Some generalized functionals and their Bessel functions

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Abstract

For each locally compact group and three of its closed subgroups with some relations among them, and their representations, we introduce a functional and a Bessel function attached to it whose behavior would allow us to study these functionals. Among the examples are Weyl integration formula for which Bessel functions are orbital integrals, as well as intertwining operators and different functionals such as Whittaker and Bessel functionals whose Bessel functions are again regular and twisted orbital integrals, as well as generalized Bessel functions which generalize the classical ones on $SL(2)$. They cover many of the integrals which have been studied in the subject so far.

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1 Introduction

In the recent proofs of Functoriality from quasisplit classical groups to GL_n [6, 7, 4, 11] the lack of the local Langlands conjecture at the ramified places was replaced by the stability of the local γ-factor under highly ramified twists. This result was proven by writing the γ-factor as a Mellin transform of a type of Bessel function [8, 9, 10]. Any generalization of this argument requires a better understanding of the asymptotics of these Bessel functions. The purpose of this paper is to generalize the notion of a Bessel function to a setting where analogous asymptotic expansions are known. Our generalization is such that orbital integrals become Bessel functions for a particular setting and one hopes that the germ expansions of orbital integrals near singular points will generalize to the general Bessel functions

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we define here. What we do in fact is generate the Bessel functions by means of some general functionals, which we define in this paper.

We will consider certain functionals associated to a locally compact group G, three of its closed subgroups H, H', and S, as well as representations $(\sigma, \mathcal{H}_\sigma)$ of H and $(\sigma', \mathcal{H}_{\sigma'})$ of H' and a function Θ on S. If we let $(I(\sigma), V(\sigma))$ be the representation of G compactly induced from σ, then the functionals are functionals on $V(\sigma) \times \mathcal{H}_{\sigma'}$ depending on Θ as a parameter, denoted $\Lambda(f, v'; \Theta)$. These will be defined in Section 2. They are an outgrowth of constructions that have arisen in the Langlands-Shahidi method of analyzing local L-functions, so naturally they include Whittaker functionals and intertwining operators. In addition, as examples we will have functionals associated to Bessel models as well as integral representations of L-functions and functionals representing the formal degree of a representation and giving branching formulas. The formalism seems quite robust.

To each of these functionals we then associate a Bessel function. These are defined in Section 6. This definition grew out of the work in [32] where Bessel functions first appeared in the Langlands-Shahidi method. In computing the Bessel functions for our functionals we find both standard orbital integrals and twisted orbital integrals as examples of our Bessel functions. We also recover the work in [32] from our formalism. In the last section we compute one Bessel function coming from the theory of integral representations of L-functions and recover the Bessel function used for the stability of γ-factors for $SO(2n+1)$ from [8].

The current paper consists mainly of outline the formalism we have developed and then working out what the formalism gives in many examples. Our long term goal is to use this formalism to understand the Bessel functions that arise in other arithmetic situations. We would eventually like to understand the Bessel functions related to twisted γ-factors $\gamma(s, \pi \times \tau, \psi)$. One outcome would be higher rank stability results; another, based on more or less explicit computation of the Bessel function, would be to show the equality of L- and ε-factors arising from different methods. As orbital and twisted orbital integrals now occur as examples of our Bessel functions, we are interested in developing a theory of germ expansions for our Bessel functions. We hope to address this in a subsequent paper.

It is our pleasure to present these ideas in this volume dedicated to Steve Kudla. We have both known Steve for many years and have highly valued his input and insights through the years. We hope to have many more such years ahead of us.

2 The Functionals

For any locally compact topological group G, we fix three closed subgroups H, H' and S and set $D = (G, H, H', S)$ to denote this data. Later on, we shall impose some relations between these groups.

Next, we let $(\sigma, \mathcal{H}_\sigma)$ and $(\sigma', \mathcal{H}_{\sigma'})$ be a pair of irreducible representations of H and H', where $\mathcal{H}_\sigma$ and $\mathcal{H}_{\sigma'}$ denote the spaces of σ and σ'. We shall assume the representations to be continuous with respect to the topologies of $\mathcal{H}_\sigma$ and $\mathcal{H}_{\sigma'}$ which we will assume to be defined by seminorms as Fréchet spaces.

Next data that we need is a (continuous) $H \cap H'$-invariant functional $\lambda \neq 0$ on

$\mathcal{H}_\sigma \otimes \mathcal{H}_{\sigma'}$, so $\lambda \in (\mathcal{H}_\sigma \otimes \mathcal{H}_{\sigma'})'$ satisfying

$$\lambda(\sigma(h)v \otimes \sigma'(h)v') = \lambda(v \otimes v') \tag{2.1}$$

for all $v \in \mathcal{H}_\sigma$, $v' \in \mathcal{H}_{\sigma'}$ and $h \in H \cap H'$. Here V' means the continuous dual of V for every V. (The topology on $\mathcal{H}_\sigma \otimes \mathcal{H}_{\sigma'}$ is the projective topology [36, 37].)

Consider the compactly induced representation

$$I(\sigma) = I_c(\sigma) \colon = c - \mathrm{ind}_H^G \sigma.$$

Thus its space is

$$V(\sigma) = \{f \colon G \to \mathcal{H}_\sigma | f \text{ is of compact support modulo } H \text{ and } f(hg) = \sigma(h)f(g)\}$$

on which G acts by right translations. We shall assume that functions in $V(\sigma)$ are appropriately smooth. In most cases, they will be in $C^\infty(G, \mathcal{H}_\sigma)$, the space of smooth functions (locally constant or infinitely differentiable, depending on the context).

Let Θ be a function on S. Let $\tilde{S}_{\lambda,\Theta} \subset S$ be a "stabilizer" of the pair (λ, Θ) in S

$$\tilde{S}_{\lambda,\Theta} = \{\tilde{s} \in S \mid \lambda(f(\tilde{s}s) \otimes v')\Theta(\tilde{s}s) = \lambda(f(s) \otimes v')\Theta(s) \text{ for all } s \in S\}$$

In what follows, we will also fix a subgroup $S_{\lambda,\Theta} \subset \tilde{S}_{\lambda,\Theta}$ whose definition we will discuss later.

We now define our functionals $\Lambda(f, v'; \Theta) = \Lambda_D(f, v'; \Theta)$ as

$$\Lambda(f, v'; \Theta) = \int\limits_{S_{\lambda,\Theta} \backslash S} \lambda(f(s) \otimes v')\Theta(s) ds. \tag{2.2}$$

The measure $ds = d_r s$ is a fixed right invariant Haar measure on the homogeneous space $S_{\lambda,\Theta} \backslash S$.

We shall now make our assumptions:

$H \cap H'$ normalizes S and centralizes $S_{\lambda,\Theta}$, (2.3)

$\Theta(hsh^{-1}) = \Theta(s)$ for $h \in H \cap H'$ and $s \in S$, (2.4)

the integral in (2.2) converges for all f and v', not necessarily absolutely. (2.5)

Remark 2.1. We shall be dealing mainly with $H \cap H'$ and we may as well assume H' is a subgroup of H. Moreover (2.4) can be removed if $H' = \{e\}$.

We shall now establish the main quasi–invariance property of (2.2).

Proposition 2.2. *Let $h \in H \cap H'$ and $s_1 \in S$. Then*

$$\Lambda(I(s_1 h)f, v'; \Theta) = \frac{d(hsh^{-1})}{ds} \Lambda(f, \sigma'(h^{-1})v'; R(h^{-1}s_1^{-1}h)\Theta), \tag{2.6}$$

for all $f \in V(\sigma)$ and $v' \in \mathcal{H}_{\sigma'}$, where $R(s')\Theta(s) = \Theta(ss')$ for $s, s' \in S$.

Proof: By definition the left hand side of (2.6) is equal to

$$\int\limits_{S_{\lambda,\Theta}\backslash S} \lambda(f(ss_1h)\otimes v')\Theta(s)ds = \int\limits_{S_{\lambda,\Theta}\backslash S} \lambda(f(sh)\otimes v')\Theta(ss_1^{-1})dn, \tag{2.7}$$

using the fact that ds is right invariant. The integral (2.7) can now be written as

$$\begin{aligned}
&\int\limits_{S_{\lambda,\Theta}\backslash S} \lambda(f(h\cdot h^{-1}sh)\otimes v')\Theta(ss_1^{-1})ds \\
&\qquad = \int\limits_{S_{\lambda,\Theta}\backslash S} \lambda(f(h^{-1}sh)\otimes \sigma'(h^{-1})v')\Theta(ss_1^{-1})ds \\
&\qquad = \frac{d(hsh^{-1})}{ds}\int\limits_{S_{\lambda,\Theta}\backslash S} \lambda(f(s)\otimes \sigma'(h^{-1})v')\Theta(hsh^{-1}s_1^{-1})ds \\
&\qquad = \frac{d(hsh^{-1})}{ds}\int\limits_{S_{\lambda,\Theta}\backslash S} \lambda(f(s)\otimes \sigma'(h^{-1})v')\Theta(sh^{-1}s_1^{-1}h)ds \\
&\qquad = \frac{d(hsh^{-1})}{ds}\Lambda(f,\sigma'(h^{-1})v';R(h^{-1}s_1^{-1}h)\Theta).
\end{aligned}$$

Here, we have used both (2.3) and (2.4) several times. The proposition is now proved. $\square$

Remark 2.3. If we begin with the data G, H, H' with representations σ, σ' and λ as above and then take $S=\{e\}$ and $\Theta = 1$ we can obtain λ as the associated Λ. Given $v\in\mathcal{H}_\sigma$, take $f_v \in V(\sigma)$ such that $f_v(e)=v$. Then

$$\Lambda(f_v, v';\Theta) = \lambda(v\otimes v').$$

3 First Examples

We start by some rather basic but important examples.

3.1 Formal degree

Let G be a p–adic reductive group and take $H=H'=S=G$. By (2.4) Θ is a class function on G. Let $\sigma=\pi$ be an irreducible unitary supercuspidal representation of G with central character ω_π. Take $\sigma'=\tilde{\pi}$, the contragredient of π. For our linear functional λ we take the conjugate of the duality pairing of $\mathcal{H}_\pi$ and $\mathcal{H}_{\tilde{\pi}}$, namely $\lambda(v\otimes\tilde{v}) = \overline{\langle v,\tilde{v}\rangle}$.

By definition $f\in V(\sigma)$ satisfies $f(gx)=\pi(g)f(x)$, $g\in G$, $x\in G$. If $f(e)=v$ then

$$\lambda(f(g)\otimes\tilde{v}) = \overline{\langle\pi(g)v,\tilde{v}\rangle},$$

where $\tilde{v} = v' \in \mathcal{H}(\tilde{\pi})$. Assume v and $\tilde{v}$ are chosen so that $\langle v, \tilde{v}\rangle = 1$. Let $Z = Z_G$ be the center of G and take $S_{\lambda,\Theta} = Z$. The measure $ds = dg$ is now a Haar measure on G. Assume the function Θ on G satisfies $\Theta(zg) = \omega_\pi(z)\Theta(g)$. Then

$$\Lambda(f, v'; \Theta) = \int\limits_{Z\backslash G} \overline{\langle \pi(g)v, \tilde{v}\rangle}\ \Theta(g)dg. \tag{3.1}$$

Let

$$R = \int\limits_{Z\backslash G} \overline{\langle \pi(g)v, \tilde{v}\rangle}\ \pi(g)dg.$$

Then by [17], Page 94,

$$\mathrm{trace}(R) = \int\limits_{Z\backslash G} |\langle \pi(g)v, \tilde{v}\rangle|^2 dg = d(\pi)^{-1},$$

where $d(\pi)$ is the formal degree of π. We thus have

Proposition 3.1. *With notation as above, one has*

$$\Lambda(f, v'; \Theta) = d(\pi)^{-1}\ \textit{mult.}(\chi_\pi, \Theta),$$

where mult.(χ_π, Θ) *is the multiplicity of the character* χ_π *of* π *in the class function* Θ. *In particular, if* $\Theta = \chi_\pi$,

$$\Lambda(f, v'; \Theta) = d(\pi)^{-1}.$$

3.2 Branching rules

Again let G be a p–adic reductive group. Assume $S = H' \subset H = G$. Then (2.3) is clearly satisfied. Moreover Θ is a class function on H'. Let $\sigma = \pi$ be an irreducible supercuspidal representation of H and take σ' to be a constituent of $\tilde{\pi}|H'$ which we assume appears discretely. Again $f \in V(\pi)$ satisfies $f(gx) = \pi(g)f(x)$. Let $v = f(e)$. Finally, let $\langle\ ,\ \rangle$ be the pairing between π and its contragredient $\tilde{\pi}$ and set

$$\lambda(v \otimes v') :\ = \overline{\langle v, v'\rangle}.$$

Take $S_{\lambda,\Theta} = Z' = Z_{H'}$. We then have:

Proposition 3.2. *Assume* $\Theta = \chi_{\sigma'}$, *the character of* σ' *and* $\langle v, v'\rangle = \langle f(e), v'\rangle = 1$. *Then*

$$\Lambda(f, v'; \Theta) = \int\limits_{Z'\backslash H'} \overline{\langle \pi(h')v, v'\rangle}\chi_{\sigma'}(h')dh' = \textit{mult.}(\sigma' : \tilde{\pi}|H')d(\sigma')^{-1}.$$

3.3 The case $H = H'$

When $H = H'$, one notes that a non–zero λ satisfying (2.1) amount to a non–zero H–invariant pairing between $\mathcal{H}_\sigma$ and $\mathcal{H}_{\sigma'}$. Thus σ' is the topological (continuous) dual of σ and we use $\langle\ ,\ \rangle$ to denote λ as $\langle v, v'\rangle = \lambda(v \otimes v')$.

