DUAL GROUPS AND LANGLANDS FUNCTORIALITY

J.W. COGDELL

Langlands never separated the Langlands conjectures for GL_n from his general principle of functoriality [30]. In particular, he formulated a correspondence between certain Galois representations and admissible or automorphic representations for any connected reductive algebraic group G. For GL_n there was a correspondence between certain n-dimensional Galois representations, that is, representations into $\mathrm{GL}_n(\mathbb{C})$, and admissible representations of $\mathrm{GL}_n(k)$ or automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ [4]. For general G we understand what to replace the automorphic side with: admissible representations of $G(k)$ or automorphic representations of $G(\mathbb{A})$. But what replaces the target $\mathrm{GL}_n(\mathbb{C})$ on the Galois side? Based on the Satake parameterization of unramified representations [33] and his classification of representations of algebraic tori [24] Langlands introduced his idea of a dual group, now known as the Langlands dual group or L-group , ${}^LG$ to play the role of $\mathrm{GL}_n(\mathbb{C})$. The role of the n-dimensional Galois representations is taken by certain admissible homomorphisms of the Galois group into this L-group. For the purposes of functoriality, it is most convenient to view these local and global correspondences for G as giving an arithmetic parameterization of the admissible or automorphic representations of G in terms of these admissible Galois homomorphisms to ${}^LG$.

Langlands principle of functoriality states that any L-homomorphism ${}^LH \to {}^LG$ should determine a *transfer* or *lifting* of admissible or automorphic representations of H to admissible or automorphic representations of G. Once one has a parameterization, then this is conceptually done by composing the parameterizing homomorphism for the representation of H with the L-homomorphism to obtain a parameterizing homomorphism for a representation of G. If one takes $H = \{1\}$, then ${}^LH$ is simply the Galois group or a closely related group and one in essence recovers the local or global Langlands correspondence for G from this principle of functoriality.

There have been many fundamental examples of functoriality established by trace formula methods: cyclic base change, cyclic automorphic induction, lifting between inner forms. Recently however there has been much progress in global functorialities to GL_n obtained using the converse theorem for GL_n. These include the tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 by Ramakrishnan [31] and from $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6 by Kim and Shahidi [21], the symmetric cube and symmetric fourth power lifts from GL_2 to GL_4 and GL_5 by Kim and Shahidi [20, 21, 22], and the lifting from split classical groups to GL_N with Kim, Piatetski-Shapiro, and Shahidi [5, 6].

In this paper we first describe the construction of the L-group and the formulation of the local and global Langlands conjectures for a general reductive group G [2]. We next outline Langlands' principle of functoriality and its relation to the local and global Langlands correspondences. We then turn to examples. We briefly consider some of the examples of functoriality mentioned above that were established using the trace formula. We then give a more detailed description of the new functorialities to GL_n and how one uses the converse theorem as a means for establishing these liftings.

I would like to thank the referee for helping to clarify certain issues related to this paper.

1. The Dual Group

Begin with G a connected reductive algebraic group defined over k, k a local or global field. Let $\overline{k}$ be a separable algebraic closure of k and $\mathcal{G}_k = Gal(\overline{k}/k)$ the Galois group. Over $\overline{k}$, G becomes split and is classified by its root data [2, 36]. Take in $G_{/\overline{k}}$ a Borel subgroup B and maximal torus T, both defined and split over $\overline{k}$. Let $X = \mathfrak{X}^*(T)$ denote the set of $\overline{k}$-rational characters of T, $\Phi = \Phi(G,T) \subset X$ the root system associated to G and T, and $\Delta \subset \Phi$ the set of simple roots corresponding to B. Dual to the triple (X, Φ, Δ) we have the triple $(X^\vee, \Phi^\vee, \Delta^\vee)$ consisting of the lattice $X^\vee = \mathfrak{X}_*(T)$ of co-characters, or $\overline{k}$-rational one-parameter subgroups, the co-root system $\Phi^\vee$, and the simple co-roots $\Delta^\vee$. The quadruple $\Psi(G) = (X, \Phi, X^\vee, \Phi^\vee)$ is the root data for G over $\overline{k}$ and the quadruple $\Psi_0(G) = (X, \Delta, X^\vee, \Delta^\vee)$ is the based root data for G over $\overline{k}$ [2, 36]. The basic structure for connected reductive $\overline{k}$-groups is the following [36].

Theorem 1.1. *The root data $\Psi(G)$ determines G up to $\overline{k}$–isomorphism.*

For the relative structure theory, there is a split exact sequence

$$1 \longrightarrow Int(G) \longrightarrow Aut(G) \longrightarrow Aut(\Psi_0(G)) \longrightarrow 1.$$

A splitting is given by making a choice of root vector x_α for each $\alpha \in \Delta$, which then defines a splitting $(G, B, T, \{x_\alpha\}_{\alpha\in\Delta})$ of G and gives a canonical isomorphism

$$Aut(\Psi_0(G)) \to Aut(G, B, T, \{x_\alpha\}) \subset Aut(G).$$

If G is defined over k, there is an action of $\mathcal{G}_k$ on $G_{/\overline{k}}$ giving the k-structure. Hence we have homomorphisms

$$\mathcal{G}_k \to Aut(G_{/\overline{k}}) \to Aut(\Psi_0(G)).$$

So $G_{/k}$ determines the two pieces of data consisting of the root data $\Psi(G)$, determining the group over $\overline{k}$, and the homomorphism $\mathcal{G}_k \to Aut(\Psi_0(G))$.

To define ${}^LG$ one simply dualizes this structure theory. Let $\Psi_0(G)^\vee = (X^\vee, \Delta^\vee, X, \Delta)$ be the dual based root data. This defines a connected reductive algebraic group ${}^LG^0$ over $\mathbb{C}$. We can transfer the Galois structure since

$$Aut(\Psi_0({}^LG^0)) = Aut(\Psi_0(G)^\vee) = Aut(\Psi_0(G))$$

and a splitting of the exact sequence above for ${}^LG^0$ gives a map $\mu : \mathcal{G}_k \to Aut(\Psi_0({}^LG^0)) \to Aut({}^LG^0)$ which fixes the corresponding splitting $({}^LG^0, {}^LB^0, {}^LT^0, \{x_{\alpha^\vee}\}_{\alpha^\vee\in\Delta^\vee})$ of ${}^LG^0$ and hence a $\mathcal{G}_k$ action on the complex reductive group ${}^LG^0$ which encodes some of the original k-structure of G.

