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Functoriality for the quasisplit classical groups

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Functoriality is one of the most central questions in the theory of automorphic forms and representations [**3, 6, 31, 32**]. Locally and globally, it is a manifestation of Langlands' formulation of a non-abelian class field theory. Now known as the Langlands correspondence, this formulation of class field theory can be viewed as giving an arithmetic parametrization of local or automorphic representations in terms of admissible homomorphisms of (an appropriate analogue) of the Weil-Deligne group into the L-group. When this conjectural parametrization is combined with natural homomorphisms of the L-groups it predicts a transfer or lifting of local or automorphic representations of two reductive algebraic groups. As a purely automorphic expression of a global non-abelian class field theory, global functoriality is inherently an arithmetic process.

Global functoriality from a quasisplit classical group G to GL_N associated to a natural map on the L-groups has been established in many cases. We recall the main cases:

(i) For G a split classical group with the natural embedding of the L-groups, this was established in [**10**] and [**11**].

(ii) For G a quasisplit unitary group with the L-homomorphism associated to stable base change on the L-groups, this was established in [**29**],[**26**], and [**27**].

(iii) For G a split general spin group, this was established in [**5**].

In this paper we consider simultaneously the cases of quasisplit classical groups G. This includes all the cases mentioned in (i) and (ii) above as well as the new case of the quasisplit even special orthogonal groups. Similar methods should work for the quasisplit GSpin groups, and this will be pursued by Asgari and Shahidi as a sequel to [**5**].

As with the previous results above, our method combines the Converse Theorem for GL_N with the Langlands-Shahidi method for controlling the L-functions of the quasisplit classical groups. One of the crucial ingredients in this method is the use of the "stability of local γ–factors" to finesse the lack of the Local Langlands Conjecture at the ramified non-archimedean places. The advance that lets us now

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handle the quasisplit orthogonal groups is our general stability result in [**14**]. In the past, the stability results were established on a case-by-case basis as needed. The general stability result in [**14**] now lets us give a uniform treatment of all quasisplit classical groups. As before, once we have established the existence of functoriality for the quasisplit classical groups, the descent results of Ginzburg, Rallis, and Soudry [**16, 42**] then give the complete characterization of the image of functoriality in these cases.

This paper can be considered as a survey of past results, an exposition of how to apply the general stability result of [**14**], and the first proof of global functoriality for the quasisplit even orthogonal groups. We have included an appendix containing specific calculations for the even quasisplit orthogonal groups. We will return to local applications of these liftings in a subsequent paper.

Finally, we would like to thank the referee who helped us improve the readability of the paper.

Dedication. The first two authors would like to take this opportunity to dedicate their contributions to this paper to their friend and coauthor, Freydoon Shahidi. The collaboration with Freydoon has been a high point in our careers and we feel it is fitting for a paper reflecting this to appear in this volume in his honor.

1. Functoriality for quasisplit classical groups

Let k be a number field and let $\mathbb{A}_k$ be its ring of adeles. We fix a non-trivial continuous additive character ψ of $\mathbb{A}_k$ which is trivial on the principal adeles k. We will let G_n denote a quasisplit classical group of rank n defined over k. More specifically, we will consider the following cases.

(i) *Odd orthogonal groups.* In this case $G_n = \mathrm{SO}_{2n+1}$, the split special orthogonal group in $2n+1$ variables defined over k, i.e., type B_n. The connected component of the L-group of G_n is ${}^LG_n^0 = \widehat{G}_n = \mathrm{Sp}_{2n}(\mathbb{C})$ while the L-group is the direct product ${}^LG_n = \mathrm{Sp}_{2n}(\mathbb{C}) \times W_k$.