We first note that since H normalizes S, HS is a closed subgroup of G. In particular, if $H \cap S = \{e\}$, then HS is a semi–direct product. Take $f \in V(\sigma)$ and choose a compact subset C of G such that $\mathrm{supp}(f) \subset HC$. Since $S \cap H = \{1\}$, S embeds into $H\backslash G$ and we may therefore assume that the integration over S is over the compact set $S \cap \overline{C}$, where $\overline{C}$ is the image C in $H\backslash G$.

Since $s \mapsto \langle f(s), v'\rangle$ is continuous on S, the integral converges so long as Θ is locally bounded. We collect this as follows.

Proposition 3.3. *Assume $H \cap S = \{e\}$. Then $\Lambda(f, v'; \Theta)$ converges absolutely for all $f \in V(\sigma)$ and every locally bounded function Θ.*

3.3.1 Example

Assume we are in a setting where $G = \mathbf{G}(k)$ is the set of k–points of a reductive group over a local field k and let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$. Let us take $H = H' = M = \mathbf{M}(k)$ and $S = N = \mathbf{N}(k)$, so that $H \cap S = \{e\}$. Then, taking $\Theta = 1$, the map

$$\Phi \colon f(\cdot) \mapsto \int\limits_N f(n\ \cdot)dn,$$

defined by

$$\langle \Phi(f), \tilde{v}\rangle = \Lambda(f, \tilde{v}; \mathbf{1})$$

for all $\tilde{v}$ in $\mathcal{H}(\tilde{\sigma})$, converges and defines a surjection from $c-\mathrm{ind}_M^G \sigma$ onto $\mathrm{Ind}_{MN}^G \sigma \otimes \mathbf{1}$. Note that the last space is naturally isomorphic to $c - \mathrm{Ind}_{MN}^G \sigma \otimes \mathbf{1}$ since $P\backslash G$ is compact. We remark that using our setting this can be vastly generalized to other inductions and in particular when Θ is any character of N.

3.4 The case of trivial H'

When $H' = \{e\}$, the conditions (2.3) and (2.4) are automatically satisfied for all S and Θ. Moreover, σ' being the trivial representation and $\mathcal{H}_{\sigma'} = \mathbb{C}$, allow us to identify $\mathcal{H}_\sigma \otimes \mathcal{H}_{\sigma'} \simeq \mathcal{H}_\sigma$ and thus λ will be a continuous functional on $\mathcal{H}_\sigma$. The resulting functional $\Lambda(f, z; \Theta)$, with $z \in \mathbb{C} = \mathcal{H}_{\sigma'}$, is then

$$\Lambda(f, z; \Theta) = \int\limits_{S_{\lambda,\Theta}\backslash S} z\lambda(f(s))\Theta(s)ds.$$

Note that $\Lambda(f, 1; \Theta)$ is the integral of $(\lambda \cdot f)\Theta$ over a quotient of an arbitrary closed subgroup of G and therefore may be looked at as a period integral. We will discuss this in Section 3.5.3.

3.5 Some global examples

Here we will assume G is either the adelic points of a connected reductive group $\mathbf{G}$ over a global field k or a covering of such a group, such as a metaplectic group.

3.5.1 Automorphy

Let $H = H' = G$. Let σ be an irreducible admissible representation of $H = G$ and let σ' be an irreducible admissible L^2-*automorphic* representation of $H' = G$, i.e., σ' appears in $L^2(\mathbf{G}(k)\backslash G; \omega)$, where $G = \mathbf{G}(\mathbb{A}_k)$ and ω is a unitary automorphic character of the center of G. Then

σ is L^2-automorphic if and only if there exists a σ' and $\lambda \neq 0$ satisfying (2.1).

Now assume $S = \{e\}$, $\Theta = \mathbf{1}$, and given v, take $f \in V(\sigma)$ with $f(e) = v$. Then we are in the setting of Remark 2.3 and

$$\lambda(v \otimes v') = \Lambda(f, v'; \mathbf{1}).$$

Thus σ is L^2-automorphic if and only if $\Lambda(f, v'; \mathbf{1}) \neq 0$ for some σ', $v' \in \mathcal{H}_{\sigma'}$ and $f \in V(\sigma) \simeq \mathcal{H}_\sigma$.

3.5.2 Eisenstein series

The pairings that arise in the theory of Eisenstein series and the decomposition of $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}))$ can be viewed as examples of our functionals. We again take $G = H = H' = \mathbf{G}(\mathbb{A})$ and we now take σ of the form

$$\sigma = \mathrm{Ind}_{MN}^{G}(\sigma_M \otimes \mathbf{1}),$$

where $\mathbf{P} = \mathbf{MN}$ is a parabolic subgroup of $\mathbf{G}$ and σ_M is a cuspidal representation of $M = \mathbf{M}(\mathbb{A})$. The induced representation σ may be reducible, but we will assume that it has a unique irreducible quotient. To conform with Langlands and Arthur [1, 3], we will assume $k = \mathbb{Q}$. Given $h \in \mathcal{H}_\sigma$, we consider the corresponding Eisenstein series

$$E_h(g) = \sum_{\gamma \in \mathbf{P}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{Q})} h(\gamma g).$$

Fix a sufficiently regular element $T \in \mathfrak{a}_0$, the real Lie algebra of $\mathbf{A}_0$, a maximal split torus of $\mathbf{G}$ contained in $\mathbf{M}$. If Λ^T is the Arthur–Langlands truncation operator attached to T, then $\Lambda^T E_h$ is square integrable on $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1$, where $\mathbf{G}(\mathbb{A})^1$ is the kernel of all the unramified characters of G. We then define

$$\lambda_T(h \otimes \varphi) = \langle \Lambda^T E_h, \overline{\varphi} \rangle, \tag{3.2}$$

where $\langle \ \ , \ \ \rangle$ denotes the L^2–inner product for $L^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)$ and we have $\varphi \in \mathcal{H}_{\sigma'} \subset \mathcal{A}^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})^1)$, the space of L^2-automorphic forms, so φ is K–finite and its K–span is a finite sum of irreducible representations of V_P. Here $K = \prod_v K_v$ is a maximal compact subgroup of G and V_P is the globally induced representation of G from $L^2_{\mathrm{disc}}(\mathbf{M}(\mathbb{Q})\backslash\mathbf{M}(\mathbb{A})^1)$ as explained in [3]. By choosing $S=\{e\}$ and $\Theta=1$ as usual we have that $\lambda_T(h\otimes\varphi)=\Lambda(f_h,\varphi;1)$.

The inner product (3.2) has been computed by Langlands in terms of the constant terms of the Eisenstein series E_h and $\overline{\varphi}$, if φ itself is an Eisenstein series (see [3], Proposition 15.3, Page 86), and

$$\langle\Lambda^T E_h,\overline{\varphi}\rangle=\langle\Lambda^T E_h,\Lambda^T\overline{\varphi}\rangle$$

by basic properties of Λ^T. In general, we cannot assume that the pairing will vanish unless φ is an Eisenstein series as the theory CAP representations shows otherwise [24].

3.5.3 Period integrals

We now show how period integrals for cusp forms can be put into our setting.

We take $H=G=\mathbf{G}(\mathbb{A})$ and $\sigma=\pi$, a cuspidal automorphic representation of G. We take $H'=\{e\}$ so $\sigma'=\mathbb{C}$ as above. We take λ, which in the setting of Section 3.4 is now a continuous functional on $\mathcal{H}_\pi$, to be the evaluation at identity, i.e., $\lambda(\varphi)=\varphi(e)$, so that

$$\lambda(\pi(g)\varphi)=\varphi(g) \tag{3.3}$$

for every cusp form $\varphi\in\mathcal{H}_\pi\subset L^2_0(\mathbf{G}(k)\backslash G^1)$, where L^2_0 denotes the subspace of cuspidal L^2-automorphic forms. Given $\varphi\in\mathcal{H}_\pi$, we choose $f\in V(\pi)$, the space of $I(\pi)$, such that $f(e)=\varphi$, or (3.3) holds.

Let $\mathbf{S}$ be an arbitrary closed subgroup of $\mathbf{G}$ and $S=\mathbf{S}(\mathbb{A})$. Then, given any function Θ on $\mathbf{S}(k)\backslash\mathbf{S}(\mathbb{A})$, we may take $S_{\lambda,\Theta}=\mathbf{S}(k)$ so that we have

$$\Lambda(f,1;\Theta)=\int\limits_{\mathbf{S}(k)\backslash\mathbf{S}(\mathbb{A})}\varphi(s)\Theta(s)ds.$$

When $\Theta\equiv 1$, this is clearly a period integral for φ along the cycle S in the classical sense.

3.5.4 Integral representations

Related to the global period integral discussed here are the periods occurring in integral representations of L-functions. As a basic example we obtain the Hecke integral for the standard L-function for $\mathbf{G}=GL_2$. So we take $G=H=\mathbf{G}(\mathbb{A})$ and $\sigma=\pi$ a cuspidal representation of $GL_2(\mathbb{A})$. $H'=\{e\}$ so that $\mathcal{H}_{\sigma'}=\mathbb{C}$. Then for $\varphi\in\mathcal{H}_\pi$ we take

$$\lambda(\varphi\otimes 1)=\lambda(\varphi)=\varphi(e)$$

which becomes the linear functional on $\mathcal{H}_\pi$ given by evaluation at the identity. For S we take $\mathbb{A}^\times$ embedded as a subgroup of G in the upper left corner and $\Theta(a)=|a|^s$. (Now, as usual, we let $a\in\mathbb{A}^\times=S$ and $s\in\mathbb{C}$.) Then $S_{\lambda,\Theta}=k^\times$ and

$$\Lambda(\varphi,1;|\cdot|^s)=\int_{k^\times\backslash\mathbb{A}^\times}\varphi\begin{pmatrix}a & \\ & 1\end{pmatrix}|a|^s d^\times a$$

is the Hecke integral representing the standard L-function for GL_2 [18] We will return to a more complete discussion of integral representations as examples of our functionals in Section 8.

4 Further Examples

In this section we will discuss a family of examples which very naturally fit into this scheme and were our main motivation for this program.

4.1 The general setting

Let k be a local field and $\mathbf{G}$ a connected reductive group over k. For every k–group $\mathbf{H}$, we let $H = \mathbf{H}(k)$. We shall fix a minimal parabolic subgroup $\mathbf{P}_0$ of $\mathbf{G}$ with a Levi decomposition $\mathbf{P}_0 = \mathbf{M}_0\mathbf{N}_0$ in which $\mathbf{N}_0$ is the unipotent radical and $\mathbf{M}_0$ is a Levi factor, a splitting of the reductive quotient $\mathbf{P}_0/\mathbf{N}_0$. We let $\mathbf{A}_0$ be the split component of $\mathbf{M}_0$, a maximally split torus of $\mathbf{G}$ which is the maximal split torus in the center of $\mathbf{M}_0$. We let $W_0 = W(\mathbf{G}, \mathbf{A}_0)$ be the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. Moreover, let $\Phi^+ = \Phi^+(\mathbf{A}_0, \mathbf{N}_0)$ be the set of roots of $\mathbf{A}_0$ in Lie $(\mathbf{N}_0)$. Now, let $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup of $\mathbf{G}$. Assume $\mathbf{P} \supset \mathbf{P}_0$, $\mathbf{N} \subset \mathbf{N}_0$ and $\mathbf{M}_0 \subset \mathbf{M}$. Then there is a unique simple root $\alpha \in \Delta^+ \subset \Phi^+$ which is the unique simple root of $\mathbf{A}_0$ in Lie$(\mathbf{N}) \subset$ Lie$(\mathbf{N}_0)$. Finally let $\mathbf{A} \subset \mathbf{A}_0$ be the split component of $\mathbf{M}$. Denote by $X^*(\mathbf{A})_k$ the group of k–rational characters of $\mathbf{A}$. Set $\mathfrak{a}^*_{\mathbb{C}} = X^*(\mathbf{A})_k \otimes_{\mathbb{Z}} \mathbb{C}$. Finally, if w_ℓ and $w_{\ell,\mathbf{M}}$ are representatives of the long elements of $W(\mathbf{G}, \mathbf{A}_0)$ and $W(\mathbf{M}, \mathbf{A}_0)$, chosen as in [27], we set $w_0 = w_\ell \cdot w_{\ell,\mathbf{M}}^{-1}$.

Finally, we let $\mathbf{N}^-$ be the opposite of $\mathbf{N}$ and let $\mathbf{N}' = w_0\mathbf{N}^-w_0^{-1}$.

We now put in this data in our scheme. Let $G = \mathbf{G}(k)$, $M = \mathbf{M}(k)$, $N = \mathbf{N}(k)$, $N^- = \mathbf{N}^-(k)$ and $N' = \mathbf{N}'(k)$. Let $H' \subset M$ be a closed subgroup. Let π be an irreducible admissible representation of M and σ' an irreducible representation of H'. We will take $H = P$ and as usual extend π to $\pi \otimes \mathbf{1}$ on P. We take $S = N^-$. Clearly $H' \subset M \subset H = P$ normalizes $S = N^-$. To take care of convergence issues and more importantly as one of the objects of interest we fix $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and then deform π to $\pi_\nu = \pi \otimes \exp\langle \nu + \rho_P, H_M(\ )\rangle$, where H_M is the standard map from M into $\mathfrak{a} = \mathrm{Hom}(X(M)_k, \mathbb{R})$, satisfying

$$\exp\langle \chi, H_M(m)\rangle = |\chi(m)|_k,$$

for all $\chi \in X(M)_k$. We will take π_ν as the representation σ of $H = P$ in our scheme. Note that $\mathcal{H}_\sigma = \mathcal{H}_{\pi_\nu} = \mathcal{H}_\pi$. Then our $I(\sigma)$ will become a normalized induced representation

$$I(\sigma) = I(\nu, \pi) = \mathrm{Ind}_{MN}^G(\pi \otimes \exp\langle \nu, H_M(\ )\rangle \otimes \mathbf{1})$$

acting on the space $V(\nu, \pi)$.