Definition 1.1. *The (Langlands) dual group, or L-group, of G is*

$$ {}^{L}(G_{/k}) = {}^{L}G = {}^{L}G^{0} \rtimes \mathcal{G}_k. $$

Remarks. 1. Sometimes it is convenient use the Weil form of the dual group. Since there is a natural map $W_k \to \mathcal{G}_k$ one may form instead ${}^{L}G = {}^{L}G^{0} \rtimes W_k$, but there is no essential difference. One could also use a Weil-Deligne form for certain purposes.
2. If G' is a k-group which is isomorphic to G over $\overline{k}$, then G and G' are inner forms of each other iff ${}^{L}G$ is isomorphic to ${}^{L}G'$ over $\mathcal{G}_k$ [2]. So the dual group does not quite distinguish between k-forms; it distinguishes only up to inner forms. It does completely determine a quasi-split form.

In practice, this duality preserves the types A_n and D_n and interchanges the types B_n and C_n. In addition it interchanges the adjoint and simply connected forms of the relevant groups.

G	${}^{L}G^{0}$
GL_n	$\mathrm{GL}_n(\mathbb{C})$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C})$
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$
adjoint type	simply connected
simply connected	adjoint type

The local and global constructions are compatible. So if G is defined over a global field k, v is a place of k, and we let G_v to denote G as a group over k_v, then there are natural maps ${}^{L}G_v \to {}^{L}G$.

2. Langlands Conjectures for G

2.1. **Local Langlands Conjecture.** Let k be a local field and let W'_k be the associated Weil-Deligne group [4]. If k is archimedean, we simply take $W'_k = W_k$ to be the Weil group. Following Borel [2] a homomorphism $\phi : W'_k \to {}^{L}G$ is called *admissible* if

(i) ϕ is a homomorphism over $\mathcal{G}_k$, i.e., the following diagram commutes:

$$\begin{array}{ccc} W'_k & \xrightarrow{\phi} & {}^{L}G \\ \downarrow & & \downarrow \\ \mathcal{G}_k & = & \mathcal{G}_k \end{array}$$

(ii) ϕ is continuous, $\phi(\mathbb{G}_a)$ is unipotent in ${}^L G^0$, and ϕ maps semisimple elements to semisimple elements.
(iii) If $\phi(W_k')$ is contained in a Levi subgroup of a proper parabolic subgroup P of ${}^L G$ then P is relevant.

For all undefined concepts, such as *relevant*, we refer the reader to Borel [2]. If $G = \mathrm{GL}_n$ the admissible homomorphisms are precisely the Frobenius-semisimple complex representations of W_k' [4].

Following Borel [2] and Langlands [26] we let $\Phi(G)$ denote the set of all admissible homomorphisms $\phi : W_k' \to {}^L G$ modulo inner automorphisms by elements of ${}^L G^0$ (not to be confused with the earlier [4] use of Φ as a geometric Frobenius). Note that if G and G' are inner forms of one another, so that ${}^L G = {}^L G'$, it need not be true that $\Phi(G) = \Phi(G')$ since the condition (iii) above sees the k structures. If G is the quasi-split form, then one does have $\Phi(G') \subset \Phi(G)$.

To state the local Langlands conjecture for G there are two supplemental constructions that are needed, for which we refer the reader to Borel [2]. First, for every $\phi \in \Phi(G)$ there is a way to construct a character ω_ϕ of the center $C(G)$ of G. Next, if we let $C({}^L G^0)$ denote the center of ${}^L G^0$, then to every $\alpha \in H^1(W_k'; C({}^L G^0))$ there is associated a character χ_α of $G(k)$. If we write $\phi \in \Phi(G)$ as $\phi = (\phi_1, \phi_2)$ with $\phi_1(w) \in {}^L G^0$ and $\phi_2(w) \in \mathcal{G}_k$ then ϕ_1 is a cocycle on W_k' with values in ${}^L G^0$ and the map $\phi \mapsto \phi_1$ gives an embedding of $\Phi(G) \hookrightarrow H^1(W_k'; {}^L G^0)$. Then $H^1(W_k'; C({}^L G^0))$ acts naturally on $H^1(W_k'; {}^L G^0)$ and this action preserves $\Phi(G)$.

With these constructions, we can state the local Langlands conjecture for G [2]. As before, let $\mathcal{A}(G) = \mathcal{A}(G(k))$ denote the set of equivalence classes of irreducible admissible complex representations of $G(k)$.

Local Langlands Conjecture: *Let k be a local field. Then there is a surjective map $\mathcal{A}(G) \to \Phi(G)$ with finite fibres which partitions $\mathcal{A}(G)$ into disjoint finite sets $\mathcal{A}_\phi = \mathcal{A}_\phi(G)$ satisfying*

(i) *If $\pi \in \mathcal{A}_\phi$ then the central character ω_π of π is equal to ω_ϕ;*
(ii) *Compatibility with twisting, i.e., if $\alpha \in H^1(W_k'; C({}^L G^0))$ and χ_α is the associated character of $G(k)$ then $\mathcal{A}_{\alpha\cdot\phi} = \{\pi\chi_\alpha | \pi \in \mathcal{A}_\phi\}$;*
(iii) *One element $\pi \in \mathcal{A}_\phi$ is square integrable modulo $C(G)$ iff all $\pi \in \mathcal{A}_\phi$ are square integrable modulo $C(G)$ iff $\phi(W_k')$ does not lie in a proper Levi subgroup of ${}^L G$;*
(iv) *One element $\pi \in \mathcal{A}_\phi$ is tempered iff all $\pi \in \mathcal{A}_\phi$ are tempered iff $\phi(W_k)$ is bounded;*
(v) *If H is a reductive connected k-group and $\eta : H(k) \to G(k)$ is a k morphism with commutative kernel and co-kernel, then there is a required compatibility between decompositions for $G(k)$ and $H(k)$. Namely, η induces a natural map ${}^L\eta : {}^L G \to {}^L H$ and if we set $\phi' = {}^L\eta \circ \phi$ for $\phi \in \Phi(G)$ then any $\pi \in \mathcal{A}_\phi(G)$, when viewed as a $H(k)$ module, decomposes into a direct sum of finitely many irreducible admissible representations belonging to $\mathcal{A}_{\phi'}(H)$.*

The sets $\mathcal{A}_\phi(G)$ for $\phi \in \Phi(G)$ are called *L-packets*. The version I of the local Langlands conjecture in [4] was the specialization of this to the group GL_n. In that case, the L-packets are all singletons and the map from $\mathcal{A}(G)$ to $\Phi(G)$ was a bijection. This conjecture gives an *arithmetic parameterization* of the irreducible admissible representations of $G(k)$.