(ii) *Even orthogonal groups.* In this case either (a) $G_n = \mathrm{SO}_{2n}$, the split special orthogonal group in $2n$ variables defined over k, type D_n, or (b) $G_n = \mathrm{SO}_{2n}^*$ is the quasisplit special orthogonal group associated to a quadratic extension E/k, i.e, type ${}^2D_n$. In either case, the connected component of the L-group of G_n is ${}^LG_n^0 = \widehat{G}_n = \mathrm{SO}_{2n}(\mathbb{C})$. In the split case (a), the L-group of the product ${}^LG_n = \mathrm{SO}_{2n}(\mathbb{C}) \times W_k$, while in the quasisplit case (b), the L-group is the semi-direct product ${}^LG_n = \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_k$ where the Weil group acts through the quotient $W_k/W_E \simeq \mathrm{Gal}(E/k)$ which gives the Galois structure of SO_{2n}^*. We will need to make this Galois action more explicit. Let $\mathrm{O}_{2n}(\mathbb{C})$ denote the even orthogonal group of size $2n$. Then we have $\mathrm{Gal}(E/k) \simeq \mathrm{O}_{2n}(\mathbb{C})/\mathrm{SO}_{2n}(\mathbb{C})$. Conjugation by an element of $\mathrm{O}_{2n}(\mathbb{C})$ of negative determinant gives an outer automorphism of $\mathrm{SO}_{2n}(\mathbb{C})$ corresponding to the diagram automorphism which exchanges the roots α_n and α_{n-1} in Bourbaki's numbering [**7**] or the numbering in Shahidi [**38**]. So if we let $h' \in \mathrm{O}_{2n}(\mathbb{C})$ be any element of negative determinant then for $\sigma \in \mathrm{Gal}(E/k)$

the non-trivial element of the Galois group the action of σ on ${}^LG_n^0$ is

$$\sigma(g) = (h')^{-1}gh'.$$

We will discuss this case in more detail below when we discuss the relevant L-homomorphism and in the appendix (Section 7.1). (Note that when $n = 4$ except for the SO_8^* defined by a quadratic extension the other non-split quasisplit forms of D_4 are not considered to be classical groups.)

(iii) *Symplectic groups.* In this case $G_n = \mathrm{Sp}_{2n}$, the symplectic group in $2n$ variables defined over k, type C_n. The connected component of the L-group of G_n is ${}^LG_n^0 = \widehat{G}_n = \mathrm{SO}_{2n+1}(\mathbb{C})$ and the L-group is the product ${}^LG = \mathrm{SO}_{2n+1}(\mathbb{C}) \times W_k$.

(iv) *Unitary groups.* In this case either (a) $G_n = \mathrm{U}_{2n}$ is the even quasisplit unitary group defined with respect to a quadratic extension E/k or (b) $G_n = \mathrm{U}_{2n+1}$ is the odd quasisplit unitary group defined with respect to a quadratic extension E/k. Both are of type ${}^2A_n$. In case (a) the connected component of the L-group is ${}^LG_n^0 = \widehat{G}_n = \mathrm{GL}_{2n}(\mathbb{C})$ and the L-group is the semi-direct product ${}^LG_n = \mathrm{GL}_{2n}(\mathbb{C}) \rtimes W_k$ where the Weil group acts through the quotient $W_k/W_E \simeq \mathrm{Gal}(E/k)$ which gives the Galois structure of U_{2n}. In case (b) the connected component of the Langlands dual group is ${}^LG_n^0 = \mathrm{GL}_{2n+1}(\mathbb{C})$ and the L-group is the semi-direct product ${}^LG_n = \mathrm{GL}_{2n+1}(\mathbb{C}) \rtimes W_k$ where the Weil groups acts through the quotient $W_k/W_E \simeq \mathrm{Gal}(E/k)$ which gives the Galois structure of U_{2n+1}. We will need to make precise the Galois action. Following [**26, 27**] we let

$$J_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} \quad \text{and set} \quad J_n' = \begin{cases} \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} & \text{in case (a)} \\ \begin{pmatrix} & & J_n \\ & 1 & \\ -J_n & & \end{pmatrix} & \text{in case (b)} \end{cases}$$

so that $G_n = \mathrm{U}(J_n')$. Then if σ is the non-trivial element in $\mathrm{Gal}(E/k)$ then the action of σ on ${}^LG_n^0$ is

$$\sigma(g) = (J_n')^{-1}{}^tg^{-1}J_n',$$

the outer automorphism of ${}^LG_n$ conjugated by the form.