Assume there exists a non–zero λ satisfying (2.1). Let Θ be a function on N^- satisfying (2.4) and take $S_{\lambda,\Theta} = \{e\} \subset N^-$.

Next, we define an induced functional by

$$\int\limits_{N'} \lambda(f(w_0^{-1}n') \otimes v')\ \Theta'(n')dn', \tag{4.1}$$

where

$$\Theta'(n') = \Theta(w_0^{-1}n'w_0) = \Theta(n^-),$$

$n' \in N'$ and $n^- = w_0^{-1}n'w_0 \in N^-$. Using equation (2.4) satisfied by Θ

$$\Theta'(h^{-1}n'h) = \Theta'(n'), \tag{4.2}$$

for all $h \in w_0H'w_0^{-1}$ and $n' \in N'$. In fact,

$$\begin{aligned}
\Theta'(h^{-1}w_0n^-w_0^{-1}h) &= (Ad(w_0^{-1})\Theta')(h^{'-1}n^-h') \\
&= \Theta(h^{'-1}n^-h') \\
&= \Theta(n^-) \\
&= \Theta'(w_0n^-w_0^{-1})
\end{aligned}$$

which follows from equation (2.4), implies equation (4.2) with $n' = w_0n^-w_0^{-1}$ and $h' = w_0^{-1}hw_0$.

We now note that (4.1) is in fact equal to

$$\Lambda(R_{w_0^{-1}}f, v'; \Theta) = \int\limits_{N^-} \lambda((R_{w_0^{-1}}f)(n^-) \otimes v')\Theta(n^-)dn^-. \tag{4.3}$$

We now address the convergence condition (2.5) for the functional $\Lambda(R_{w_0^{-1}}f, v'; \Theta)$ in (4.3) or equivalently for (4.1).

We will assume $\mathcal{H}_\pi$ and $\mathcal{H}_{\sigma'}$ are topologized by seminorms $\{\|\ \ \|_i\}$ and $\{\|\ \|'_j\}$ and thus

$$|\lambda(v_1 \otimes v_2)| \le \sum_{i,j} c_{ij}^\lambda \|v_1\|_i \|v_2\|'_j.$$

with real constants c_{ij}^λ and $\sum\limits_{i,j}$ depending on λ. For example, we may assume σ to be a differentiable representation on the Frechét space $\mathcal{H}_\pi$; it could be the representation on the subspace of differentiable vectors inside a Banach or Hilbert (unitary) representation space. When k is p–adic we can just take the trivial locally convex topology on $\mathcal{H}_\pi$ and $\mathcal{H}_{\sigma'}$ for which every seminorm is continuous. On the other hand, for an archimedean k the choices matter.

We further will assume Θ is bounded on N^-. The absolute convergence of (4.1) will then become the same as that of class one intertwining operators as addressed in Lemma 2.3 of [27]. We collect these as

Proposition 4.1. *a) Assume Θ is bounded and $f \in V(\nu, \pi)$, where π is a differentiable representation on the Fréchet space $\mathcal{H}_\pi$. Then $f \in C^\infty(G, \mathcal{H}_\pi)$. Further assume the restriction of λ to $\mathcal{H}_\pi$ is continuous in this topology, where we have* *identified* $\mathcal{H}_\pi \hookrightarrow \mathcal{H}_\pi \otimes \mathcal{H}_{\sigma'}$ *by* $\mathcal{H}_\pi \simeq \mathcal{H}_\pi \otimes v'$. *Then the induced functional (4.1) or equivalently* $\Lambda(f, v'; \Theta)$ *converges absolutely for* $Re(\nu(H_\alpha))$ *sufficiently large, depending only on* π.

b) Assume the restriction of λ *to* $\mathcal{H}_\pi \otimes v'$ *is continuous for every* $v' \in \mathcal{H}_{\sigma'}$, *i.e.,* λ *is continuous in the first variable. Then* $\Lambda(f, v'; \Theta)$ *satisfies (2.6) for* $Re(\nu(H_\alpha))$ *sufficiently large depending only on* π.

Remark 4.2. For the dependence of the lower bound for $\mathrm{Re}(\nu(H_\alpha))$ on π we refer to Lemma 6.5 of [20].

Remark 4.3. We note that G can be replaced by any of its central coverings $\widetilde{G}$. One will then need to replace P and M with their full preimages $\widetilde{P}$ and $\widetilde{M}$ in $\widetilde{G}$. We note that the map $\widetilde{N}_0 \to N_0$ as well as those for N, N' and N^- all split. Although the theory of intertwining operators is not as complete as in the algebraic case, one expects the same result to hold in the covering setting as well.

We will now discuss special cases of the general setting in this section.

4.2 Intertwining operators

We will start with the case that $H = H' = M$ with $\sigma' \simeq \tilde{\pi}$, and take $\lambda(v \otimes \tilde{v}) = \langle v, \tilde{v} \rangle$, a matrix coefficient or a pairing between $\mathcal{H}_\pi$ and $\mathcal{H}_{\tilde{\pi}}$, with $v' = \tilde{v}$. We take $S = N^-$ and now $\Theta = \mathbf{1}$.

The functional $\Lambda(R_{w_0^{-1}} f, v'; \Theta)$ then becomes a matrix coefficient for the standard intertwining operator

$$\Lambda(R_{w_0^{-1}} f, \tilde{v}; \mathbf{1}) = \int_{N^-} \langle f(n^- w_0^{-1}), \tilde{v} \rangle \, dn^- = \langle A(\nu, \pi, w_0) f(e),\ \tilde{v} \rangle, \tag{4.4}$$

where $f \in V(\nu, \pi)$.

The significance of the theory of intertwining operators is well–known with many applications both in local and global theory, ranging from determining Plancherel measures, reducibility points for $I(\nu, \pi)$ and the definition of local L–functions defined in Langlands–Shahidi method [29].

4.3 Whittaker functionals

We now assume $\mathbf{G}$ is quasisplit. Then $\mathbf{P}_0$ is a Borel subgroup of $\mathbf{G}$. We use $\mathbf{U}$ to denote $\mathbf{N}_0$ and set $\mathbf{U_M} = \mathbf{U} \cap \mathbf{M}$. We then have $U_M = \mathbf{U_M}(k)$ and we let χ denote a generic character of $U = \mathbf{U}(k)$. We take $H = MN = P$ and $H' = U_M$. We take σ' to be χ^{-1}, the inverse of the generic character χ restricted to $H' = U_M$.

The function λ will need to satisfy

$$\lambda(\pi(u) v \otimes v') = \chi(u) \lambda(v \otimes v'). \tag{4.5}$$

Since $\mathcal{H}_{\sigma'} \simeq \mathbb{C}$, we may fix v' and identify $\mathcal{H}_\pi \otimes \mathcal{H}_{\sigma'} \simeq \mathcal{H}_\pi$ by

$$v \otimes v' \mapsto v$$

and in particular get a functional $\lambda_{v'}$ on $\mathcal{H}_\pi$ by $\lambda_{v'}(v) = \lambda(v \otimes v')$. By (4.5), it then satisfies the standard property of a Whittaker functional

$$\lambda_{v'}(\pi(u)v) = \chi(u)\lambda_{v'}(v)$$

for all $u \in U_M$.

Recall that $\Theta(n^-) = \Theta'(w_0 n^- w_0^{-1}) = \Theta'(n')$, where $n^- \in N^- = S$ and $n' \in N'$. We let $\Theta' = \chi^{-1}$, the restriction of χ^{-1} to $N' \subset U$, and then take Θ to be the associated character of N^-. Any character of N^- will satisfy

$$\Theta(un^-u^{-1}) = \Theta(n^-)$$

for all $u \in U_M = H'$ and thus (2.4) is automatically satisfied. The functional $\Lambda(R_{w_0^{-1}}f, v'; \Theta)$ is then the canonical induced χ-Whittaker functional

$$\int_{N^-} \lambda_{v'}((R_{w_0^{-1}}f)(n^-))\Theta(n^-)dn^- = \int_{N'} \lambda_{v'}(f(w_0^{-1}n'))\chi^{-1}(n')dn' \tag{4.6}$$

for the induced representation $I(\nu, \pi)$ (cf. [26]).

In fact, the content of equation (2.6) in this case is that $\Lambda(R_{w_0^{-1}}f, v'; \Theta)$ is simply a Whittaker functional for $I(\nu, \pi)$ with respect to $\chi(u)$ for $u \in U$.

4.4 Bessel and other functionals

Again in the setting of this section, we will take $H' \subset M$ to be any subgroup for which π has a non–zero functional transforming according to the representation σ' of H', i.e., a non–zero functional λ satisfying (2.1). One will again have an induced functional transforming according to $\sigma'(h^{-1})\Theta^{-1}(n^-)$ for $h \in H'$ and $n^- \in N^- = S$. Recall that Θ must satisfy (2.4).

One interesting example of this is the work of Friedberg and Goldberg in [12]. We will now elaborate how their work connects to our general setting.

The group G in [12] is the k–points of either $\mathbf{G} = SO_{2r+1}, U_{2r+1}, SO_{2r}$ or U_{2r} for a positive integer r, where k is a p–adic field of characteristic zero. We will call the first two cases "odd" and the other two "even". We fix the Borel subgroup $\mathbf{B} = \mathbf{TU}$ of upper triangular matrices inside these groups. We will take $H = MN$, $M = G_1 \times G(m)$, where G_1 is either $GL_n(k)$, or $GL_n(K)$, in which case $\mathbf{G}$ is unitary and defined by K/k, $[K : k] = 2$, and N is a unipotent subgroup $N \subset U$ such that $\mathbf{MN}$ is a parabolic subgroup of $\mathbf{G}$.

Write π, the irreducible representation of M, as $\pi = \pi_1 \otimes \tau$ with σ_1 on G_1 and τ on $G(m)$, with $\mathbf{G}(m) = SO_{2m+1}$, U_{2m+1}, SO_{2m}, U_{2m}. Assume τ has a Bessel model of minimal rank ℓ_1 (cf. Definitions 1.3 and 1.5 of [12]). Let $\ell_0 = 2\ell_1 + 1$ or $2\ell_1$ according as $\mathbf{G}$ is odd or even. Let $\ell = m - \ell_1$ and let U_ℓ be the subgroup of upper unipotent elements in G for which the middle $\ell_0 \times \ell_0$ block is the identity I_{ℓ_0}. We take $S = U_\ell$ and let Θ be a character of U_ℓ whose restriction χ_1^{-1} to $U_\ell \cap GL_n(E)$, $E = k$ or K accordingly, is generic while $\chi\colon = \Theta^{-1}|U_\ell \cap G(m)$ is the character χ defining the Bessel model of τ, as in page 136 of [12]. Let

$$\mathbf{M}_\ell = \left\{ \begin{pmatrix} I_{\ell+n} & & \\ & g & \\ & & I_{\ell+n} \end{pmatrix} \in \mathbf{G} \right\}$$

and $M_\ell = \mathbf{M}_\ell(k)$. If $\chi^g(u) = \chi(g^{-1}ug)$ for $g \in M_\ell$, we let

$$M_\chi = \{g \in M_\ell | \chi^g = \chi\} = Stab_{M_\ell}(\chi).$$

Let ω be the irreducible representation of M_χ so that $\omega_\chi = \omega \otimes \chi$ is the corresponding representation of the group $R_\chi = M_\chi(U_\ell \cap G(m))$, defining the Bessel model of τ.

Going back to our setting, we take

$$H' = U_1 R_\chi,$$

where $U_1 = U_\ell \cap GL_n(E)$, $E = k$ or K accordingly. Note that $H' \subset H = MN$. We take $\sigma' = \chi_1^{-1} \otimes (\tilde{\omega} \otimes \chi^{-1})$ as our representation of H'. We note that Θ satisfies (2.4). Now our functional $\Lambda(R_{w_0^{-1}} f, v'; \Theta)$ becomes the one given in equation (3.1), page 155 of [12], or rather its limit (Lemma 3.4 of [12]), if one takes $v' = \tilde{v}$.

Remark 4.4. More generalized models, such as Gelfand-Graev models or Fourier-Jacobi models, have been studied by Ginzburg, Rallis, and Soudry [34], as well as Ginzburg, Piatetski-Shapiro, and Rallis [14], and for each one of them one may consider the corresponding functionals Λ as in the cases of Whittaker and Bessel models discussed in the previous two sections. A similar theory is expected to hold when G is a covering group, such as a metaplectic group, as well [34].

5 Intertwinings and Generalized Local Coefficients

Each of our functionals is defined by a set of data. We set

$$D = (G, H, H', S, \sigma, \sigma', \lambda, \Theta, v')$$

to denote the set. If

$$D_1 = (G_1, H_1, H_1', S_1, \sigma_1, \sigma_1', \lambda_1, \Theta_1, v_1')$$

is another data, we would like to study any intertwining which exists between them.

We note that we have induced representations $I(\sigma) = c - \mathrm{ind}_H^G \sigma$ and $I(\sigma_1) = c - \mathrm{ind}_{H_1}^{G_1} \sigma_1$. To have any intertwining we have to assume $G = G_1$. Even if we assume $H = H_1$, and use the Frobenius reciprocity law

$$\mathrm{Hom}_G(I(\sigma), I(\sigma_1)) \simeq \mathrm{Hom}_H(I(\sigma), \sigma_1),$$

defined by $A \mapsto A_H$, $A_H(f) = Af(e)$, f in the space of $I(\sigma)$, not much insight will be achieved on whether there are any non–zero A.