Other than the results for GL_n, the following is known towards this conjecture.

1. If the local field k is archimedean, i.e., $k = \mathbb{R}$ or $\mathbb{C}$, then this was completely established by Langlands [26].

2. If k is non-archimedean and G is quasi-split over k and split over a finite Galois extension then one knows how to parameterize the unramified representations of $G(k)$ via the unramified admissible homomorphisms [2]. This is a rephrasing in this language of the Satake classification [33].

3. If k is non-archimedean then Kazhdan and Lusztig have shown how to parameterize those representations of $G(k)$ having an Iwahori fixed vector in terms of admissible homomorphisms of the Weil-Deligne group [19].

4. Recently, in the case of k non-archimedean of characteristic zero and G the split SO_{2n+1}, Jiang and Soudry have given the parameterization of generic representations of $\mathrm{SO}_{2n+1}(k)$ in terms of admissible homomorphisms of the Weil-Deligne group [16, 17]. They obtain this parameterization as an outgrowth of recent work on global functoriality from split SO_{2n+1} to GL_{2n}, to be discussed later, by pulling back the parameterization for $\mathrm{GL}_{2n}(k)$.

If one thinks of this version of the local Langlands conjecture as providing an arithmetic parameterization of the irreducible admissible representations of $G(k)$, then one can use this parameterization to define local L-functions associated to arbitrary $\pi \in \mathcal{A}(G)$. One needs a second parameter, namely a representation $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$, by which we mean a continuous homomorphism whose restriction to ${}^LG^0$ is a morphism of complex Lie groups. Then for any admissible homomorphism $\phi \in \Phi(G)$ the composition $r \circ \phi : W_k' \to \mathrm{GL}_n(\mathbb{C})$ is a continuous complex representation of the Weil-Deligne group as considered in [4] and to it we can associate an L-factor $L(s, r \circ \phi)$ and ε-factor $\varepsilon(s, r \circ \phi, \psi)$ for an additive character ψ of k.

Definition 2.1. *If $\pi \in \mathcal{A}_\phi$ is in the L-packet defined by the admissible homomorphism ϕ then we set*

$$L(s, \pi, r) = L(s, r \circ \phi) \qquad \text{and} \qquad \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \phi, \psi).$$

According to this definition, one cannot distinguish between the representations π lying in a given L-packet $\mathcal{A}_\phi$ in terms of their L-functions and ε-factors, hence the terminology. At present these L-functions are well defined only for those π for which the parameterization is known, for example if π is unramified.

If one takes this as the definition of the local L-functions attached to an admissible representation, then version II of the local Langlands conjecture presented in [4] would be phrased in terms of matching L- and ε-factors defined in an analytic nature, as in [3] for GL_n, with those defined here. I have not seen a formulation in these terms for general reductive groups, however in the work of Jiang and Soudry cited above this is what they achieve. To each generic representation π of $\mathrm{SO}_{2n+1}(k)$ they attach an admissible homomorphism ϕ_π such that for the standard embedding $r : \mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$ they have an equality

$$L(s, \pi \times \pi') = L(s, \pi \times \pi', r \otimes id) = L(s, (r \circ \phi_\pi) \otimes \rho_{\pi'})$$

with the similar equality of ε-factors where π' is an irreducible admissible representation of $\mathrm{GL}_m(k)$, $\rho_{\pi'}$ is the associated representation of W_k' from the local Langlands conjecture for GL_m, and $L(s, \pi \times \pi')$ is the analytic L-function defined by Shahidi [34].

2.2. **Global Langlands Conjecture.** Now take k to be a global field and $\mathbb{A}$ its ring of adeles. For G a reductive algebraic group over k, let $\mathcal{A}(G) = \mathcal{A}(G(\mathbb{A}))$ denote the set of irreducible automorphic representations of $G(\mathbb{A})$. As with GL_n, to formulate a global Langlands conjecture we would replace the Weil-Deligne group W_k' by the conjectural Langlands group $\mathcal{L}_k$ and consider the set of admissible homomorphisms $\phi : \mathcal{L}_k \to {}^LG$. These homomorphisms should then parameterize irreducible automorphic representations of $G(\mathbb{A})$ in some way. The exact form this would take is quite speculative at the moment.

Not knowing what this should look like, one still expects to have global–local compatibility. If one begins an irreducible automorphic representation $\pi = \otimes' \pi_v$ of $G(\mathbb{A})$ then, assuming the local Langlands conjecture for each local group $G(k_v)$, one can attach to π the collection $\{\phi_v\}$ of local parameters $\phi_v = \phi_{\pi_v} : W_{k_v}' \to {}^LG_v$ given by the local components π_v. If we compose these with the natural compatibility maps for the dual groups $\iota_v : {}^LG_v \to {}^LG$ one gets a collection $\{\iota_v \circ \phi_v\}$ of local parameters $\iota_v \circ \phi_v : W_{k_v}' \to {}^LG$.

Such a system of maps must come out of a global parameter $\phi : \mathcal{L}_k \to {}^LG$ for the local and global theories to be consistent. This system of local parameters can often be used as a substitute for a global parameter ϕ. For example, this collection of local data is sufficient to define the global L-function and ε-factor attached to π. If $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ then the composition $r_v = r \circ \iota_v : {}^LG_v \to \mathrm{GL}_n(\mathbb{C})$ gives representations of the local dual groups.

Definition 2.2. *If $\pi = \otimes' \pi_v$ is an irreducible automorphic representation of $G(\mathbb{A})$ and $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ we set*

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r_v) = \prod_v L(s, r \circ \iota_v \circ \phi_v)$$

and

$$\varepsilon(s, \pi, r) = \prod_v \varepsilon(s, \pi_v, r_v, \psi_v) = \prod_v \varepsilon(s, r \circ \iota_v \circ \phi_v, \psi_v)$$

where $\psi = \otimes \psi_v$ is an additive character of $\mathbb{A}$ trivial on k.