In each of these cases let δ_1 denote the first fundamental representation, or standard representation, of the connected component of the L-group. This is the defining representation of $\widehat{G}_n$ on the appropriate $\mathbb{C}^N$. As can be seen from the description of ${}^LG_n^0$ above, in each case either $N = 2n$ or $N = 2n+1$. Associated to this representation is a natural embedding of ${}^LG_n^0$ into $\mathrm{GL}_N(\mathbb{C}) = {}^L\mathrm{GL}_N^0$. There is an associated standard representation of the L-group ${}^LG_n$ on either $\mathbb{C}^N$ or $\mathbb{C}^N \times \mathbb{C}^N$ which gives rise to a natural L-homomorphism ι which we now describe.

In the case of the split classical groups, the standard representation of the L-group is still on $\mathbb{C}^N$ and is obtained by extending δ_1 to be trivial on the Weil group. This representation then determines an L-homomorphism $\iota : {}^LG_n \hookrightarrow {}^L\mathrm{GL}_N$. By Langlands' principle of functoriality [**3, 6, 9**], associated to these L-homomorphisms there should be a *transfer* or *lift* of automorphic representations from $G_n(\mathbb{A}_k)$ to $\mathrm{GL}_N(\mathbb{A}_k)$. These were the cases treated in [**10, 11**].

In the case of the quasisplit even orthogonal group, we extend the first fundamental representation of ${}^LG^0$ to an embedding of the L-group in the natural way, to obtain $\iota : {}^LG_n \hookrightarrow {}^L\mathrm{GL}_N$. So in this case we again expect a transfer from $\mathrm{SO}^*_{2n}(\mathbb{A}_k)$ to $\mathrm{GL}_{2n}(\mathbb{A}_k)$. Let us elaborate on this embedding, since although well known it is not all together straightforward. It is related to the theory of twisted endoscopy and can be found in [**1, 2, 4**]. The first fundamental representation gives an embedding of $\mathrm{SO}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ with $N = 2n$. In fact this extends to an embedding of $\mathrm{O}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_N(\mathbb{C})$. We then choose an L-homomorphism

$$\xi : W_k \to \mathrm{O}_{2n}(\mathbb{C}) \times W_k \subset \mathrm{GL}_N(\mathbb{C}) \times W_k$$

which induces the isomorphism

$$W_k/W_E \to \mathrm{Gal}(E/k) \simeq \mathrm{O}_{2n}(\mathbb{C})/\mathrm{SO}_{2n}(\mathbb{C})$$

that is, such that ξ factors through $W_k/W_E \simeq \mathrm{Gal}(E/k)$ and sends the non-trivial Galois automorphism σ to an element of negative determinant in $\mathrm{O}_{2n}(\mathbb{C})$ times σ. Let us write this as $\xi(w) = \xi'(w) \times w$ with $\xi'(w) \in \mathrm{O}_{2n}(\mathbb{C})$. Then in the construction of ${}^L\mathrm{SO}^*_{2n} = \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_k$ the Weil group acts on $\mathrm{SO}_{2n}(\mathbb{C})$ through conjugation by $\xi'(w)$. We now turn to the embedding of the L-group. If we represent elements of ${}^L\mathrm{SO}^*_{2n}$ as products $h \times w = (h \times 1)(1 \times w)$ with $h \in \mathrm{SO}_{2n}(\mathbb{C})$ and $w \in W_k$ then $\iota : \mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_k \hookrightarrow \mathrm{GL}_N(\mathbb{C}) \times W_k$ is given by $\iota(h \times 1) = h \times 1 \in \mathrm{GL}_N(\mathbb{C}) \times W_k$ and $\iota(1 \times w) = \xi(w) = \xi'(w) \times w$. One can find a more detailed description of the embedding in the appendix (Section 7.1).

In the case of unitary groups we follow the description in [**26, 27**], to which the reader can refer for more details. The standard representation of ${}^LG_n$ is now on $\mathbb{C}^N \times \mathbb{C}^N$. The action of the connected component ${}^LG^0_n$ is by

$$[g \times 1](v_1, v_2) = (gv_1, \sigma(g)v_2)$$

while the Weil group acts through the quotient $W_k/W_E \simeq \mathrm{Gal}(E/k)$ with the non-trivial Galois element acting by

$$[1 \times \sigma](v_1, v_2) = (v_2, v_1).$$

It determines an embedding ι of ${}^LG_n \simeq {}^LG^0_n \rtimes W_k$ into $(\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})) \rtimes W_k$ given by $\iota(g \times w) = (g \times \sigma(g)) \times w$, where on the right hand side, W_k acts on $\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})$ through the quotient $W_k/W_E \simeq \mathrm{Gal}(E/k)$ with $\sigma(g_1 \times g_2) = g_2 \times g_1$. The group $(\mathrm{GL}_N(\mathbb{C}) \times \mathrm{GL}_N(\mathbb{C})) \rtimes W_k$ defined in this way is the L-group of the restriction of scalars $\mathrm{Res}_{E/k}\mathrm{GL}_N$. Hence the map on L-groups we consider is that associated to stable base change $\iota : {}^LG_n \hookrightarrow {}^L(\mathrm{Res}_{E/k}\mathrm{GL}_N)$.