5.1 Defining Objects

To proceed we make some natural assumptions and consider a larger class of maps between $I(\sigma)$ and $I(\sigma_1)$. We may and will assume $H' \subset H$ and $H'_1 \subset H_1$. Our assumptions are then as follows.

(5.1) *$H' \cap S = H'_1 \cap S_1 = \{1\}$ and thus $H'S$ and H'_1S_1 are closed subgroups of G which are semi–direct products in which H' and H'_1 are normal subgroups.*

(5.2) *There exists an isomorphism $\phi : H'S \xrightarrow{\sim} H'_1S_1$, sending H' to H'_1 and S to S_1. Write $\phi(h) = h_1$ and $\phi(s) = s_1$ for $h \in H'$ and $s \in S$ with $h_1 \in H'_1$ and $s_1 \in S_1$. We transfer the measure ds to ds_1 on S_1 by $d(\phi(s)) = ds$.*

(5.3) *$\Theta_1 = \Theta \circ \phi^{-1}$. Note that if Θ satisfies (2.7) for H' and S, then so does Θ_1 for H'_1 and S_1.*

(5.4) *The representations σ' and σ'_1 commute with ϕ, i.e., there exists an isomorphism $A_\phi : \mathcal{H}_{\sigma'} \xrightarrow{\sim} \mathcal{H}_{\sigma'_1}$ such that*

$$A_\phi(\sigma'(h)v') = \sigma'_1(\phi(h))A_\phi(v'),$$

$h \in H'$, $v' \in \mathcal{H}_{\sigma'}$.

We will call such a ϕ satisfying (5.2)–(5.4) *admissible*.

We note that if $\phi(s) = s_1$ and $\phi(h) = h_1$ as above, then by (5.3)

$$(R(h_1^{-1}s_1^{-1}h_1)\Theta_1)(\tilde{s}_1) = \Theta(\tilde{s}h^{-1}s^{-1}h) = (R(h^{-1}s^{-1}h)\Theta)(\tilde{s}), \tag{5.5}$$

where $\tilde{s}_1 = \phi(\tilde{s})$ for $\tilde{s} \in S$ with $\tilde{s}_1 \in S_1$.

Example. Take $\mathbf{G} = GL_{m+n}$ with $m, n \in \mathbb{N}$. Let U_j be the subgroup of upper unipotent elements of $GL_j(k)$. Take $H' = U_m \times U_n$ and $S = N^- = \mathbf{N}^-(k)$ where $\mathbf{N}^-$ the unipotent radical of the maximal parabolic subgroup whose Levi subgroup is $GL_m \times GL_n$ and contains the opposite unipotent group $\mathbf{U}^-_{m+n}$. Then $H'N^- = U^-_{m+n}$. Change the role of m and n and take the subgroups $H'_1 = U_n \times U_m$ and $S_1 = N_1^-$ of the corresponding lower unipotent subgroup. Let w_ℓ be a representative for the long element of the Weyl group of $\mathbf{G}$ which fixes the splittings in $\mathbf{U}_{m+n}$ and $\mathbf{U}^-_{m+n}$. Let $\phi(u) = w_0{}^t u^{-1} w_0^{-1}$ for $u \in U^-_{m+n}$. When $n \neq m$, $H' \neq H'_1$ and $N^- \neq N_1^-$, but $H'N^- = H'_1N_1^-$ and ϕ is a group isomorphism from H' and N^- onto H'_1 and N_1^-, respectively. In the context of Section 4.3, we can use

$$\chi(u) = \chi(u_{12} + u_{23} + \ldots + u_{m+n-1,m+n})$$

for σ' and in what follows get the local coefficients and thus root numbers for pairs of representations on $GL_m(k)$ and $GL_n(k)$.

5.2 Intertwinings and local coefficients

Our intertwinings are now maps $A : I(\sigma) \to I(\sigma_1)$ satisfying

$$A \circ I(hs) = I_1(\phi(hs)) \circ A \tag{5.6}$$

for all $h \in H'$, $s \in S$. We denote the subspace of such maps by $\mathrm{Hom}^{\phi}_{H'S}(I(\sigma), I(\sigma_1))$.

Assume $H'S = H'_1S_1 \subset G$. Every element $hs \in H'S$ can now be also written as $hs = h_1s_1$ with $h_1 = \phi(h)$ and $s_1 = \phi(s)$. Then every $A \in \mathrm{Hom}_G(I(\sigma), I(\sigma_1))$ belongs to $\mathrm{Hom}^{\phi}_{H'S}(I(\sigma), I(\sigma_1))$ for any admissible ϕ.

Take $A \in \mathrm{Hom}^{\phi}_{H'S}(I(\sigma), I(\sigma_1))$ and f in the space $V(\sigma)$ of $I(\sigma)$ and consider $\Lambda_1(Af, \Theta_1, v'_1)$. Then for each $h \in H'$ and $s \in S$ we have

$$\Lambda_1(AI(hs)f, v'_1; \Theta_1) = \Lambda_1(I_1(h_1s_1)Af, v'_1; \Theta_1). \tag{5.7}$$

By Proposition 2.2, (5.7) equals

$$\frac{d(h_1s_1h_1^{-1})}{ds_1}\, \Lambda_1(Af, \sigma'_1(h_1^{-1})v'_1; R(h_1^{-1}s_1^{-1}h_1)\Theta_1). \tag{5.8}$$

Define a new functional $\Lambda_1 \cdot A$ by

$$(\Lambda_1 \cdot A)(f, v'; \Theta) = \Lambda_1(Af, A_{\phi}(v'); \Theta_1), \tag{5.9}$$

where $\Theta_1 = \Theta \cdot \phi^{-1}$. Then combining (5.7) and (5.8), (5.4) and (5.5) imply

$$\begin{aligned}(\Lambda_1 \cdot A)(I(hs)f, v'; \Theta) &= (d(h_1s_1h_1^{-1})/ds_1)\Lambda_1(Af, \sigma'_1(h_1^{-1})A_{\phi}(v'); R(h_1^{-1}s_1^{-1}h_1)\Theta_1)\\ &= \frac{d(h_1s_1h_1^{-1})}{ds_1}\, \Lambda_1(Af, A_{\phi}(\sigma'(h^{-1})v'); R(h_1^{-1}s_1^{-1}h_1)\Theta_1)\\ &= \frac{d(hsh^{-1})}{ds}\, (\Lambda_1 \cdot A)(f, \sigma'(h^{-1})v'; R(h^{-1}s^{-1}h)\Theta),\end{aligned}$$

using

$$d(\phi(hsh^{-1})) = d(h_1s_1h_1^{-1}).$$

Thus $\Lambda_1 \cdot A$ satisfies the same invariance properties as Λ.

The generalized local coefficients can now be defined when the space of all functionals $\Lambda(-, -; \Theta)$ satisfying (2.6) is one dimensional.

Proposition 5.1. *Fix ϕ and A_{ϕ} as in (5.2) and (5.4). Take $A \in Hom^{\phi}_{H'N^-}(I(\sigma), I(\sigma_1))$. Assume the space of functionals $\Lambda(-, -; \Theta)$ satisfying (2.6) is one dimensional. Then there exists a constant $C(A, \phi, A_{\phi}, \sigma, \sigma'; \Theta)$ such that*

$$C(A, \phi, A_{\phi}, \sigma, \sigma'; \Theta)(\Lambda_1 \cdot A) = \Lambda, \tag{5.10}$$

for any $A \in Hom^{\phi}_{H'N^-}(I(\sigma), I(\sigma_1))$.

Remark 5.2. This gives a very general definition of a local coefficient for any set of data. It avoids many issues which show up with the definition of these coefficients in special cases [26, 28, 29], e.g., compatibility (cf.[29]), by identifying different datum by means of ϕ and A_{ϕ} in a natural way.

6 Computing the Functionals; Generalized Bessel Functions

We will continue with the harmless assumption that $H' \subset H$. Conditions (2.3) and (2.4) allow us to compute these functionals by means of the group action coming from (2.3). We simply break the integral as one over H' and the other over the orbits of S under the action of H' which we may denote by $H' \backslash S$. It must be understood that H' is not necessarily a subgroup of S.

6.1 Generalized Bessel functions

We will first address the integration over H'. We have fixed λ, v' and Θ subject to condition (2.1), (2.3), (2.4) and (2.5). Given $s \in S$, define the centralizer of s in H' by

$$H'_s = \{h \in H' | h^{-1}sh = s\}.$$

We then define the *generalized Bessel function* $B_f(s)$ attached to a function $f \in V(\sigma)$ by

$$B_f(s) = \int\limits_{H'_s \backslash H'} \lambda(f(h^{-1}sh) \otimes v') d\dot{h} = \int\limits_{H'_s \backslash H'} \lambda(f(sh) \otimes \sigma'(h)v') d\dot{h}, \tag{6.1}$$

$d\dot{h} = dh/dh_s$, where dh and dh_s are suitable invariant measures on H' and H'_s. If we let $\overline{S}$ denote the space $H' \backslash S$ of H'-orbits in S, then we find

$$\begin{aligned}
\Lambda(f, v'; \Theta) &= \int_{S_{\lambda,\Theta} \backslash S} \lambda(f(s) \otimes v') \Theta(s)\ ds \\
&= \int_{S_{\lambda,\Theta} \backslash \overline{S}} \left(\int_{H'_s \backslash H'} \lambda(f(h^{-1}sh) \otimes v') dh \right) \Theta(s)\ d\overline{s} \qquad (6.2) \\
&= \int_{S_{\lambda,\Theta} \backslash \overline{S}} B_f(s) \Theta(s)\ d\overline{s}.
\end{aligned}$$

Note that since H' centralizes $S_{\lambda,\Theta}$ by (2.3), $S_{\lambda,\Theta}$ injects into $\overline{S}$ as a subgroup.

If the convergence in (2.5) is absolute, then by Fubini (6.1) will also converge absolutely. This may not be the case for a given data, but it is possible that given f, the integration can be accomplished if one integrates over a suitably large compact subset Y of S. We can therefore write

$$\Lambda_Y(f, v'; \Theta) = \int\limits_{S_{\lambda,\Theta} \backslash S} \lambda(f(s) \otimes v') \varphi_Y(s) \Theta(s) ds, \tag{6.3}$$

where φ_Y is the characteristic function of the set Y. This happens when k is a p–adic local field and λ is either a Whittaker or a Bessel functional for the corresponding representation σ. The generalized Bessel function will now be called a *generalized partial Bessel function*. It will be given by

$$\begin{aligned} B_{f,Y}(s) &= \int\limits_{H'_s\backslash H'} \lambda(f(h^{-1}sh)\otimes v')\varphi_Y(h^{-1}sh)d\dot{h} \\ &= \int\limits_{H'_s\backslash H'} \lambda(f(sh)\otimes \sigma'(h)v')\varphi_Y(h^{-1}sh)d\dot{h}. \end{aligned} \tag{6.4}$$

so that

$$\Lambda_Y(f,v';\Theta) = \int_{S_{\lambda,\Theta}\backslash\overline{S}} B_{f,Y}(s)\Theta(s)\ d\overline{s}.$$

Even the convergence of (6.4) is non–trivial [8, 32, 9, 10].

6.2 Orbital integrals

A beautiful example of this appears as that of the convergence of orbital integrals for regular semisimple elements of reductive groups, if we analyze the case in Section 3.1. Recall that in this situation $G = H = H' = S$ and $S_{\lambda,\Theta} = Z$. The functional (3.1) is

$$\Lambda(f,v';\Theta) = \int\limits_{Z\backslash G} \overline{\langle\pi(g)v,\tilde{v}\rangle}\Theta(g)dg = \int\limits_{Z\backslash G} \tilde{f}(g)\Theta(g)dg, \tag{6.5}$$

where $\tilde{f}(g) = \overline{\langle\pi(g)v,\tilde{v}\rangle}$ is in $C_c^\infty(Z\backslash G,\overline{\omega}_\pi)$, while $\Theta(zg) = \omega_\pi(z)\Theta(g)$, $\omega_\pi\overline{\omega}_\pi = 1$.

The Bessel function $B_f(s)$ in this situation is an integral over the conjugacy class of $s \in S = G$. Up to a set of measure zero, we may realize the set of equivalence classes of G under conjugation as the disjoint union of regular conjugacy classes of Cartan subgroups of G. Let $\{T_i\}$ be a complete set of conjugacy classes of maximal tori in G. Then we may take $s = \gamma \in T_i$ a regular element. Then the stabilizer $G_\gamma = T_i$ and

$$B_f(\gamma) = \int\limits_{G_\gamma\backslash G} \lambda(f(h^{-1}\gamma h)\otimes v')d\dot{h} = \int_{T_i\backslash G} \tilde{f}(h^{-1}\gamma h)d\dot{h} = I(\gamma,\tilde{f}).$$

$I(\gamma,\tilde{f})$ is the *orbital integral* of a regular semisimple element $\gamma \in T_i$, where the measure $d\dot{g}$ is defined by

$$\int\limits_G \varphi(g)dg = \int\limits_{T_i\backslash G} (\int\limits_{T_i} \varphi(\gamma h)d\gamma)d\dot{h},$$

By the Selberg principle, $I(\gamma,\tilde{f}) = 0$ unless T_i is elliptic (Lemma 45 of [17]).

