Theorem 13.6 (Rigidity)** *Let $\pi_1$ and $\pi_2$ be two globally generic cuspidal representations of $SO_{2n+1}(\mathbb{A})$. Suppose there is a finite set of places $S$ such that $\pi_{1,v} \simeq \pi_{2,v}$ for all $v \notin S$. Then $\pi_{1,v} \simeq \pi_{2,v}$ for all $v$, that is. $\pi_1 \simeq \pi_2$.*

More importantly, having this Local Converse Theorem allow one to pull back local results from $GL_{2n}$ to $SO_{2n+1}$ through global functoriality. As a first result, Jiang and Soudry are able to complete the local functoriality from $SO_{2n+1}(k_v)$ to $GL_{2n}(k_v)$ for generic representations at those places where it was not previously known. The first step is the following.

Let $\mathcal{A}_0^g(SO_{2n+1})$ denote the set of irreducible generic super-cuspidal representations of $SO_{2n+1}(k_v)$ up to equivalence and let $\mathcal{A}_\ell(GL_{2n})$ denote the set of all $\Pi_v$ of $GL_{2n}(k_v)$ of the form

$$\Pi_v \simeq \mathrm{Ind}(\Pi_{1,v} \otimes \cdots \otimes \Pi_{d,v})$$

where each $\Pi_{i,v}$ is an irreducible super-cuspidal self dual representation of some $GL_{2n_i}(k_v)$ such that $L(s, \Pi_{i,v}, \wedge^2)$ has a pole at $s = 0$ and $\Pi_{i,v} \not\simeq \Pi_{j,v}$ for $i \neq j$.

**Theorem 13.7 (Local functoriality)** *There exists a unique bijection taking $\mathcal{A}_0^g(SO_{2n+1}) \to \mathcal{A}_\ell(GL_{2n})$, denoted $\pi_v \mapsto \Pi_v = \Pi(\pi_v)$ such that*

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v)$$
$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

*for all irreducible super-cuspidal representations of $GL_d(k_v)$ for all $d$.*

Using the way in which one builds a general generic representation from generic super-cuspidal ones, they later extended this theorem to all of $\mathcal{A}^g(SO_{2n+1})$, the set of all irreducible admissible generic representations of $SO_{2n+1}(k_v)$.

Through local functoriality they were then able to pull back the local Langlands conjecture (or arithmetic Langlands parameterization) for $\mathcal{A}_0^g(SO_{2n+1})$. Let $\Phi_0^g(SO_{2n+1})$ denote the set of admissible, completely reducible, multiplicity free representations $\rho : W_{k_v} \to Sp_{2n}(\mathbb{C}) = {}^L SO_{2n+1}$ which are symplectic (so each irreducible constituent must be symplectic).

**Theorem 13.8 (Local Langlands conjecture)** *There exists a unique bijection $\Phi_0^g(SO_{2n+1}) \to \mathcal{A}_0^g(SO_{2n+1})$ denoted $\rho_v \mapsto \pi_v = \pi(\rho_v)$ such that*

$$L(s, \rho_v \otimes \phi_v) = L(s, \pi(\rho_v) \times \pi'(\phi_v))$$
$$\varepsilon(s, \rho_v \otimes \phi_v, \psi_v) = L(s, \pi(\rho_v) \times \pi'(\phi_v), \psi_v)$$

*for all irreducible admissible representations $\phi_v : W_{k_v} \to GL_d(k_v)$, that is $\phi_v \in \Phi_0(GL_d)$.*

In the same subsequent paper, they extended this result to all of $\mathcal{A}^g(SO_{2n+1})$ as well.

There are other local consequences to be found in their papers, such as the existence of a unique generic member of each tempered $L$-packet for $SO_{2n+1}$, but we shall end here.

## 5 Further applications

Independent of the descent theory, Kim has proved several results along the lines of Jiang and Soudry's independently. Even though we do not have complete local descent results in the cases of the other classical groups, we can still apply Kim's techniques and explicitly compute the local functorial lifts $\Pi_v$ of $\pi_v$ at the places $v \in S$. With these local lifts at hand, which were needed for the more complete Ramanujan results mentioned in Section 10.3, we have been able to prove several results on the $p$-adic local representation theory of the split classical groups. These include a proof of Mœglin dimension relation for generic discrete series representations $\pi_v$, the basic analysis of the conductor $f(\pi_v)$ of a generic representation, and the holomorphy and non-vanishing of the local intertwining operators in the region $Re(s) \geq 0$ for constituents $\pi_v$ of globally generic cuspidal representations.

## 6 References

[1] J.W. Cogdell, H.H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to $GL_N$*. Publ, Math. IHES **93** (2001), 5–30.

[2] J.W. Cogdell, H.H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the classical groups*. Preprint (2003).

[3] D. Ginzburg, S. Rallis, and D. Soudry, *Generic automorphic forms on $SO(2n+1)$: functorial lift to $GL(2n)$, endoscopy, and base change*. Internat. Math. Res. Notices **2001**, no. 14, 729–764.

[4] D. Jiang and D. Soudry, *The local converse theorem for $SO(2n+1)$ and applications*. Ann. of Math. **157** (2003), 743–806.

[5] D. Jiang and D. Soudry, *Generic representations and local Langlands reciprocity law for p-adic* $SO_{2n+1}$. Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002) (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.), Johns Hopkins University Press, Baltimore, to appear.

[6] H. Kim, *Applications of Langlands' functorial lift of odd orthogonal groups.* Trans. Amer. Math. Soc. **354** (2002), 2775–2796.

[7] D. Soudry, *On Langlands functoriality from classical groups to* $GL_n$. Asterisque, to appear. [IHP Lectures, Spring 2000]