To give a unified presentation of these functorialities, we let

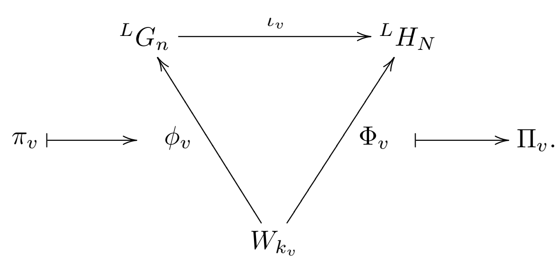
$$H_N = \begin{cases} \mathrm{GL}_N & \text{if } G_n \text{ is orthogonal or symplectic} \\ \mathrm{Res}_{E/k}\mathrm{GL}_N & \text{if } G_n \text{ is unitary} \end{cases}$$

where $N = 2n$ or $2n+1$ as described above. Then the functorialities that we will establish are from G_n to H_N given in the following table. The embedding ι of L-groups is that described above.

G_n	$\iota : {}^LG_n \hookrightarrow {}^LH_N$	H_N
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C}) \times W_k \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_k$	GL_{2n}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C}) \times W_k \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_k$	GL_{2n}
SO^*_{2n}	$\mathrm{SO}_{2n}(\mathbb{C}) \rtimes W_k \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times W_k$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C}) \times W_k \hookrightarrow \mathrm{GL}_{2n+1}(\mathbb{C}) \times W_k$	GL_{2n+1}
U_{2n}	$\mathrm{GL}_{2n}(\mathbb{C}) \rtimes W_k \hookrightarrow (\mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{GL}_{2n}(\mathbb{C})) \rtimes W_k$	$\mathrm{Res}_{E/k}\mathrm{GL}_{2n}$
U_{2n+1}	$\mathrm{GL}_{2n+1}(\mathbb{C}) \rtimes W_k \hookrightarrow (\mathrm{GL}_{2n+1}(\mathbb{C}) \times \mathrm{GL}_{2n+1}(\mathbb{C})) \rtimes W_k$	$\mathrm{Res}_{E/k}\mathrm{GL}_{2n+1}$

By Langlands' principle of functoriality, as explicated in [**3, 6, 9**], associated to these L-homomorphisms there should be a *transfer* or *lift* of automorphic representations from $G_n(\mathbb{A}_k)$ to $H_N(\mathbb{A}_k)$. To be more precise, for each place v of k we have the local versions of the L–groups, obtained by replacing the Weil group W_k with the local Weil group W_{k_v}. The natural maps $W_{k_v} \to W_k$ make the global and local L-groups compatible. We will not distinguish between our local and global L-groups notationally. Our global L-homomorphism ι then induces a local L-homomorphism, which we will denote by $\iota_v : {}^LG_n \to {}^LH_N$.

Let $\pi = \otimes' \pi_v$ be an irreducible automorphic representation of $G_n(\mathbb{A}_k)$. For v a finite place of k where π_v is unramified, and if necessary the local quadratic extension E_w/k_v is also unramified, the unramified arithmetic Langlands classification or the Satake classification [**6, 35**] implies that π_v is parametrized by an unramified admissible homomorphism $\phi_v : W_{k_v} \to {}^LG_n$ where W_{k_v} is the Weil group of k_v. By composing with $\iota_v : {}^LG_n \hookrightarrow {}^LH_N$ we have an unramified admissible homomorphism $\Phi_v = \iota_v \circ \phi_v : W_{k_v} \longrightarrow {}^LH_N$ and this defines an irreducible admissible unramified representation Π_v of $H_N(k_v)$ [**17, 18**]. Then Π_v is the *local functorial lift* of π_v. The process is outlined in the following *local functoriality diagram.*