Then by Lemma 42 of [17] the functional $\lambda(f,v';\Theta)$ in (6.5) is given by the Weyl integration formula

$$\Lambda(f,v';\Theta) = \int_{Z\backslash G} \tilde{f}(g)\Theta(g)dg = \sum_{\{T_i\}} \int\limits_{Z\backslash T_i} I(\gamma,\tilde{f})\Theta(\gamma)[W(T_i)]^{-1}|D(\gamma)|d\gamma.$$

The sum of integrals

$$\sum_{\{T_i\}} \int_{Z\backslash T_i}$$

represents the integral over $Z\backslash\overline{S}$, the equivalence classes of $S = G$ under the action of $H' = G$, which here is only conjugation within G. The measure

$$d\bar{s} = [W(T_i)]^{-1}|D(\gamma)|d\gamma$$

is defined through the Jacobian of the map

$$\begin{aligned} T_i\backslash G \times T_i' &\longrightarrow (T_i')^G \\ (h, \gamma) &\mapsto h^{-1}\gamma h \end{aligned}$$

in the Weyl integration formula. Here T_i' is the subset of regular elements in T_i and $d\gamma$ is the normalized measure on $Z\backslash T_i$ for which Vol $(Z\backslash T_i) = 1$.

We collect this in the following proposition.

Proposition 6.1. *Let π be an irreducible unitary supercuspidal representation of a p–adic connected reductive group G and assume we are in the setting of Section 3.1 and Proposition 3.1. Let $\tilde{f}(g) = \overline{\langle \pi(g)v, \tilde{v}\rangle}$, where $\langle \pi(g)v, \tilde{v}\rangle$ is a matrix coefficient of π with $\langle v, \tilde{v}\rangle = 1$. Let T_i be an elliptic Cartan subgroup of G and let $\gamma \in T_i$ be a regular element. Then for the functional $\Lambda(f, \tilde{v}; \Theta)$ in (3.1), the generalized Bessel function is the orbital integral*

$$B_f(\gamma) = \int_{T_i\backslash G} \tilde{f}(h^{-1}\gamma h)d\dot{h} = I(\gamma, \tilde{f}) \tag{6.6}$$

which converges absolutely for any $\tilde{f} \in C_c^\infty(Z\backslash G, \overline{\omega}_\pi)$. The functional $\Lambda(f, \tilde{v}; \Theta)$ is given by the Weyl integration formula

$$\Lambda(f, \tilde{v}; \Theta) = \sum_{\{T_i\}} \int_{Z\backslash T_i} I(\gamma, \tilde{f})\Theta(\gamma)[W(T_i)]^{-1}|D(\gamma)|d\gamma. \tag{6.7}$$

Now, for each $\gamma \in T_i$, Lemma 19 of [17] implies the existence of an open neighborhood $\omega_\gamma \subset T_i$ such that the integration in (6.6) can be achieved by integrating over a compact subset of $T_i\backslash G$ depending on the support of $\tilde{f}$ and ω_γ. Covering $Z\backslash T_i$ by open neighborhoods $Z \cap \omega_\gamma\backslash\omega_\gamma$, $\gamma \in T_i$, we may assume these neighborhoods are finite in number. Thus the integration in (6.6) is over a compact set in $T_i\backslash G$ depending only on $\tilde{f}$ alone or the corresponding generalized Bessel function, the orbital integral $I(\gamma, \tilde{f})$, is in fact a partial one which converges, in fact, absolutely.

Remark 6.2. As already referred to a number of times, this example appears in [17], giving the formal degree and the orthogonality as the subject matter of Part VIII of this write–up of Harish–Chandra's early and important work on p–adic groups by van Dijk. In it, the connection with conjugacy is paramount and fundamental. A much deeper example of these connections can be seen when one addresses these functionals for intertwining operators as discussed in Section 4.2. In the case of classical groups, the connection comes through the theory of endoscopy, a very deep subject with definitive consequences in Langlands functoriality conjecture [23, 21, 3]. This connection is well–studied in a number of papers, particularly in [31] and will be addressed next.

6.3 Intertwining operators

The next cases are the Bessel functions appearing in the cases discussed in Section 4. Let us first address the case of intertwining operators, i.e., Section 4.2. Recall that now $H = H' = M$ is a maximal Levi subgroup of G.

What is interesting for intertwining operators is the residue at the pole $\nu = 0$. The work addressed in [31] covers fully the cases of quasisplit groups when $S = N^-$ is abelian. As explained in [31], the general theory of prehomogeneous vector spaces then shows that $M\backslash N^-$ will have only a finite number of (open) orbits whose union has a complement of measure zero in N^-. To put ourselves in the situation of that paper, we assume in addition that $\mathbf{P}$ is self associate, so that $w_0 N^- w_0^{-1} = N$, π is cuspidal and that $w_0(\pi) \simeq \pi$, so it is possible for the intertwining operator to have a pole at $\nu = 0$. In addition, by a lemma of Rallis, we may assume that the support of f lies in PN^-. Then from (4.4) our functional has an expression

$$\Lambda(R_{w_0^{-1}} f, \tilde{v}; 1) = \int_{N^-} \langle R_{w_0^{-1}} f(n^-), \tilde{v}\rangle \, dn^- = \langle A(\nu, \pi, w_0) f(e), \tilde{v}\rangle. \tag{6.8}$$

Decomposing this functional in terms of the Bessel function as in (6.2), and using that M has a finite number of open orbits on N^-, we have

$$\Lambda(R_{w_0^{-1}} f, \tilde{v}; 1) = \sum_i B_{R_{w_0^{-1}} f}(\tilde{n}_i^-)$$

where

$$B_{R_{w_0^{-1}} f}(\tilde{n}_i^-) = \int_{M_{\tilde{n}_i^-}\backslash M} \langle f(m^{-1}\tilde{n}_i^- m w_0^{-1}), \tilde{v}\rangle \, d\dot{m} \tag{6.9}$$

and the $\tilde{n}_i^-$ represent the finite open orbits of M on N^-. Hence the residue of the intertwining operator will come from the residues of the Bessel functions at $\nu = 0$. Since f is supported on PN^-, we next write $\tilde{n}_i^- w_0^{-1} = w_0^{-1} n_i = n_i' m_i n_i^- \in NMN^-$. If we let $M_i = M_{m_i}^t = \{m \in M \mid w_0(m)^{-1} m_i m = m_i\}$ denote the twisted centralizer of m_i, then the calculations in [31] show that the Bessel function is represented by formula (2.2) of [31], namely

$$\begin{aligned} B_{R_{w_0^{-1}} f}(\tilde{n}_i^-) = \int_{M_i\backslash M} & q^{\langle \nu, H_M(w_0(m)^{-1} m_i m)\rangle} \\ & \times \sum_{m_0 \in M_{n_i}\backslash M_i} \langle \pi(w_0(m)^{-1} m_i m) f((m_0 m)^{-1} n_i^- m_0 m), \tilde{v}\rangle \, d\dot{m}. \end{aligned}$$

For the purpose of computing the residue of the Bessel function, we may assume there exists a Schwartz function Φ on the Lie algebra $\mathfrak{n}^-$ so that $f(\exp X) = \Phi(X)f(e)$. Also, since π is assumed supercuspidal, there exists $\tilde{f} \in C_c^\infty(M)$ which represents the matrix coefficient in the sense that

$$\langle \pi(m)f(e), \tilde{v}\rangle = \int_{\tilde{A}} \tilde{f}(am)\omega_\pi(a)\ da$$

where $\tilde{A}$ is the center of M. Following [31], we see that the Bessel function is then given by the expression in (2.10) of that paper, namely

$$B_{R_{w_0^{-1}}f}(\tilde{n}_i^-) = \int_{M_i\backslash M}\int_{\tilde{A}'\backslash\tilde{A}} \theta_\nu(m)\tilde{f}(zw_0(m)^{-1}m_i m)q^{\langle \nu, H_M(w_0(m)^{-1}m_i m)\rangle}\omega_\pi(z)^{-1}\ dz\ d\dot{m}$$

where $\tilde{A}' = \{w_0(a)^{-1}a \mid a \in \tilde{A}\}$ and

$$\theta_\nu(m) = \int_{Z(G)\backslash\tilde{A}} \sum_{m_0} \Phi(Ad(am_o m)X_i^-)q^{\langle \nu, w_0(a)^{-1}a\rangle}\ da.$$

As noted in [31], the pole is carried by the function $\theta_\nu(m)$. We may take the residue by taking the residue of $\theta_\nu(m)$ at $\nu = 0$. The pole is simple and the residue is a non-zero constant which is independent of π. Hence, from [31], we finally obtain

$$\operatorname*{Res}_{\nu=0} B_{R_{w_0^{-1}}f}(\tilde{n}_i^-) = c_i\int_{M_i\backslash M}\int_{\tilde{A}'\backslash\tilde{A}} \tilde{f}(zw_0(m)^{-1}m_i m)\omega_\pi(z)^{-1}\ dz\ d\dot{m}.$$

This last expression can be reformulated in terms of twisted orbital integrals for $\tilde{f}$. Let $\theta = Ad(w_0)|_M$ be the involution of M obtained from the action of w_0 and now let

$$M_{m_0}^\theta = \{m \in M | \theta(m)^{-1}m_0 m = m_0\}.$$

denote the θ-twisted stabilizer of $m_0 \in M$. Then the θ-twisted orbital integral of $\tilde{f}$ is defined by

$$\Phi_\theta(m_0, \tilde{f}) = \int_{M_{m_0}^\theta\backslash M} \tilde{f}(\theta(m)^{-1}m_0 m)d\dot{m}. \tag{6.10}$$

We then have an expression of the residue of our Bessel function at $\nu = 0$ in terms of θ-twisted orbital integrals.

Proposition 6.3. *Let $\Lambda(R_{w_0^{-1}}f, \tilde{v}; 1)$ be the functional associated to the intertwining operator $A(\nu, \pi, w_0)$ as in (6.8). In the case of abelian N^-, the residue of the associated Bessel function $B_{R_{w_0^{-1}}f}(\tilde{n}_i^-)$ at $\nu = 0$ is given by integrals of the θ-twisted orbital integrals $\Phi_\theta(m_0, \tilde{f})$ defined above, namely*

$$\operatorname*{Res}_{\nu=0} B_{R_{w_0^{-1}}f}(\tilde{n}_i^-) = c_i\int_{\tilde{A}'\backslash\tilde{A}} \Phi_\theta(zm_i, \tilde{f})\omega_\sigma^{-1}(z)dz. \tag{6.11}$$

This proposition gives an interpretation of Theorem 2.5 of [31] in terms of our Bessel functions. Again as in the case of regular conjugacy, it is convergent, even absolutely (cf.[2]). Similarly for (6.10).

On the other hand when the orbits are not finite in number, while the residues of the generalized Bessel functions remain twisted orbital integrals, the integration over the orbits becomes a lot more involved and complicated. Appropriate Jacobians must be calculated as soon as a "continuous" set of representatives for $H'\backslash N^-$ is fixed. In the case of classical groups, they can be given by regular elements in different Cartan subgroups in GL_n and $\mathbf{G}_m$, where $\mathbf{M} = GL_n \times \mathbf{G}_m$ is the corresponding maximal Levi subgroup inside $\mathbf{G}$. The representatives will be given as a pair (x, y) with x and y in appropriate Cartan subgroups of $GL_n \times \mathbf{G}_m$, related to each other through the norm map between GL_n and $\mathbf{G}_m$ as developed by Kottwitz and Shelstad [21] and in [30]. The problem has been studied by Shahidi, Goldberg–Shahidi, Spallone and Shahidi–Spallone in a series of papers [27, 15, 16, 35, 33]. Complete results in terms of functorial transfers are finally emerging [33]. Even in the simple case of the Levi factor $GL_2 \times SO_2^*$ inside SO_6^*, the quasisplit form of SO_6, defined by K/k, $[K : k] = 2$, studied in [33], the result is deep and subtle. The residue becomes the pairing of the character τ on $SO_2^*(k) = K^1$ with that of the representation π_1 of $GL_2(k)$, matched through the norm map which in this case is simply, $x \mapsto x/\beta(x)$, i.e., that of Hilbert's Theorem 90, from K^* to K^1. Here $\pi = \pi_1 \otimes \tau$ and β is the non–trivial element of $\mathrm{Gal}(K/k)$. Moreover K^1 is the subgroup of elements of norm 1 in K^*. When combined with Labesse–Langlands character identities [22], this leads to a beautiful direct (local) determination of the vanishing and non–vanishing of these residues, giving definitive local information for example on reducibility of $I(\nu, \pi)$. The general case, although much more subtle, should follow a similar path.

6.4 Whittaker functionals

The true Bessel functions or their partial version appear in Sections 4.3 and 4.4, and therefore the name.