To define the full L-function as above requires the solution of the local Langlands conjecture at all places, something only known for GL_n. However, for any irreducible automorphic representation π there is a finite set of places $S = S(\pi)$ such that for all $v \notin S$ the representation π_v is unramified and hence the local parameterization problem has been solved. Then the partial L-function

$$L^S(s,\pi,r)=\prod_{v\notin S} L(s,\pi_v,r_v)$$

is always well defined and Langlands has shown that this Euler product is always absolutely convergent in a right half plane [25].

3. Functoriality

As one can tell from his recent writings [29, 30] Langlands has always viewed the "principle of functoriality" as central to his view of automorphic representations. It encompasses what is referred to above as the "local and global Langlands conjectures" as special cases of this principle.

Let k denote either a local or global field and let H and G be two connected reductive groups defined over k. We have defined their associated dual groups ${}^LH$ and ${}^LG$. A homomorphism $u : {}^LH \to {}^LG$ is called an *L-homomorphism* if (i) it is a homomorphism over $\mathcal{G}_k$, that is, we have the commutation of the following diagram

$$\begin{array}{ccc} {}^LH & \xrightarrow{u} & {}^LG \\ \downarrow & & \downarrow \\ \mathcal{G}_k & = & \mathcal{G}_k \end{array}$$

(ii) u is continuous, and (iii) the restriction of u to ${}^LH^0$ is a complex analytic homomorphism $u : {}^LH^0 \to {}^LG^0$.

If in addition G is quasi-split, then for any admissible homomorphism $\phi \in \Phi(H)$ the composition $u \circ \phi$ is again an admissible homomorphism in $\Phi(G)$. So the map $\phi \mapsto u \circ \phi$ defines a map $\Phi(u) : \Phi(H) \to \Phi(G)$. If k is a global field and v a place of k then, since $\mathcal{G}_{k_v}$ can be viewed naturally as a subgroup of $\mathcal{G}_k$, we can view ${}^LG_v$ as a subgroup of ${}^LG$. Then, upon restriction to ${}^LH_v$, u will induce an L-homomorphism of the local dual groups $u_v : {}^LH_v \to {}^LG_v$ and hence a local map $\Phi(u_v) : \Phi(H_v) \to \Phi(G_v)$.

The principle of functoriality can now be roughly formulated as follows [30].

The Principle of Functoriality: *If k is a local (respectively global) field, H and G connected reductive k-groups with G quasi-split, then to each L-homomorphism $u : {}^LH \to {}^LG$ there is associated a* transfer *or* lifting *of admissible (resp. automorphic) representations of H to admissible (resp. automorphic) representations of G.*

If we assume the local and global Langlands conjectures, so that we have an arithmetic parameterization of $\mathcal{A}(H)$ and $\mathcal{A}(G)$ then this process of lifting is easy to describe.

3.1. **Local functoriality.** First, take k to be a local field, $u : {}^LH \to {}^LG$ a local L-homomorphism. If we take $\pi \in \mathcal{A}(H)$ an irreducible admissible representation of $H(k)$ then this is parameterized by an admissible homomorphism $\phi = \phi_\pi : W_k' \to {}^LH$. In fact, ϕ parameterizes an entire local L-packet $\mathcal{A}_\phi(H)$. If we compose ϕ with u we obtain $\phi' = \Phi(u)(\phi) = u \circ \phi \in \Phi(G)$, an admissible homomorphism of W_k' to ${}^LG$. Then ϕ' parameterizes a local L-packet $\mathcal{A}_{\phi'}(G)$ and this L-packet (or sometimes any element Π of it) is the functorial lift (or transfer, or Langlands lift, or ...) of π or of the packet $\mathcal{A}_\phi(H)$.

In general, we then "understand" the local functoriality in the cases where we understand the local parameterization:

1. $k = \mathbb{R}$ or $\mathbb{C}$, H any connected reductive k-group and G any quasi-split connected reductive k-group.

2. k a non-archimedean local field, $H = \mathrm{GL}_m$ and $G = \mathrm{GL}_n$ (and related examples – see Section 4).

3. Suppose that k is non-archimedean with ring of integers $\mathcal{O}$. Suppose both H and G are quasi-split and there is a finite extension K of k such that both H and G split over K and have an $\mathcal{O}$ structure so that both $H(\mathcal{O})$ and $G(\mathcal{O})$ are special maximal compact subgroups. Let π be an unramified representation of $H(k)$ with a non-trivial $H(\mathcal{O})$ vector and unramified parameter $\phi = \phi_\pi \in \Phi(H)$. Then for any L-homomorphism $u : {}^LH \to {}^LG$ the parameter $\phi' = u \circ \phi$ is unramified and defines an L-packet $\mathcal{A}_{\phi'}(G)$ which contains a (unique) representation Π of $G(k)$ which is unramified with respect to $G(\mathcal{O})$ [2]. Π is called the natural unramified lift of π.

3.2. **Global functoriality.** If we now consider k a global field, then, in principle, functorial lifting should work as it does in the local situation in terms of global parameterization. But now we are again at a disadvantage since we don't really understand the parameterizing group $\mathcal{L}_k$. In its stead, we fall back on the desired local-global compatibility. So let H be a connected reductive k-group, G a quasi-split connected reductive k-group and $u : {}^LH \to {}^LG$ an L-homomorphism. For each place v of k we have the associated local L-homomorphism $u_v : {}^LH_v \to {}^LG_v$ described above. Now let $\pi \in \mathcal{A}(G)$, $\pi = \otimes' \pi_v$, be an irreducible automorphic representation of $H(\mathbb{A})$. If v is archimedean then by the work of Langlands we know how to parameterize π_v with a local parameter $\phi_v : W_{k_v}' \to {}^LH_v$. If v is a non-archimedean place, then for almost all v the local group H_v is quasi-split, split over a finite extension of k_v, and the representation π_v is unramified with respect to a special maximal compact subgroup. So we are in the situation where we have a local parameter $\phi_v : W_{k_v}' \to {}^LH_v$ for π_v. In either of these situations, we can form a local lift Π_v as a representation of $G(k_v)$ associated to the parameter $\phi_v' = u_v \circ \phi_v$, that is, a local lift as defined above.