Similarly, if v is an archimedean place, then by the arithmetic Langlands classification π_v is determined by an admissible homomorphism $\phi_v : W_v \longrightarrow {}^LG_n$ where

W_v is the local Weil group of k_v [**6, 30**]. The composition $\iota_v \circ \phi_v$ is an admissible homomorphism of W_v into ${}^L H_N$ and hence determines a representation Π_v of $H_n(k_v)$ via the same diagram. This is again the *local functorial lift* of π_v. Note that in either case we have an equality of local L-functions

$$L(s, \Pi_v) = L(s, \Phi_v) = L(s, \iota_v \circ \phi_v) = L(s, \pi_v, \iota_v)$$

as well as equalities for the associated ε–factors (if ψ_v is unramified as well at the finite place in question).

An irreducible automorphic representation $\Pi = \otimes' \Pi_v$ of $H_N(\mathbb{A}_k)$ is called a *functorial lift* of π if for every archimedean place v and for almost all non-archimedean places v for which π_v is unramified we have that Π_v is a local functorial lift of π_v. In particular this entails an equality of (partial) Langlands L-functions

$$L^S(s, \Pi) = \prod_{v \notin S} L(s, \Pi_v) = \prod_{v \notin S} L(s, \pi_v, \iota_v) = L^S(s, \pi, \iota),$$

where S is the (finite) complement of the places where we know the local Langlands classification, so the ramified places.

We will let B_n denote a Borel subgroup of G_n and let U_n denote the unipotent radical of B_n. The abelianization of U_n is a direct sum of copies of k and we may use ψ to define a non-degenerate character of $U_n(\mathbb{A}_k)$ which is trivial on $U_n(k)$. By abuse of notation we continue to call this character ψ.

Let π be an irreducible cuspidal representation of $G_n(\mathbb{A}_k)$. We say that π is *globally generic* if there is a cusp form $\varphi \in V_\pi$ such that φ has a non-vanishing ψ-Fourier coefficient along U_n, i.e., such that

$$\int_{U_n(k) \backslash U_n(\mathbb{A}_k)} \varphi(ug)\psi^{-1}(u)\, du \neq 0.$$

Cuspidal automorphic representations of GL_n are always globally generic in this sense. For cuspidal automorphic representations of the classical groups this is a condition. In general the notion of being globally generic may depend on the choice of splitting of the group. However, as is shown in the Appendix to [**11**], given a π which is globally generic with respect to some splitting there is always an "outer twist" which is globally generic with respect to a fixed splitting. This outer twist provides an abstract isomorphism between globally generic cuspidal representations and will not effect the L- or ε-factors nor the notion of the functorial lift. Hence we lose no generality in considering cuspidal representations that are globally generic with respect to our fixed splitting.

The principal result that we will prove in this paper is the following.

THEOREM 1.1. *Let k be a number field and let π be an irreducible globally generic cuspidal automorphic representation of a quasisplit classical group $G_n(\mathbb{A}_k)$ as above. Then π has a functorial lift to $H_N(\mathbb{A}_k)$ associated to the embedding ι of L-groups above.*

The low-dimensional cases of this theorem are already well understood. In the split cases, they were discussed in [**11**]. Thus we will concentrate primarily on the cases where $n \geq 2$, except for the quasisplit orthogonal groups where we restrict to $n \geq 4$.

2. The Converse Theorem

In order to effect the functorial lifting from G_n to H_N we will use the Converse Theorem for GL_N [**12, 13**] as we did in [**10, 11**]. Let us fix a number field K and a finite set S of finite places of K. For the case of G_n orthogonal or symplectic, the target for functoriality is $\mathrm{GL}_N(\mathbb{A}_k)$ and we will need $K = k$. However, in the case of unitary G_n, the target of functoriality is $\mathrm{Res}_{E/k}\mathrm{GL}_N(\mathbb{A}_k) \simeq \mathrm{GL}_N(\mathbb{A}_E)$ and we will need to apply the converse theorem for $K = E$.