We return to the situation of Section 4.3. So $G = \mathbf{G}(k)$, $P_0 = TU$ a Borel subgroup and $P = MN$ is a standard maximal parabolic subgroup. We fix a non-degenerate character χ of U. We then have $H = MN = P$ and $H' = U_M = U \cap M$. For σ' we take the one dimensional representation of U_M transforming by the restriction of χ^{-1}. Then $\lambda_{v'}(v) = \lambda(v \otimes v')$ is a χ-Whittaker functional on $\mathcal{H}_\pi$. Then $S = N^-$ and $\Theta(n^-) = \Theta'(w_0 n^- w_0^{-1})$ where Θ' was the restriction of χ^{-1} to N'. Then the Whittaker functional on $I(\nu, \pi)$ is given by

$$\Lambda(R_{w_0^{-1}} f, v' : \Theta) = \int_{N^-} \lambda_{v'}(R_{w_0^{-1}} f(\tilde{n}^-))\Theta(\tilde{n}^-)\, d\tilde{n}^-.$$

The associated Bessel function for this functional is then

$$B_{R_{w_0^{-1}} f}(\tilde{n}^-) = \int_{U_{M,\tilde{n}^-}\backslash U_M} \lambda_{v'}(f(u^{-1}\tilde{n}^- u w_0^{-1}))\, d\dot{u}.$$

Using that $\lambda_{v'}$ is a Whittaker functional on $\mathcal{H}_\pi$ and then conjugating through by w_0^{-1} we have

$$\begin{aligned}\lambda_{v'}(f(u^{-1}\tilde{n}^- u w_0^{-1})) &= \lambda_{v'}(\sigma(u^{-1})f(\tilde{n}^- u w_0^{-1}))\\ &= \chi(u)^{-1} f(w_0^{-1} n w_0^{-1}(u))\end{aligned}$$

where $\tilde{n}^- w_0^{-1} = w_0^{-1} n$. Since $w_0^{-1}(U_{M,\tilde{n}^-}) = U_{M,n}$ then we can change variables in the definition of the Bessel function and find

$$B_{R_{w_0^{-1}} f}(n^-) = \int_{U_{M,n}\backslash U_M} \chi(w_0(u))^{-1} \lambda_{v'}(f(w_0^{-1} n u))\ d\dot{u}.$$

Let us now assume that P is self associate and that χ and w_0 are compatible, so that $\chi(w_0(u)) = \chi(u)$. Then this becomes

$$B_{R_{w_0^{-1}} f}(n^-) = \int_{U_{M,n}\backslash U_M} \chi(u)^{-1} \lambda_{v'}(f(w_0^{-1} n u))\ d\dot{u}.$$

At this point, let us assume that $w_0^{-1} n = n' m' n^- \in PN^-$. Then we have

$$\begin{aligned}B_{R_{w_0^{-1}} f}(\tilde{n}^-) &= \int_{U_{M,n}\backslash U_M} \chi(u)^{-1} \lambda_{v'}(f(n' m' n^- u))\ d\dot{u}\\ &= \int_{U_{M,n}\backslash U_M} \chi(u)^{-1} \lambda_{v'}(f(m' n^- u))\ d\dot{u}\\ &= \int_{U_{M,n}\backslash U_M} \chi(u)^{-1} \lambda_{v'}(f(m' u u^{-1} n^- u))\ d\dot{u}\end{aligned}$$

We can now write

$$\lambda_{v'}(f(m' u u^{-1} n^- u)) = \lambda_{v'}(\pi(m' u) f(u^{-1} n^- u)) = W_{f(u^{-1} n^- u)}(m' u)$$

so that

$$B_{R_{w_0^{-1}} f}(\tilde{n}^-) = \int_{U_{M,n}\backslash U_M} W_{f(u^{-1} n^- u)}(m' u) \chi(u)^{-1}\ d\dot{u}.$$

For applications to stability, as in considering the poles of intertwining operators above, it suffices to restrict to f which are supported on PN^- and compactly supported on N^- mod P. On the other hand f is locally constant when restricted to N^-. So let us fix a open compact subset $Y \subset N^-$ on which f is constant. Let $Y^* \subset N^-$ be the inverse image of Y under the map $N^- \longrightarrow N^-$ given by

$$\tilde{n}^- \mapsto \tilde{n}^- w_0^{-1} = w_0^{-1} n = n' m' n^- \mapsto n^-.$$

Then to Y^* we have associated a partial Bessel function as in (6.4)

$$\begin{aligned}B_{R_{w_0^{-1}} f, Y^*}(\tilde{n}^-) &= \int_{U_{M,n}\backslash U_M} W_{f(u^{-1} n^- u)}(m' u) \varphi_Y(u^{-1} n^- u) \chi(u)^{-1}\ d\dot{u}\\ &= \int_{U_{M,n}\backslash U_M} W_{v_Y}(m' u) \varphi_Y(u^{-1} n^- u) \chi(u)^{-1}\ d\dot{u}\end{aligned}$$

where $v_Y = f(u^{-1}n^-u)$ for $u^{-1}n^-u \in Y$, which now depends only on Y. Note that our original Bessel function can be written as a finite sum of these partial Bessel functions. They are as in the classical setting, integrals of Whittaker functions against characters of the defining unipotent group.

These objects appear in the problem of stability of root numbers as studied in different papers [8, 9, 10]. They show themselves in a general setting in equation (6.21), page 2103 of [32] and one part of this program hopes to prove the stability of root numbers appearing in the Langlands–Shahidi method through the examination of our functionals. In fact, the inverse of the local coefficient $C_\chi(s,\pi)$ in [32] just simply becomes one of our functionals (see equation (6.58) of [32], as well as Section 7 below). This has been successfully treated in certain important cases needed for the proof of functoriality from classical or $GSpin$ groups to $GL(n)$ for which we refer to [6, 7, 4, 11].

7 Local coefficients and stability

We would now like to relate this formalism to the expression of the usual local coefficient as a Mellin transform of a partial Bessel function as in [32] and [10]. We begin by specializing Section 4 to this situation.

Let k be a non-archimedean local field. As in Section 4.3 we take $\mathbf{G}$ a quasisplit reductive algebraic group over k, and $G = \mathbf{G}(k)$. Let $\mathbf{P}_0 = \mathbf{TU}$ be a minimal parabolic subgroup and now χ denote a generic character of $U = \mathbf{U}(k)$. We let $\mathbf{P} = \mathbf{MN}$ be a self-associate standard maximal parabolic subgroup and will take $H = \mathbf{P}(k) = P$. Self associate implies that $N' = N$ in the notation of Section 4.1. As σ we take π_ν as in Section 4.1, but with π an irreducible admissible χ-generic representation of $M = \mathbf{M}(k)$ and $\nu = s\tilde{\alpha}$ as in Section 2 of [32], so $I(\sigma) = I(s,\pi)$ in the notation of [32]. Then $H' = U_M = U \cap M$ and $\sigma' = \chi^{-1}$ is the inverse of the generic character χ restricted to U_M. Then fixing $v' \in \mathcal{H}_{\sigma'} = \mathbb{C}$ we have $\lambda_{v'}(v) = \lambda(v \otimes v')$ is χ-Whitaker functional on $\mathcal{H}_{\pi_s}$. We take $S = N^-$ and Θ as Section 4.3, i.e, choose Θ so that $\Theta'(n) = \Theta(w_0 n w_0^{-1})$ is the restriction of χ^{-1} to N. Then

$$\begin{aligned}\Lambda(R_{w_0^{-1}}f, v'; \Theta) &= \int\limits_{N^-} \lambda_{v'}((R_{w_0^{-1}}f)(n^-))\Theta(n^-)dn^- \\ &= \int\limits_{N} \lambda_{v'}(f(w_0^{-1}n))\chi^{-1}(n)dn \\ &= \lambda_\chi(s,\pi)(f)\end{aligned}$$

which is the χ-Whittaker functional on $I(s,\pi)$ which occurs in [32].

As in Section 4.2, the standard intertwining operator $A(s,\pi) : V(s,\pi) \to V(-s, w_0(\pi))$ is given by

$$A(s,\pi)(f) = \int_N f(w_0^{-1}ng)\ dn. \tag{7.1}$$

and the local coefficient, generalized in Section 5, is defined by

$$\lambda_\chi(s,\pi) = C_\chi(s,\pi)\lambda_\chi(-s, w_0(\pi))A(s,\pi).$$

or, in the current scheme,

$$\Lambda(R_{w_0^{-1}}f, v'; \Theta) = C_\chi(s,\pi)\Lambda(R_{w_0^{-1}}Af, v'; \Theta)$$

where $A = A(s,\pi)$.

For the purposes of stability, we need to express the local coefficient $C_\chi(s,\pi)$, or as we did in practice its inverse $C_\chi(s,\pi)^{-1}$, itself as one of our functionals, so we can rely on the Bessel function to compute it. To this end we first rewrite the above as

$$C_\chi(s,\pi)^{-1}\lambda_\chi(s,\pi) = \lambda_\chi(-s, w_0(\pi))A(s,\pi). \tag{7.2}$$

To obtain an expression for $C_\chi(s,\pi)^{-1}$ itself we make a judicious choice of $f \in V(s,\pi)$ so that $\lambda_\chi(s,\pi)(f) \equiv 1$. For this purpose, we first choose $f \in V(s,\pi)$ so that the restriction of f to PN^- has compact support modulo P, where N^- is the unipotent radical of the parabolic subgroup P^- which is opposite to P. Let N_0^- be a compact subset of N^- which contains the support of $f|_{PN^-}$ modulo P. We then let

$$\mu = \int_{N_0^-} \chi^{-1}(w_0^{-1}n_1^- w_0) f(n_1^-)\, dn_1^- \in \mathcal{H}_{\pi_s}.$$

Then, as shown on page 2108 of [32], $\lambda_\chi(s,\pi)(R_{w_0}f) = W_\mu(e)$. If we scale f, and hence μ, so that $W_\mu(e) = 1$, then when we evaluate (7.2) at f the left hand side becomes simply $C_\chi(s,\pi)^{-1}$. This expresses $C_\chi(s,\pi)$ as one of our functionals, namely

$$C_\chi(s,\pi)^{-1} = \lambda_\chi(-s, w_0(\pi))A(s,\pi)(R_{w_0}f) = \Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f, v'; \Theta). \tag{7.3}$$

By (6.52) of [32] this can be written as

$$C_\chi(s,\pi)^{-1} = \int_N W_\mu(m)\varphi_{N_0^-}(\overline{n})q^{\langle\rho, H_M(m)\rangle}\chi'(\overline{n})\, dn \tag{7.4}$$

where we have written $w_0^{-1}n = mn'\overline{n} \in PN^-$ with $\chi'(\overline{n}) = \chi(w_0^{-1}\overline{n}w_0)$ (see [32] for details and unexplained notations).

For the purpose of proving stability in [10], the expression in (7.4) was expressed in terms of a partial Bessel function in [32]. Interestingly, this is not the Bessel function coming from the functional in (7.3). Instead, we need a new expression for $C_\chi(s,\pi)^{-1}$ as one of our functionals. Note that in the expression (7.3) there are actually two integrations over N (or its conjugate N^-) that take place, one in the expression for the Whittaker functional $\lambda_\chi(s,\pi)$ and one in the expression for the intertwining operator $A(s,\pi)$. In the functional appearing in (7.3) the subgroup S is the N^- coming from the Whittaker functional. The N in expression (7.4) is the N from the intertwining operator. Hence we must somehow "flip" the expression in (7.3) to interchange the two occurrences of N in the functional expressing $C_\chi(s,\pi)^{-1}$ to obtain the "correct" Bessel function for stability.

We begin with the expression in (7.3). Using that $A(s,\pi)$ is an intertwining operator we find

$$\begin{aligned}\Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f, v'; \Theta) &= \Lambda(A(s,\pi)f, v'; \Theta)\\ &= \int_{N^-} \lambda(A(s,\pi)f(n^-)\otimes v')\Theta(n^-)\, dn^-\end{aligned}$$

As in [32] by the general theory of Whittaker functions this integral stabilizes and with the choice of N_0^- made in [32] and used above this is

$$\Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f,v';\Theta)=\int_{N_0^-}\lambda(A(s,\pi)f(n^-)\otimes v')\Theta(n^-)\ dn^-$$

We assume $Re(s)$ is sufficiently large so that the intertwining operator is given by the integral (7.1). Then

$$\begin{aligned}\Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f,v';\Theta)&=\int_{N_0^-}\lambda(\int_N f(w_0^{-1}n_1n^-)\ dn_1\otimes v')\Theta(n^-)\ dn^-\\&=\int_{N_0^-}\lambda(\int_{N^-} f(n_1^-w_0^{-1}n^-)\ dn_1^-\otimes v')\Theta(n^-)\ dn^-\end{aligned}$$

Since $Re(s)>0$ we can use Fubini's Theorem to interchange the order of integration and obtain

$$\Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f,v';\Theta)=\int_{N^-}\lambda(\int_{N_0^-} f(n_1^-w_0^{-1}n^-)\Theta(n^-)dn^-\otimes v')\ dn_1^-.$$

If we now consider the inner integration, we see that if we set

$$f_\mu=\int_{N_0^-}\Theta(n^-)R_{n^-}f\ dn^-=\int_{N_0^-}\chi^{-1}(w_0^{-1}n^-w_0)R_{n^-}f\ dn^-\in V(s,\pi)$$

then f_μ is that function in the induced representation with $f_\mu(e)=\mu\in\mathcal{H}_{\pi_s}$. Then

$$\Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f,v';\Theta)=\int_{N^-}\lambda(R_{w_0^{-1}}f_\mu(n_1^-)\otimes v')\ dn_1^-.$$

The right hand side of this expression is now another of our functionals. We still have G, H, and $\sigma=\pi_s$ as before. Also $H'=U_M$ and σ' is the one dimensional representation transforming by χ^{-1} as before and λ is the same functional on $\mathcal{H}_{\pi_s}\otimes\mathcal{H}_{\chi^{-1}}$ as before. We retain that $S=N^-$, but in the new functional we have $\Theta\equiv 1$. So, in this situation we see that

$$\Lambda(R_{w_0^{-1}}f_\mu,v';1)=\int_{N^-}\lambda(R_{w_0^{-1}}f_\mu(n^-)\otimes v')\ dn^-. \tag{7.5}$$

So we have the equality

$$\Lambda(R_{w_0^{-1}}A(s,\pi)R_{w_0}f,v';\Theta)=\Lambda(R_{w_0^{-1}}f_\mu,v';1)$$

and we have "flipped" the two N^-. If we combine this with (7.3) we obtain a new expression of $C_\chi(s,\pi)^{-1}$ as one of our functionals, namely

$$C_\chi(s,\pi)^{-1}=\Lambda(R_{w_0^{-1}}f_\mu,v';1). \tag{7.6}$$

We now compute the Bessel function for this functional. We first recall a conventions from [32]. Namely, we can choose χ and w_0 to be compatible, i.e., for