Definition 3.1. *Let H be a connected reductive k-group and let $\pi = \otimes' \pi_v$ be an irreducible automorphic representation of $H(\mathbb{A})$. Let G be a quasi-split connected reductive k-group and let $u : {}^LH \to {}^LG$ be an L-homomorphism. Then an automorphic representation $\Pi = \otimes' \Pi_v$ of $G(\mathbb{A})$ is a (weak) functorial lift of π (with respect to u) if for all archimedean places and* *almost all finite places where π_v is unramified we have that Π_v is a local functorial lift with respect to u_v as described above. Π is a (strong) functorial lift of π if Π_v is a local functorial lift of π_v for all places of k.*

Note that as a consequence of this definition, if π is an automorphic representation of $H(\mathbb{A})$, $u : {}^LH \to {}^LG$ an L-homomorphism, and Π a functorial lift of π to an automorphic representations of $G(\mathbb{A})$, then for every representation $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ we have an equality of L-functions and ε-factors

$$L^S(s, \pi, r \circ u) = L^S(s, \Pi, r) \qquad \varepsilon^S(s, \pi, r \circ u, \psi) = \varepsilon^S(s, \Pi, r, \psi)$$

where S is the finite (possibly empty) set of places where we do not know how to locally lift π_v.

In fact, we need to do this on the level of L-packets. This is easy enough to formulate, but given the partial state of our knowledge, there seems to be little gained in doing this at this time. But the ambiguity in the local lifts and hence the global lifts coming from the phenomenon of local and global L-packets should always be kept in mind.

4. Examples

We have noted that Langlands views functoriality as encompassing the local and global Langlands conjectures and their consequences, such as the strong Artin conjecture. One reason for this is the following example.

Consider the case where $H = \{1\}$. Begin with k a local field. Since there is a natural map from the Weil-Deligne group W_k' to $\mathcal{G}_k$ we may consider the Weil-Deligne form of the L-group: ${}^LG = {}^LG^0 \rtimes W_k'$. Then ${}^LH = W_k'$. If we take for example $G = \mathrm{GL}_n$ then $u : {}^LH \to {}^LG$ is an admissible homomorphism in $\Phi(G)$ or a complex representation of the Weil-Deligne group and functoriality for these groups encompasses the local Langlands conjectures. If one takes k a global field and leaves ${}^LG$ as the Galois form of the L-group, then again taking $H = \{1\}$ and $G = \mathrm{GL}_n$ we obtain a global Langlands conjecture for GL_n.

The other examples of functoriality I wish to discuss fall into what I view as two types: Galois theoretic and group theoretic. The first include base change, automorphic induction, and lifting between inner forms. The second are all liftings to GL_n and include the tensor product liftings, symmetric powers liftings, and liftings from classical groups. I will not touch on the important class of liftings known as *endoscopic*, even though some of the example we discuss can be interpreted as examples of (possibly twisted) endoscopy. Endoscopic liftings are those in which the L-homomorphism $u : {}^LH \to {}^LG$ realizes ${}^LH^0$ as the fixed points of an involution in ${}^LG^0$, or a twisted such. The significance of these liftings come primarily from their necessity in understanding the trace formula, which we are not in a position to discuss. Instead, we refer the reader to the work of Langlands [28, 30] and of Kottwitz and Shelstad [23] and the references therein.

4.1. **Galois theoretic examples.** In these examples, the L-homomorphisms have their origins in Galois theory.

1. *Base change (or automorphic restriction).* Suppose that K is a finite extension of k. Then on the level of Weil groups we have $W_K \subset W_k$ so that any representation of W_k gives a representation of W_K by restriction. The analogous lifting on the level of admissible or automorphic representations is the following. Let H be connected, reductive and split over k. Then we may consider H as a group over K as well and if we let $G = R_{K/k}(H)$ be Weil's restriction of scalars from K to k, so $G(k) = H(K)$, then G is the group over k determined by $H_{/K}$. There is then a natural embedding

$$u : {}^LH = {}^LH^0 \times \mathcal{G}_k \to (\prod_{\mathcal{G}_K\backslash\mathcal{G}_k} {}^LH^0) \rtimes \mathcal{G}_k = {}^LG,$$

where $\mathcal{G}_k$ acts on $\prod {}^LH^0$ via permutations of the index set, which is the diagonal map on ${}^LH^0$ and the identity on $\mathcal{G}_k$. In the case where k is a local field, then the induced map $\Phi(u) : \Phi(H) \to \Phi(G)$ is indeed the restriction map, viewing W_K' as an open subgroup of W_k'. Functoriality coming from this L-homomorphism would begin with a representation π of $H(k)$ or $H(\mathbb{A}_k)$ and produce a representation of $G(k) = H(K)$ or $H(\mathbb{A}_K)$ called the *base change* of π. This program has been carried out when $H = \mathrm{GL}_n$ and the extension K/k is solvable, first for $n = 2$ by Langlands [27] and then general n by Arthur and Clozel [1]. Their technique was the twisted trace formula. In addition, when $H = \mathrm{GL}_2$ Jacquet, Piatetski-Shapiro, and Shalika have obtained a non–normal cubic base change by converse theorem methods [14].

2. *Automorphic induction.* We still take K a finite separable extension of k of degree d, so that $W_K \subset W_k$. If one starts with a representation of W_K then one obtains a representation of W_k simply by induction. The analogous lifting on the level of admissible or automorphic representations is now the following. Take $H = R_{K/k}(\mathrm{GL}_n)$ to be $\mathrm{GL}_n(K)$ viewed as a k-group as above and let $G = \mathrm{GL}_{dn}(k)$. Now one has an L-homomorphism

$$u : {}^LH = (\prod_{\mathcal{G}_K\backslash\mathcal{G}_k} \mathrm{GL}_n(\mathbb{C})) \rtimes \mathcal{G}_k \to {}^LG = \mathrm{GL}_{dn}(\mathbb{C}) \times \mathcal{G}_k$$

by sending ${}^LH^0 = \mathrm{GL}_n(\mathbb{C}) \times \cdots \times \mathrm{GL}_n(\mathbb{C})$ into ${}^LG^0 = \mathrm{GL}_{dn}(\mathbb{C})$ as block diagonal matrices and extending to an L-homomorphism by letting $\mathcal{G}_k$ act on $\mathrm{GL}_{dn}(\mathbb{C})$ via permutation matrices from $\mathfrak{S}_d$. The local or global functorialities coming from such an L-homomorphism are called *automorphic induction.* The map $\Phi(u)$ on the sets of admissible homomorphisms should be induction. Again, when the extension K/k is solvable this was analyzed locally and globally by Arthur and Clozel [1] using the twisted trace formula, preceded by Jacquet and Langlands for $n = 2$ [15]. Henniart and Herb, building on earlier work by Kazhdan in the $n = 1$ case [18], gave the first definition and analysis of local automorphic induction for GL_n in terms of local character identities [13]. This work uses a simpler version of the trace formula than either [1] or [18] and allows fields of positive characteristic.