For each integer m, let

$$\mathcal{A}_0(m) = \{\tau \mid \tau \text{ is a cuspidal representation of } \mathrm{GL}_m(\mathbb{A}_K)\}$$

and

$$\mathcal{A}_0^S(m) = \{\tau \in \mathcal{A}_0(m) \mid \tau_v \text{ is unramified for all } v \in S\}.$$

We set

$$\mathcal{T}(N-1) = \coprod_{m=1}^{N-1} \mathcal{A}_0(m) \quad \text{and} \quad \mathcal{T}^S(N-1) = \coprod_{m=1}^{N-1} \mathcal{A}_0^S(m).$$

If η is a continuous character of $K^\times\backslash\mathbb{A}_K^\times$, let us set

$$\mathcal{T}(S;\eta) = \mathcal{T}^S(N-1)\otimes\eta = \{\tau = \tau'\otimes\eta \;:\; \tau' \in \mathcal{T}^S(N-1)\}.$$

Theorem 2.1 (Converse Theorem). *Let $\Pi = \otimes'\Pi_v$ be an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A}_K)$ whose central character ω_Π is invariant under $K^\times$ and whose L-function $L(s,\Pi) = \prod_v L(s,\Pi_v)$ is absolutely convergent in some right half-plane. Let S be a finite set of finite places of K and let η be a continuous character of $K^\times\backslash\mathbb{A}_K^\times$. Suppose that for every $\tau \in \mathcal{T}(S;\eta)$ the L-function $L(s,\Pi\times\tau)$ is* nice, *that is, it satisfies*

1. *$L(s,\Pi\times\tau)$ and $L(s,\widetilde{\Pi}\times\tilde{\tau})$ extend to entire functions of $s \in \mathbb{C}$,*
2. *$L(s,\Pi\times\tau)$ and $L(s,\widetilde{\Pi}\times\tilde{\tau})$ are bounded in vertical strips, and*
3. *$L(s,\Pi\times\tau)$ satisfies the functional equation*

$$L(s,\Pi\times\tau) = \varepsilon(s,\Pi\times\tau)L(1-s,\widetilde{\Pi}\times\tilde{\tau}).$$

Then there exists an automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A}_K)$ such that $\Pi_v \simeq \Pi'_v$ for almost all v. More precisely, $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.

In the statement of the theorem, the twisted L- and ϵ-factors are defined by the products

$$L(s,\Pi\times\tau) = \prod_v L(s,\Pi_v\times\tau_v) \qquad \varepsilon(s,\Pi\times\tau) = \prod_v \varepsilon(s,\Pi_v\times\tau_v,\psi_v)$$

of local factors as in [**12, 10**].

To motivate the next few sections, let us describe how we will apply this theorem to the problem of Langlands lifting from G_n to H_N. We begin with our globally generic cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $G_n(\mathbb{A}_k)$.

If G_n is an orthogonal or symplectic group, then for each place v we need to associate to π_v an irreducible admissible representation Π_v of $H_N(k_v) = \mathrm{GL}_N(k_v)$ such that for every $\tau \in \mathcal{T}(S;\eta)$ we have

$$L(s, \pi_v \otimes \tau_v, \iota_v \otimes \iota'_v) = L(s, \Pi_v \times \tau_v)$$
$$\varepsilon(s, \pi_v \otimes \tau_v, \iota_v \otimes \iota'_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v)$$

where ι' is the identity map on $\mathrm{GL}_m(\mathbb{C})$, or more accurately from the L-group ${}^L\mathrm{GL}_m = \mathrm{GL}_m(\mathbb{C}) \times W_k$ to $\mathrm{GL}_m(\mathbb{C})$ given by projection on the first factor, and similarly ι now represents the representation of ${}^LG_n$ given by ι followed by the projection onto the first factor of ${}^LH$, or the connected component of the identity, i.e., the associated map to $\mathrm{GL}_N(\mathbb{C})$.