$u \in U_M$ we have $\chi(u) = \chi(w_0(u)) = \chi(w_0^{-1}uw_0)$. We also note that if $w_0^{-1}n = n^-w_0^{-1}$ then $U_{M,n^-} = w_0(U_{M,n})$. By definition the Bessel function is given by

$$B_{R_{w_0^{-1}}f_\mu}(n^-) = \int_{U_{M,n^-}\backslash U_M} \lambda(R_{w_0^{-1}}f_\mu(u^{-1}n^-u) \otimes v')\ d\dot{u}.$$

We conjugate by w_0^{-1} within f_μ to obtain

$$B_{R_{w_0^{-1}}f_\mu}(n^-) = \int_{U_{M,n^-}\backslash U_M} \lambda(f_\mu(w_0^{-1}w_0^{-1}(u)^{-1}nw_0^{-1}(u)) \otimes v')\ d\dot{u}.$$

Since $w_0^{-1}(U_{M,n^-}) = U_{M,n}$ we can rewrite this as an integral over $U_{M,n}\backslash U_M$

$$B_{R_{w_0^{-1}}f_\mu}(n^-) = \int_{U_{M,n}\backslash U_M} \lambda(f_\mu(w_0^{-1}u^{-1}nu) \otimes v')\ d\dot{u}$$

where we have used $n = w_0^{-1}(n^-)$ as above. If we write $n^-w_0^{-1} = w_0^{-1}n = n'm\overline{n} \in NMN^-$ as above, then under conjugation by u we find

$$w_0^{-1}u^{-1}nu = [w_0(u)^{-1}n'w_0(u)] \cdot [w_0(u)^{-1}mu] \cdot [u^{-1}\overline{n}u].$$

Since $f_\mu \in V(s,\pi)$ it is left invariant under N and transforms by π_s under M. On the other hand, since $f|_{PN^-}$ has support in N_0^- mod P and the passage from f to f_μ involves an integral over N_0^-, which we take to be a subgroup since N^- is exhausted under compact open subgroups, $f_\mu(w_0^{-1}u^{-1}nu)$ will vanish unless $u^{-1}\overline{n}u \in N_0^-$, and then by the definition of f_μ it transforms by the character Θ^{-1} under N_0^-. Together, these considerations give

$$\begin{aligned} f_\mu(w_0^{-1}u^{-1}nu) &= \pi_s(w_0(u)^{-1}mu)q^{\langle\rho,H_M(w_0(u)^{-1}mu)\rangle}f_\mu(e)\Theta^{-1}(u^{-1}\overline{n}u)\varphi_{N_0^-}(u^{-1}\overline{n}u) \\ &= q^{\langle\rho,H_M(w_0(u)^{-1}mu)\rangle}\pi_s(w_0(u)^{-1}mu)\mu\ \chi(w_0^{-1}u^{-1}\overline{n}uw_0)\varphi_{N_0^-}(u^{-1}\overline{n}u) \\ &= q^{\langle\rho,H_M(m)\rangle}\pi_s(w_0(u)^{-1}mu)\mu\ \chi'(\overline{n})\varphi_{N_0^-}(u^{-1}\overline{n}u) \end{aligned}$$

If we insert this into the Whittaker functional and use that χ and w_0 are compatible, we find

$$\lambda(f_\mu(w_0^{-1}u^{-1}nu) \otimes v') = q^{\langle\rho,H_M(m)\rangle}\chi^{-1}(u)W_\mu(mu)\chi'(\overline{n})\varphi_{N_0^-}(u^{-1}\overline{n}u)$$

and so finally

$$\begin{aligned} B_{R_{w_0^{-1}}f_\mu}(n^-) &= \int_{U_{M,n}\backslash U_M} q^{\langle\rho,H_M(m)\rangle}\chi^{-1}(u)W_\mu(mu)\chi'(\overline{n})\varphi_{N_0^-}(u^{-1}\overline{n}u)\ d\dot{u} \\ &= q^{\langle\rho,H_M(m)\rangle}\chi'(\overline{n})\int_{U_{M,n}\backslash U_M} W_\mu(mu)\chi^{-1}(u)\varphi_{N_0^-}(u^{-1}\overline{n}u)\ d\dot{u} \end{aligned}$$

which essentially agrees with the Bessel function in [32].

Proposition 7.1. *Let $\Lambda(R_{w_0^{-1}} f_\mu, v'; 1)$ be the functional (7.5) associated to the local coefficient $C_\chi(s,\pi)$ in (7.6). Then, with the notation as above, the associated Bessel function is*

$$B_{R_{w_0^{-1}} f_\mu}(n^-) = q^{\langle \rho, H_M(m)\rangle} \chi'(\overline{n}) \int_{U_{M,n}\backslash U_M} W_\mu(mu)\chi^{-1}(u)\varphi_{N_0^-}(u^{-1}\overline{n}u)\ d\dot{u}. \tag{7.7}$$

If we return to (7.6), then the general decomposition of the functional into an integral of the associated Bessel function as in (6.2) now gives an integral representation for the inverse of the local coefficient.

Corollary 7.2. *The (inverse of) the local coefficient can be expressed as*

$$C_\chi(s,\pi)^{-1} = \int_{U_M\backslash N} \left[\int_{U_{M,n}\backslash U_M} W_\mu(mu)\chi^{-1}(u)\varphi_{N_0^-}(u^{-1}\overline{n}u)\ d\dot{u} \right] q^{\langle \rho, H_M(m)\rangle} \chi'(\overline{n})\ d\dot{n}. \tag{7.8}$$

This expression agrees with (6.55) in [32], which was the starting point for the proof of stability of the local coefficient in [10].

8 Integral representations

Many integral representations of L-functions fit into this scheme, in fact we have found this possible for every example we have looked at. In this section we will discuss three representative examples. As we will see, they all fit into the scheme with $H' = \{e\}$ and $\sigma' = \mathbb{C}$, so we will make this blanket assumption for this section. Then λ is simply a functional on $\mathcal{H}_\sigma$. On the other hand, the role of S and Θ are expanded and in fact often Θ will come from functions in a representation τ of S.

Let k be a global field and $\mathbb{A}$ its ring of adeles and ψ a non-trivial character of $\mathbb{A}$ trivial on k.

8.1 Rankin-Selberg convolutions for GL_n

We refer the reader to [5] for the basics of this integral representation and any unexplained notation.

Let us begin with the case where $n > m$. We then take $G = H = GL_n(\mathbb{A})$ and take $\sigma = \pi$ to be a cuspidal automorphic representation of $GL_n(\mathbb{A})$ with $\mathcal{H}_\pi$ its automorphic realization. As noted, we take $H' = \{e\}$ and $\sigma' = \mathbb{C}$, so that $\lambda : \mathcal{H}_\pi \to \mathbb{C}$ is a linear functional on $\mathcal{H}_\pi$. In the notation of [5], Section 2, we take $\lambda(\varphi) = \mathbb{P}^n_m \varphi(e)$. (If $m = n-1$ then $\lambda(\varphi) = \varphi(e)$.) This is a (normalized) partial Whittaker functional on $\mathcal{H}_\pi$. We take $f = f_\varphi \in V(\pi)$ such that $f_\varphi(e) = \varphi$.

For S we take $GL_m(\mathbb{A})$, embedded in $GL_n(\mathbb{A})$ in the upper left corner. Let $\tau = \pi'$ be a cuspidal automorphic representation of $GL_m(\mathbb{A})$. Then if we take $\varphi' \in \mathcal{H}_{\pi'}$ a cusp form on $GL_m(\mathbb{A})$, then we set $\Theta(h) = \varphi'(h)|\det(h)|^{s-1/2}$. Then we have $S_{\lambda,\Theta} = GL_m(k)$ and

$$\Lambda(f_\varphi, 1; \Theta) = \int_{GL_m(k)\backslash GL_m(\mathbb{A})} \mathbb{P}^n_m \varphi \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\, dh = I(s;\varphi,\varphi')$$

which is the global integral in the integral representation for $L(s, \pi \times \pi')$.

In the case of $m = n$, the set up is the same, except now $G = H = S = GL_n(\mathbb{A})$. The functional λ on $\mathcal{H}_\pi$ is $\lambda(\varphi) = \varphi(e)$. The most significant difference is that now we take Θ to be of the form $\Theta(g) = \varphi'(g)E(g, \Phi; s, \eta)$, where $\varphi' \in \mathcal{H}'_\pi$ is a cusp form on $GL_n(\mathbb{A})$ and $E(g, s; \Phi, \eta)$ is a mirabolic Eisenstein series as in Section 2.3 of [5]. Then

$$\Lambda(f_\varphi, 1; \Theta) = \int_{GL_n(k)\backslash GL_n(\mathbb{A})} \varphi(g)\varphi'(g)E(g, s; \Phi, \eta)\; dg = I(;\varphi, \varphi', \Phi)$$

which is the global integral for $L(s, \pi \times \pi')$ from [5] in this case.

The local integrals follow the example of the global ones. Briefly, let v be a place of k. Take $G = H = GL_n(k_v)$ and $\sigma = \pi_v$ an irreducible admissible generic representation of $GL(n(k_v)$. We realize this in its Whittaker model, i.e., $\mathcal{H}_{\pi_v} = \mathcal{W}(\pi_v, \psi_v)$. As always, $H' = \{e\}$ and $\sigma' = \mathbb{C}$. Then $\lambda : \mathcal{W}(\pi_v, \psi_v) \to \mathbb{C}$ is $\lambda(W) = W(e)$. Moreover $S = GL_m(k_v)$ and we take $\tau = \pi'_v$ an irreducible admissible generic representation of $GL_m(k_v)$, realized in its Whittaker model. Then, as above, for Θ we take either $\Theta(h) = W'_v(h)|\det(h)|^{s-(n-m)/2}$ in the case $n > m$, where $W'_v \in \mathcal{W}(\pi'_v, \psi_v^{-1})$, while if $n = m$ we take $\Theta(h) = W'_v(h)\Phi_v(e_n h)|\det(h)^s$. Then once again we have

$$\Lambda(f_{W_v}, 1; \Theta) = \begin{cases} \Psi(s; W_v, W'_v) & n > m \\ \Psi(s; W_v, W'_v, \Phi_v) & n = m \end{cases}$$

where again the Ψ are the local integrals for $L(s, \pi_v \times \pi'_v)$ as in Section 3 of [5].

8.2 Exterior square for GL_{2n}

The reference for this construction is [19].

For the global integrals representing the exterior square we will take $G = H = GL_{2n}(\mathbb{A})$ and $\sigma = \pi$ a cuspidal representation of G. As always in these constructions $H' = \{e\}$ and $\sigma' = \mathbb{C}$. So λ becomes a continuous linear functional on $\mathcal{H}_\pi$. For $\varphi \in \mathcal{H}_\pi$ a cusp form, we take

$$\lambda(\varphi) = \int_{M_n(k)\backslash M_n(\mathbb{A})} \varphi \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \psi(tr(x))dx.$$

Let $f_\varphi \in I(\pi)$ such that $f_\varphi(e) = \varphi \in \mathcal{H}_\pi$.

For the subgroup S we take $GL_n(\mathbb{A})$ embedded in G by $g \mapsto \begin{pmatrix} g & \\ & g \end{pmatrix}$. For Θ we take a mirabolic Eisenstein series $E(g, s; \Phi, \chi)$ on S. Then we have $S_{\lambda,\Theta} = GL_n(k)$.

Then our functional $\Lambda(f,1;\Theta)$ becomes the global integral representing the exterior square L-function from [19]

$$\Lambda(f_\varphi,1;\Theta)=\int_{GL_n(k)\backslash GL_n(\mathbb{A})}\lambda\left(f_\varphi\begin{pmatrix} g & \\ & g\end{pmatrix}\right)E(g,s;\Phi,\chi)dg$$
$$=\int_{GL_n(k)\backslash GL_n(\mathbb{A})}\int_{M_n(k)\backslash M_n(\mathbb{A})}\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi(tr(x))dx\ E(g,s;\Phi,\chi)dg.$$

For the local integrals, we take $G=H=GL_{2n}(k_v)$ and $\sigma=\pi_v$ a local component of π and take $\mathcal{H}_{\pi_v}=\mathcal{W}(\pi_v,\psi_v)$ the Whittaker model of π_v. For $W\in\mathcal{H}_{\pi_v}$ we define the local functional by

$$\lambda_v(W)=\int_{\mathfrak{p}_{o,n}(k_v)\backslash M_n(k_v)}W\left(\sigma\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\psi_v(tr(x))dx$$

where $\mathfrak{p}_{o,n}$ is the set of upper triangular matrices in $M_n(k_v)$ and σ is (now) an appropriate permutation matrix (see [19]). We take $f=f_W\in I(\pi_v)$ with $f_W(e)=W\in\mathcal{H}_{\pi_v}$. Once again $S=GL_n(k_v)$ is embedded as a subgroup of $GL_{2n}(k_v)$ as above. Θ_v is the unfolded version of the mirabolic Eisenstein series, namely $\Theta_v(g_v)=\Phi_v(\epsilon_n g_v)\chi_v(g_v)|\det(g_v)|^s$ with (now) $s\in\mathbb{C}$, $\Phi_v\in\mathcal{S}(K_v^n)$, and $\epsilon_n=(0,\ldots,0,1)\in k_v^n$. The stabilizer group is $S_{\lambda_v,\Theta_v}=N_n(k_v)$ the upper triangular maximal unipotent subgroup of S. Then our local functional $\Lambda_v(f_W,1;\Theta_v)$ is the local integral of Jacquet and Shalika

$$\Lambda_v(f_W,1;\Theta_v)=\int_{N_n(k_v)\backslash GL_n(k_v)}\lambda_v\left(f_W\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\Phi_v(\epsilon_n g)\chi_v(g)|\det(g)|^s dg$$
$$=\int\int W\left(\sigma\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} g & \\ & g\end{pmatrix}\right)\psi_v(tr(x))dx\ \Phi_v(\epsilon_n g)\chi_v(g)|\det(g)|^s dg$$

with the outer integral over $N_n(k_v)\backslash GL_n(k_v)$ and the inner integral over $\mathfrak{p}_{o,n}(k_v)\backslash M_n(k_v)$.