3. *Inner forms.* Let G be connected, reductive, and quasi-split over a local or global k and let H be an inner form of G. Then ${}^LH = {}^LG$, the identity map $u : {}^LH \to {}^LG$ is an L-homomorphism, and we should have a corresponding lifting. Note that if k is a local field

we have $\Phi(H) \subset \Phi(G)$, while if k is a global field we in fact have $H_v = G_v$ for almost all places so that $\Phi(H_v) = \Phi(G_v)$. In the case of $G = \mathrm{GL}_2$ and $H = D^\times$ the multiplicative group of a rank 2 division algebra over k the lifting from representations of $D^\times$ to representations of GL_2 is the so-called Jacquet-Langlands correspondence, established in [15]. If we take $G = \mathrm{GL}_n$ and $H = \mathrm{GL}_m(D)$ where D is a central simple division algebra of rank d with $dm = n$ then the local functoriality has been analysed by Rogawski [32] in the case $m = 1$ and by Deligne, Kazhdan, and Vigneras [10] utilizing the trace formula.

4.2. **Group theoretic examples.** In this set of examples, the groups H involved are all split and the target group G is always a general linear group GL_n, so the Galois theory plays little role. The L-homomorphism is a natural map from group theory. There has been much progress in this family of functorialities recently based on using the converse theorem for GL_n as the primary tool for establishing global functorialities to GL_n.

1. *Tensor products.* Let k be either a local of global field and let $H = \mathrm{GL}_m \times \mathrm{GL}_n$. Then ${}^LH^0 = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ and ${}^LH = {}^LH^0 \times \mathcal{G}_k$. If we take $G = \mathrm{GL}_{mn}$ then ${}^LG^0 = \mathrm{GL}_{mn}(\mathbb{C})$ and ${}^LG = {}^LG^0 \times \mathcal{G}_k$. The simple tensor product map $\otimes : \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{mn}(\mathbb{C})$, extended by the identity map on $\mathcal{G}_k$, defines an L-homomorphism $u_\otimes : {}^LH \to {}^LG$. The associated functoriality is the *tensor product lifting.* Note that if k is a local field, then the local lifting is now understood in principle since the local parameterization problem (local Langlands conjecture) for GL_n has been solved. So the interesting question is the global functoriality. This has been recently solved in the cases of $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 by Ramakrishnan [31] and $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6 by Kim and Shahidi [21].

2. *Symmetric powers.* Let k be either a local or global field and let $H = \mathrm{GL}_2$, so ${}^LH^0 = \mathrm{GL}_2(\mathbb{C})$ and ${}^LH = {}^LH^0 \times \mathcal{G}_k$. We take $G = \mathrm{GL}_{n+1}$ for $n \geq 1$, so ${}^LG^0 = \mathrm{GL}_{n+1}(\mathbb{C})$ and ${}^LG = {}^LG^0 \times \mathcal{G}_k$. For each $n \geq 1$ there is the natural symmetric n-th power map $sym^n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$. If we extend this symmetric power map by the identity map on the Galois group we obtain an L-homomorphism $sym^n : {}^LH \to {}^LG$. The associated functoriality is the *symmetric power lifting* from representations of GL_2 to representations of GL_{n+1}. Once again, if k is a local field the local symmetric powers liftings are understood in principle thanks to the solution of the local Langlands conjecture for GL_n. So once again the interesting functoriality is the global one. The global symmetric square lifting, so GL_2 to GL_3, is an old theorem of Gelbart and Jacquet [11]. Recently, Kim and Shahidi have shown the existence of the global symmetric cube lifting from GL_2 to GL_4 [21, 22] and then Kim followed with the global symmetric fourth power lifting from GL_2 to GL_5 [20, 22]. The achievement of symmetric power functoriality for all n would lead to a proof of the Ramanujan conjecture for GL_2.

3. *Classical groups.* Again, k is either a local or global field. Take H to be a split classical group over k, more specifically, the split form of either SO_{2n+1}, Sp_{2n}, or SO_{2n}. The connected component of the L-group are then $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, or $\mathrm{SO}_{2n}(\mathbb{C})$ and there are natural embeddings into an appropriate general linear group.

H	${}^LH^0$	$u^0 : {}^LH^0 \hookrightarrow {}^LG^0$	${}^LG^0$	G
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$	$\mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{SO}_{2n+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$	$\mathrm{GL}_{2n+1}(\mathbb{C})$	GL_{2n+1}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$	$\mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}

These homomorphisms extend to L-homomorphisms by extending them with the identity map on the Galois groups. Associated to each should be a lifting of admissible or automorphic representations from $\mathcal{A}(H)$ to $\mathcal{A}(G)$. In collaboration with Kim, Piatetski-Shapiro, and Shahidi, we established a weak global lift for generic cuspidal representations from SO_{2n+1} to GL_{2n} over a number field k using converse theorem methods [5]. Soon thereafter, Ginzburg, Rallis, and Soudry showed that our weak lift was indeed a strong lift and characterized the image [12]. The results of Jiang and Soudry on the local Langlands conjecture for SO_{2n+1} over a p-adic field cited above [16, 17] were then obtained as a local consequence of this global functoriality. Recently we have been able to extend our functoriality results to the other split classical groups as well [6].

We would like to explain the converse theorem method for obtaining global functorialities to general linear groups. We begin with a group H defined over a number field k. Take $\pi = \otimes\pi_v$ a cuspidal representation of $H(\mathbb{A})$. For each local place v we apply local functoriality to construct a local representation Π_v of $G(k_v) = \mathrm{GL}_N(k_v)$ for an appropriate N. If we are in example 1 or 2 above, we can do this for all v since the local Langlands conjecture is known for $\mathrm{GL}_n(k_v)$ [4]. For the cases of the classical groups we can perform this at all archimedean places v and at the non-archimedean places v where π_v is unramified. The method is simply composing the local parameter map ϕ_v for π_v with the L-homomorphism as described above. In the case of classical groups we must finesse the local liftings at the remaining places v to construct a local lift Π_v. But assume for now that we understand the local lifts at all places. Then by construction we have an equality of local L-factors

$$L(s, \pi_v, r_v) = L(s, r_v \circ \phi_v) = L(s, u_v \circ \phi_v) = L(s, \Pi_v, \iota_v)$$

with a similar equality for local ε-factors. Here we may take $r = u^0$ viewed as a complex representation $r : {}^LH \to \mathrm{GL}_N(\mathbb{C})$ and $\iota : {}^LG \to \mathrm{GL}_N(\mathbb{C})$ is just projection onto the first factor ${}^LG^0$. Hence, if we set $\Pi = \otimes'\Pi_v$ as an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A})$ then we globally have

$$L(s, \pi, r) = L(s, \Pi, \iota) \qquad \text{and} \qquad \varepsilon(s, \pi, r) = \varepsilon(s, \Pi, \iota).$$