If G_n is a unitary group, then for each place v we need to associate to π_v an irreducible admissible representation Π_v of $H_N(k_v) = \mathrm{GL}_N(E_v)$, where $E_v = E \otimes k_v$ is either an honest quadratic extension or the split quadratic algebra over k_v, such that for every $\tau \in \mathcal{T}(S;\eta)$ we have

$$L(s, \pi_v \otimes \tau_v, \iota_v \otimes \iota'_v) = L(s, \Pi_v \times \tau_v)$$
$$\varepsilon(s, \pi_v \otimes \tau_v, \iota_v \otimes \iota'_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

Now τ_v must be viewed as a representation of $\mathrm{GL}_m(E_v)$, i.e., of $\mathrm{Res}_{E/k}\mathrm{GL}_m(k_v)$. If E_v/k_v is an honest quadratic extension, then $G_n(k_v)$ is an honest local unitary group and $H_N(k_v) \simeq \mathrm{GL}_N(E_v)$. If v splits in E, so $E_v \simeq E_{w_1} \oplus E_{w_2}$ with $E_{w_i} \simeq k_v$, then $G_n(k_v) \simeq \mathrm{GL}_N(k_v)$ and $H_N(k_v) \simeq \mathrm{GL}_N(E_v) \simeq \mathrm{GL}_N(k_v) \times \mathrm{GL}_N(k_v)$. In this case $\Pi_v = \Pi_{1,v} \otimes \Pi_{2,v}$, an outer tensor product, and similarly $\tau_v = \tau_{1,v} \otimes \tau_{2,v}$ and we have a product of two factors on each side. A more detailed description for this case can be found in [**26, 27**].

For archimedean places v and those non-archimedean v where π_v is unramified, we take Π_v to be the local functorial lift of π_v described above. For those places v where π_v is ramified, we will finesse the lack of a local functorial lift using the stability of γ–factors as described in Section 4 below. This will allow us to associate to π_v a representation Π_v of $H_N(k_v)$ at these places as well. The process involves the choice of a highly ramified character η_v of $H_1(k_v)$. If we then take $\Pi = \otimes' \Pi_v$, this is an irreducible representation of $H_N(\mathbb{A}_k)$. With the choices above we will have

$$L(s, \pi \otimes \tau, \iota \otimes \iota') = L(s, \Pi \times \tau)$$
$$\varepsilon(s, \pi \otimes \tau, \iota \otimes \iota') = \varepsilon(s, \Pi \times \tau)$$

for $\mathrm{Re}(s) >> 0$ and all $\tau \in \mathcal{T}(S;\eta)$ for a suitable fixed character η of $H_1(\mathbb{A}_k)$. This is our candidate lift. The theory of L-functions for $G_n \times H_m$, which we address in the next section, will then guarantee that the twisted L-functions $L(s, \pi \otimes \tau, \iota \otimes \iota')$ are nice for all $\tau \in \mathcal{T}(S;\eta)$. Then the $L(s, \Pi \times \tau)$ will also be nice and Π satisfies the hypotheses of the Converse Theorem. Hence there exists an irreducible automorphic representation Π' of $H_N(\mathbb{A}_k)$ such that $\Pi_v \simeq \Pi'_v$ for all archimedean v and almost all finite v where π_v is unramified. Hence Π' is a functorial lift of π.

3. L-functions for $G_n \times H_m$

Let π be a globally generic cuspidal representation of $G_n(\mathbb{A}_k)$ and τ a cuspidal representation of $H_m(\mathbb{A}_k)$, with $m \geq 1$. We let ι and ι' be the representations of the L–groups defined in Sections 1 and 2 respectively. To effect our lifting, we must control the analytic properties of the twisted L-functions $L(s, \pi \otimes \tau, \iota \otimes \iota')$. This we do by the method of Langlands and Shahidi, as we outline here.

The L-functions $L(s, \pi \otimes \tau, \iota \otimes \iota')$ are the completed L-functions as defined in [**39**] via the theory of Eisenstein series. If we let $M_{n,m}$ denote $G_n \times H_m$ with $m \geq 1$, then this appears as the Levi factor of a maximal self-associate parabolic subgroup $P_{n,m} = M_{n,m}N_{n,m}$ of G_{n+m} associated to the root α_m as in [**38**]. The representation $\iota \otimes \iota'$ then occurs in the adjoint action of ${}^LM_{n,m}$ on the Lie algebra ${}^L\mathfrak{n}_{n,m}$ as the representation $\tilde{r}_1$ of [**38**]. Then these L-functions can be defined and controlled by considering the induced representation $I(s, \pi \otimes \tau)$ described in [**38, 39**] since $\pi \otimes \tau$ is a cuspidal representation of $M_{n,m}(\mathbb{A}_k)$. The local factors are then defined in [**39**] via the arithmetic Langlands classification for archimedean places, through the Satake parameters for finite unramified places, as given by the poles of the associated γ–factor (or local coefficient) if π_v and τ_v are tempered, by analytic extension if π_v and τ_v are quasi-tempered, and via the representation theoretic Langlands classification otherwise. Since it is only these representations that we will be considering, we will abbreviate our notation by suppressing the L–homomorphism, so for example