8.3 Doubling

Our reference here is [25].

We take $G(V)$ to be the group of isometries of a vector space V over k equipped with a non-degenerate symmetric or symplectic form. So either $G(V)=O(V)$ or $G(V)=Sp(V)$. Let $W=V\oplus(-V)$ where $-V$ represents the same vector space as V but with the negative of the original symmetric or symplectic form. So $G(W)$ is a split classical group and we have an embedding $\iota: G(V)\times G(V)\to G(W)$. Let $V^d\subset W$ be the "diagonal" subspace $V^d=\{(v,v)\in V\oplus(-V)=W\mid v\in V\}$; then V^d is a totally isotropic subspace of W. Let $P\subset G(W)$ be the parabolic subgroup stabilizing V^d.

Back to our scheme, we take $G=H=G(W,\mathbb{A})$ the adelic points of $G(W)$. For σ we take an induced representation of the form $\sigma=\mathrm{Ind}_{P(\mathbb{A})}^G(\delta_P^s)$. As before $H'=\{e\}$ and $\sigma'=\mathbb{C}$. Then λ will be a linear functional on this induced representation. For a section $F\in\mathrm{Ind}_{P(\mathbb{A})}^G(\delta_P^s)$ we set $\lambda(F)=E_F(e;s)$, the evaluation of the associated Eisenstein series at the identity e of G. (See section 5 of [25] for the definition of E_F. Note that they use f for the section instead of our F.) We also take the associated $f = f_F \in V(\sigma)$ so that $f_F(e) = E_F$.

For S we take $G(V,\mathbb{A}) \times G(V,\mathbb{A})$ embedded in G by ι given above. We take $\tau = \pi \otimes \pi$, where π is a cuspidal automorphic representation of $G(V,\mathbb{A})$ and let φ_1, $\varphi_2 \in \mathcal{H}_\pi$ be two cusp forms on $G(V,\mathbb{A})$. We finally take $\Theta(g_1,g_2) = \varphi_1(g_1)\varphi_2(g_2)$ and $S_{\lambda,\Theta} = G(V,k) \times G(V,k)$. Then $\Lambda(f_F,1;\Theta)$ gives the basic integral in the doubling method:

$$\Lambda(f_F,1;\Theta) = \int_{G(V,k)\times G(V,k)\backslash G(V,\mathbb{A})\times G(V,\mathbb{A})} E_F(\iota(g_1,g_2);s)\varphi_1(g_1)\varphi_2(g_2)dg_1\ dg_2.$$

Locally, the situation is a bit different, more so than in the GL_n case before. This is due to the basic identity in Section 1 of [25]. We fix a place v of k. We will still take $G = H = G(W,k_v)$ and σ the local induced representation $\sigma = \mathrm{Ind}_{P(k_v)}^G(\delta_P^s)$. As before, $H' = \{e\}$ and $\sigma' = \mathbb{C}$. Then λ is a functional on the space of the induced representation. If now F_v is a local section, then we take $\lambda(F_v) = F_v(e)$. Then $f = f_{F_v} \in V(\sigma)$ is as always taken so that $f(e) = F_v$. The difference is that now $S = G(V,k_v)$, only one copy of the classical group, embedded as $g \mapsto \iota(g,e) \in \iota(G(V,k_v) \times G(V,k_v)) \subset G(W,k_v)$. Still $\tau = \pi_v$ an irreducible admissible representation of $G(V,k_v)$, a local component of the above. Now for Θ we simply take a matrix coefficient for π_v, i.e., $\Theta(g) = \langle \pi_v(g)\varphi_{1,v}, \varphi_{2,v}\rangle$ for $\varphi_{i,v} \in \mathcal{H}_{\pi_v}$. Then $S_{\lambda,\Theta} = \{e\}$ and now our $\Lambda(f_{F_v},1;\Theta)$ gives the associated local integral for doubling, namely

$$\Lambda(f_{F_v},1;\Theta) = \int_{G(V,k_v)} F_v(\iota(g,e);s)\langle \pi_v(g)\varphi_{1,v}, \varphi_{2,v}\rangle\ dg.$$

8.4 Other examples

We hope the paradigm is now clear for expressing various integral representations in this scheme. It seems clear that the integrals appearing in [14] and [34], as well as [13] in this volume, can be expressed in this manner, since they all involve two automorphic forms, a cusp form and an Eisenstein series, on a group and a subgroup. We would take $G = H$ to be the larger group and S the subgroup. As always $H' = \{e\}$. The general paradigm is to take the form on the larger group and take a twisted period to "project" it to a form on an intermediate group. Evaluating this period at e would correspond to λ. This is then restricted and integrated against the form on the smaller group, S, which would then be our Θ. Thus $\Lambda(f,1;\Theta)$ would formally yield the integral representation. This seems to be the general pattern for placing integral representations in our scheme.

8.5 Bessel functions

The theory of Bessel functions, as developed in Section 6, seems to have little content in the situation of integral representations, at least as formulated here. Since $H' = \{e\}$, then if we have a functional of the form

$$\Lambda(f,1;\Theta) = \int_{S_{\lambda,\Theta}\backslash S} \lambda(f(s))\Theta(s)\ ds \tag{8.1}$$

then

$$B_f(s) = \lambda(f(s))$$

and the space of orbits $\overline{S}$ is merely S itself. So the expression in (8.1) is already in the form (6.2), i.e.,

$$\int_{S_{\lambda,\Theta}\backslash S} \lambda(f(s))\Theta(s)\ ds = \int_{S_{\lambda,\Theta}\backslash S} B_f(s)\Theta(s)\ ds.$$

However, the original proof of stability of γ-factors via Bessel functions took place in terms of integral representations. This was the content of [8]. In the next section we would like to revisit that work in light of the formalism developed here.

9 γ-factors and stability

We will consider the stability of the local γ-factor for SO_{2n+1} as in [8]. We refer to [8] for the general setup and any (inadvertently) undefined notation. Let k be a non-archimedean local field (of characteristic 0 for safety). This stability comes out of the local functional equation for the integral representation for the standard L-function for the split $SO_{2n+1}(k_v)$ twisted by a multiplicative character. Let us set this up in terms of our functionals.

For the basic local integrals we take $G = H = SO_{2n+1}(k)$ preserving the form represented by $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. The representation $\sigma = \pi$ is an irreducible generic representation of G realized in its Whittaker model, so $\mathcal{H}_\pi = \mathcal{W}(\pi, \psi)$ with ψ a non-trivial additive character of k. As is standard for integral representations, $H' = \{e\}$ and $\sigma' = \mathbb{C}$. So λ is a continuous functional on $\mathcal{H}_\pi$. To describe it we let

$$\beta = \begin{pmatrix} & & 1 \\ & -I_{2n-1} & \\ 1 & & \end{pmatrix},$$

a Weyl element of G. For each $t \in k^{2n-1}$ we set

$$x(t) = \begin{pmatrix} 1 & t & \frac{1}{2}\langle t, t\rangle \\ & I_{2n-1} & -t^* \\ & & 1 \end{pmatrix}$$

where $\langle t, t\rangle$ is the inner product induced on k^{2n-1} and t^* is the dual column vector defined by $tt^* = \langle t, t\rangle$. Let $e_1, \ldots, e_{2n-1}$ be the standard basis of k^{2n-1} then we set

$$\begin{aligned} X_\beta^- &= \{x(t) \mid t \in k^{2n-1}\} \\ X_1 &= \{x(t) \mid t \in Span(e_1, \ldots, e_n)\} \\ X_2 &= \{x(t) \mid t \in Span(e_{n+1}, \ldots, e_{2n-1}\} \\ U &= \beta X_2 \beta. \end{aligned}$$

Then the functional λ on $\mathcal{H}_\pi = \mathcal{W}(\pi,\psi)$ is

$$\lambda(W) = \int_U W(u)\ du.$$

The subgroup S is then a one dimensional torus (here we deviate from the notation of [8] a bit)

$$S = \left\{ \underline{h} = \begin{pmatrix} h & & \\ & I_{2n-1} & \\ & & h^{-1} \end{pmatrix} \mid h \in k^\times \right\} \simeq k^\times$$

and $\Theta(h) = \mu(h)|h|^{s-1/2}$ where μ is a character of $k^\times$ and $s \in \mathbb{C}$. The stabilizer $S_{\lambda,\Theta} = \{e\}$ is trivial and

$$\begin{aligned} \Lambda(f_W, 1; \Theta) &= \int_S \lambda(f_W(h))\mu(h)|h|^{s-1/2} d\underline{h} \\ &= \int_S \int_U W(u\underline{h}) du\ \mu(h)|h|^{s-1/2} d\underline{h} \\ &= \int_S \int_U W(\underline{h}u) du\ \mu(h)|h|^{s-n+1/2} d\underline{h} \\ &= I(W,\mu,s) \end{aligned}$$

where $I(W,\mu,s)$ is the integral representing $L(s,\pi\times\mu)$ as in [8].

The local γ-factor is defined by the local functional equation. In this instance, it relates two of our functionals, $\Lambda(f_W,1;\Theta)$ and $\Lambda(f_{R_\beta W},1;\Theta')$ where $\Theta'(h) = |h|\Theta^{-1}(h) = \mu^{-1}(h)|h|^{\frac{3}{2}-s}$, i.e.,

$$\gamma(s,\pi\times\mu,\psi)\Lambda(f_W,1;\Theta) = \Lambda(f_{R_\beta W},1;\Theta').$$

This is proven by viewing the two functionals as quasi-invariant functionals on $\mathcal{H}_\pi = \mathcal{W}(\pi,\psi)$ and proving a suitable uniqueness theorem for such functionals.

What we would now like to do is express $\gamma(s,\pi\times\mu,\psi)$ itself as one of our functionals. This is done by choosing a suitable (family of) W which makes $\Lambda(f_W,1;\Theta) \equiv 1$. This is done in Section 2 of [8]. Beginning with $W \in \mathcal{W}(\pi,\psi)$ such that $W(e) = 1$ there is constructed a compact open subgroup $X_{1,N} \subset X_1$ of the form

$$X_{1,N} = \left\{ x(t) \in X_1 \mid t = \sum_{i=1}^n t_i e_1 \quad \text{with} \quad |t_i| \le q^{N_i} \right\},$$

where we let $N = (N_1,\ldots,N_n) \in \mathbb{Z}_+^n$, such that if we let $X_{\beta,N}^- = X_2 X_{1,N} = X_{1,N} X_2$ then

$$\gamma(s,\pi\times\mu,\psi) = \int_S \int_{X_{\beta,N}^-} W(\underline{h}\beta x)\psi^{-1}(x) dx\ \mu^{-1}(h)|h|^{\frac{3}{2}-s-n} d\underline{h}$$

by Proposition 2.1 of [8]. So if we let

$$W_N(g) = \int_{X_{1,N}} W(gx)\psi^{-1}(x) dx \in \mathcal{H}_\pi = \mathcal{W}(\pi,\psi)$$

then

$$\gamma(s, \pi \times \mu, \psi) = \Lambda(f_{R_\beta W_N}, 1; \Theta') \tag{9.1}$$

As noted in Section 8.5, since we have $H' = \{e\}$ in this context, then the Bessel function is merely the term coming from the functional λ on $\mathcal{H}_\pi$. In this situation we then see

$$\begin{aligned} B_{f_{R_\beta W_N}}(h) &= \lambda(f_{R_\beta W_N}(h)) \\ &= \int_U W_N(\underline{h}x\beta)\ dx \\ &= \int_{X^-_{\beta,N}} W(\underline{h}\beta x)\psi^{-1}(x)dx. \end{aligned}$$

But this last expression is precisely the partial Bessel function $j_{v,Y}(\underline{h})$ associated to the open subgroup $Y = X^-_{\beta,N} \subset X^-_\beta$ and the vector v such that $W = W_v$ in Section 3 of [8], i.e.,

$$B_{f_{R_\beta W_N}}(h) = j_{v,Y}(\underline{h}).$$

Proposition 9.1. *Let $\Lambda(f_{R_\beta W_N}, 1; \Theta')$ be the functional associated to $\gamma(s, \pi\times\mu, \psi)$ in (9.1). Then the associated Bessel function is*

$$B_{f_{R_\beta W_N}}(h) = \int_{X^-_{\beta,N}} W(\underline{h}\beta x)\psi^{-1}(x)dx = j_{v,Y}(\underline{h}) \tag{9.2}$$

with the notation as above.

So even though the Bessel function $B_{f_{R_\beta W_N}}(h)$ is arrived at in a manner that seems "trivial" in the context of Section 6, it does recover the Bessel function used in [8]. If we combine (9.1) with the decomposition of the functional in terms of its Bessel function as in (6.2) we obtain an integral representation of the γ-factor.

Corollary 9.2. *The local γ-factor has an integral representation*

$$\gamma(s, \pi \times \mu, \psi) = \int_S j_{v,Y}(\underline{h})\mu^{-1}(h)|h|^{\frac{3}{2}-s-n} d\underline{h}. \tag{9.3}$$

This recovers Proposition 4.1 of [8], from which the proof of stability then progressed.

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