Additionally, if $\pi' = \otimes\pi'_v$ is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m \leq N-2$ then we similarly have

$$L(s, \pi_v \times \pi'_v, r_v \otimes \iota_v) = L(s, \Pi_v \times \pi'_v, \iota_v \otimes \iota_v)$$

and hence

$$L(s, \pi \times \pi', r \otimes \iota) = L(s, \Pi \times \pi', \iota \otimes \iota) = L(s, \Pi \times \pi')$$

with similar equalities for local and global ε-factors. As outlined in [3], to apply the converse theorem for GL_N we must control the analytic properties of the twisted L-functions $L(s,\Pi\times\pi')=L(s,\Pi\times\pi',\iota\otimes\iota)$ for a sufficient family of cuspidal twists π'. But from our equality of L- and ε-factors, we have that these are controlled by the analytic properties of the *automorphic* L-functions $L(s,\pi\times\pi',r\otimes\iota)$ for the group $H(\mathbb{A})$ with twisting by $\mathrm{GL}_m(\mathbb{A})$. So once sufficient analytic control of these L-functions is known, one simply applies the converse theorem [3] for GL_N and concludes that Π is automorphic. In most cases to date, this analytic control of the $L(s,\pi\times\pi',r\otimes\iota)$ has been achieved by the Langlands-Shahidi method of analyzing the L-functions through the Fourier coefficients of Eisenstein series.

Let us now revisit our examples above in light of this sketch.

1. *Tensor products.* In the case of Ramakrishnan [31], so the functoriality from $\mathrm{GL}_2\times\mathrm{GL}_2$ to GL_4, $\pi=\pi_1\otimes\pi_2$ with each π_i a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ and Π is to be an automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. To apply the converse theorem from [9] Ramakrishnan needs to control the analytic properties of $L(s,\Pi\times\pi')$ for π' cuspidal representations of $\mathrm{GL}_1(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{A})$, that is , the Rankin triple product L-functions

$$L(s,\pi\times\pi',r\otimes\iota)=L(s,\pi_1\times\pi_2\times\pi').$$

This he was able to do using a combination of the integral representation for this L-function due to Garrett and then Rallis and Piatetski-Shapiro and the work of Shahidi on the Langlands-Shahidi method. The case of Kim and Shahidi [21] is similar, now with π_2 a cuspidal representation of $\mathrm{GL}_3(\mathbb{A})$. However, since the lifted representation Π is to be an automorphic representation of $\mathrm{GL}_6(\mathbb{A})$, to apply the converse theorem of [9] they must control the analytic properties of

$$L(s,\Pi\times\pi')=L(s,\pi_1\times\pi_2\times\pi')$$

where now π' must run over appropriate cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$ with $m=1,2,3,4$. The control of these triple products is an application of the Langlands-Shahidi method of analysing L-functions and involves coefficients of Eisenstein series on GL_5, Spin_{10}, and the simply connected E_6 and E_7.

2. *Symmetric powers.* The original symmetric square lifting of Gelbart and Jacquet indeed used the converse theorem for GL_3 [11]. One needs only control twists by characters (automorphic forms on GL_1) and the L-function that one must control is the symmetric square L-function for GL_2 since

$$L(s,\Pi)=L(s,\pi,sym^2).$$

This they were able to do via an integral representation due to Shimura. For Kim and Shahidi, the symmetric cube and fourth power liftings were deduced from the functorial $\mathrm{GL}_2\times\mathrm{GL}_3$ tensor product lift above and the exterior square lift for GL_4 [20, 21, 22].

3. *Classical groups.* Here there is a secondary problem. If we begin with a generic cuspidal representation $\pi=\otimes\pi_v$ of $H(\mathbb{A})$, then there is a finite set of finite places S at which one does not know the local parameterization for π_v in terms of admissible homomorphisms, and hence one does not know what the correct local lift Π_v should be. In this case, one is able to take an *arbitrary* local lift Π_v at those places, so long as it has trivial central

character. To compensate, one applies the form of the converse theorem for GL_N in which one fixes a single highly ramified idele class character η, the ramification depending on the original representation π of $H(\mathbb{A})$ and the constructed representation Π of $\mathrm{GL}_N(\mathbb{A})$ (and actually only on the local components at the places $v \in S$), and then twists by all cuspidal representations π' of $\mathrm{GL}_m(\mathbb{A})$, $m \leq N-1$, of the form $\pi' = \tau \otimes \eta$ where τ is *unramified at all* $v \in S$ [3, 5]. This highly ramified twist plays two roles. First, it helps to control global poles of the twisted L-functions $L(s, \pi \times \pi')$ for $H(\mathbb{A})$ and secondly it allows one to match the local L- and ε-factors at those $v \in S$ through the *stability of the local* γ*-factors under highly ramified twists* [3, 8, 5]. So for these limited twists one indeed has

$$L(s, \pi \times \pi') = L(s, \pi \times \pi', r \otimes \iota) = L(s, \Pi \times \pi', \iota \otimes \iota) = L(s, \Pi \times \pi')$$

with similar equalities for ε factors. Since we are able to control the analytic properties of the $L(s, \pi \times \pi')$ via the Langlands-Shahidi method for our family of π' we may apply the converse theorem for GL_N and conclude the existence of an automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A})$ such that $\Pi_v = \Pi'_v$ for all $v \notin S$, that is, a weak lift Π' of π.

Every step in this argument is now valid for general split classical group of the type we are considering. Originally the local stability of γ-factors was known only for SO_{2n+1} [8, 5]. Now, thanks to recent results of Shahidi expressing his local coefficients as Mellin transforms of Bessel functions [35], the techniques of [8] can be used to establish the stability of the local γ-factors for the other split classical groups as well. This then allows us to extend the functoriality results of [5] to these cases [6].

In the case of SO_{2n+1}, once we have the weak lift then the theory of Ginzburg, Rallis, and Soudry [12] allows one to show that this weak lift is indeed a strong lift in the sense that the local components Π_v at those $v \in S$ are completely determined – there is in fact no possible ambiguity. In conjunction with this they are able to completely characterize the image. Once one knows that these lifts are rigid, then one can begin to define a local lift by setting the lift of π_v to be the Π_v determined globally. This is the content of the papers of Jiang and Soudry [16, 17]. In essence they show that this local lift satisfies the relations on L-functions that one expects from functoriality and then uses this lift to pull back the parameter ϕ_{Π_v} of the local $\mathrm{GL}_N(k_v)$ representation, which we know exists by the local Langlands conjecture, to obtain a parameter ϕ_{π_v} of the correct type, that is, $\phi_{\pi_v} : W'_{k_v} \to {}^L H_v$ and thus deducing the local Langlands conjecture for $H(k_v)$. We refer you to their papers for more detail and precise statements. We expect similar results will be forthcoming for the other split classical groups.

References

[1] J. Arthur and L. Clozel, *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula.* Annals of Math. Studies **120**, Princeton Univ. Press, Princeton, 1989.
[2] A. Borel, Automorphic L-functions. *Proc. Symp. Pure Math.* **33**, Part 2, (1979), 27–61.
[3] J.W. Cogdell, Analytic theory of L-functions for GL_n, this volume.
[4] J.W. Cogdell, Langlands conjectures for GL_n, this volume.
[5] J.W. Cogdell, H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, On lifting from classical groups to GL_n. *Publ. Math. IHES* **93** (2001), 5–30.

[6] J.W. Cogdell, H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, On lifting from classical groups to GL_n, II. In preparation.

[7] J.W. Cogdell and I.I. Piatetski-Shapiro, Converse theorems for GL_n, I. *Publ. Math. IHES* **79** (1994), 157–214.

[8] J.W. Cogdell and I.I. Piatetski-Shapiro, Stability of gamma factors for $\mathrm{SO}(2n+1)$. *manuscripta math.* **95** (1998) 437–461.

[9] J.W. Cogdell and I.I. Piatetski-Shapiro, Converse theorems for GL_n, II. *J. reine angew. Math.* **507** (1999), 165–188.

[10] P. Deligne, D. Kazhdan, and M-F. Vigneras, Représentations des algèbras centrales simples p-adiques. *Représentations des groupes réductifs sur un corps local*, J. Bernstein, P. Deligne, D. Kazhdan, and M-F. Vigneras, editors, Hermann, Paris, 1984, 33–117.

[11] S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3). *Ann. Sci. École Norm. Sup. (4)* **11** (1978), 471–542.

[12] D. Ginzburg, S. Rallis, and D. Soudry, Generic automorphic forms on $\mathrm{SO}(2n+1)$: functorial lift to $\mathrm{GL}(2n)$, endoscopy, and base change. *Internat. Math. Res. Notices* **2001**, no. 14, 729–764.

[13] G. Henniart and R. Herb, Automorphic induction for $\mathrm{GL}(n)$ (over local non-Archimedean fields). *Duke Math. J.* **78** (1995), 131–192.

[14] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika, Relèvement cubique non normal. *C. R. Acad. Sci. Paris Sr. I Math.* **292** (1981), 567–571.

[15] H. Jacquet and R. Langlands, *Automorphic Forms on* GL(2). Lecture Notes in Math. **114**, Springer Verlag, Berlin, 1970.

[16] D. Jiang and D. Soudry, The local converse theorem for $SO(2n+1)$ and applications. *Annals of Math.*, to appear.

[17] D. Jiang and D. Soudry, Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}. To appear in a volume in honor of J. Shalika.

[18] D. Kazhdan, On lifting. *Lie Group Representations II*, Lecture Notes in Mathematics **1041**, Springer-Verlag, New York, 1984, 209–249.

[19] D. Kazhdan and G. Lusztig, Proof of the Deligne-Langlands conjecture for Hecke algebras. *Invent. Math.* **87** (1987), 153–215.

[20] H. Kim, Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. *Preprint* (2000).

[21] H. Kim and F. Shahidi, Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and functorial symmetric cube for GL_2. *Ann. of Math.*, to appear.

[22] H. Kim and F. Shahidi, Cuspidality of symmetric powers with applications. *Duke Math. J.*, to appear.

[23] R. Kottwitz and D. Shelstad, *Foundations of Twisted Endoscopy.* Astérisque **255**, SMF, Paris,1999.

[24] R.P. Langlands, The representation theory of abelian algebraic groups. *Pacific J. Math. - Olga Taussky-Todd Volume* **61** (1998) 231–250.

[25] R.P. Langlands, *Euler Products.* Yale University Press, New Haven, 1971.

[26] R.P. Langlands, On the classification of irreducible representations of real algebraic groups. *Representation Theory and Harmonic Analysis*, P. Sally and D. Vogan editors. AMS, Providence, 1989, 101–170.

[27] R.P. Langlands, *Base Change for* GL(2). Annals of Math. Studies **96**, Princeton University Press, Princeton, 1980.

[28] R.P. Langlands, *Les Débuts d'une Formule des Traces Stable.* Publications Mathématiques de l'Université Paris VII **13**. Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983.

[29] R.P. Langlands, Where stands functoriality today? *Representation theory and automorphic forms (Edinburgh, 1996). Proc. Sympos. Pure Math.* **61** (1997), 457–471.

[30] R.P. Langlands, Endoscopy and beyond. Notes for a lecture delivered at the IAS, March 30, 2000.

[31] D. Ramakrishnan, Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2). *Ann. of Math. (2)* **152** (2000), 45–111.

[32] J. Rogawski, Representations of $GL(n)$ and division algebras over a p-adic field. *Duke Math. J.* **50** (1983), 161–196.

[33] I. Satake, Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields. *Publ. Math. IHES* **18** (1963), 5–69.

[34] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. *Ann. of Math. (2)* **132** (1990), 273–330.
[35] F. Shahidi, Local coefficients as Mellin transforms of Bessel functions; Towards a general stability. *IMRN*, to appear.
[36] T. Springer, *Linear Algebraic Groups.* Progress in Mathematics Vol. 9, Birkhäuser, Boston, 1981.

Department of Mathematics, Oklahoma State University, Stillwater OK 74078

E-mail address: cogdell@math.okstate.edu