$$L(s, \pi \times \tau) = \prod_v L(s, \pi_v \times \tau_v) = \prod_v L(s, \pi_v \otimes \tau_v, \iota_v \otimes \iota'_v) = L(s, \pi \otimes \tau, \iota \otimes \iota')$$

with similar conventions for the ε– and γ–factors.

The global theory of these twisted L-functions is now quite well understood.

THEOREM 3.1. *Let S be a non-empty set of finite places of k. Let $K = k$ when G_n is orthogonal or symplectic or $K = E$ if G_n is a unitary group associated to the quadratic extension E/k and continue to let S denote the corresponding set of places of K. Let η be a character of $K^\times \backslash \mathbb{A}_K^\times$ such that, for some $v \in S$, either the square η_v^2 is ramified if $K = k$, or if $K = E$ then for the places w of E above v we have both η_w and $\eta_w \overline{\eta}_w$ are ramified. Then for all $\tau \in \mathcal{T}(S; \eta)$ the L-function $L(s, \pi \times \tau)$ is* nice, *that is,*

1. *$L(s, \pi \times \tau)$ is an entire function of s,*
2. *$L(s, \pi \times \tau)$ is bounded in vertical strips of finite width, and*
3. *we have the functional equation*

$$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1 - s, \tilde{\pi} \times \tilde{\tau}).$$

Proof: (1) In all cases this follows from the more general Proposition 2.1 of [**28**]. Note that in view of the results of Muić [**34**] and of [**8**], the necessary result on normalized intertwining operators, Assumption 1.1 of [**28**], usually referred to as Assumption A [**24**], is valid in all cases as proved in [**24, 25**]. Note that this is the only part of the theorem where the twisting by η is needed. The specific ramification stated comes from [**11**] or [**27**].

(2) The boundedness in vertical strips of these L-functions is known in wide generality, which includes the cases of interest to us. It follows from Corollary 4.5 of [**15**] and is valid for all $\tau \in \mathcal{T}(N-1)$, provided one removes neighborhoods of the finite number of possible poles of the L-function.

(3) The functional equation is also known in wide generality and is a consequence of Theorem 7.7 of [**39**]. It is again valid for all $\tau \in \mathcal{T}(N-1)$. □

4. Stability of γ-factors

This section is devoted to the formulation of the stability of the local γ-factors for generic representations of the quasisplit groups under consideration. This result is necessary for defining a suitable local lift at the non-archimedean places where we do not have the local Langlands conjecture at our disposal.

For this section, let k denote a p-adic local field, that is, a non-archimedean local field of characteristic zero. Let G_n now be a quasisplit classical group of the types defined in Section 1, but now over k. These will correspond to the local situations that arise in our global problem, with the exception of the global unitary groups at a place which splits in the defining quadratic extension (see Remark 4.1 below).

4.1. Stability. Let π be a generic irreducible admissible representation of $G_n(k)$ and let η be a continuous character of $H_1(k) \simeq k^\times$ (resp. $E^\times$ in the local unitary case). Let ψ be a fixed non-trivial additive character of k. Let $\gamma(s, \pi\times\eta, \psi)$ be the associated γ-factor as defined in Theorem 3.5 of [**39**]. These are defined inductively through the local coefficients $C_\psi(s, \pi\otimes\eta)$ of the local induced representations analogous to those given above. They are related to the local L- and ε-factors by

$$\gamma(s, \pi\times\eta, \psi) = \frac{\varepsilon(s, \pi\times\eta, \psi) L(1-s, \tilde{\pi}\times\eta^{-1})}{L(s, \pi\times\eta)}.$$

We begin by recalling the main result of [**14**], with a slight shift in notation for consistency.

Theorem 4.1. *Let G be a quasisplit connected reductive algebraic group over k such that the Γ–diagram of G_D is of either type $B_{n+1}, C_{n+1}, D_{n+1}, {}^2A_{n+1}$ or ${}^2D_{n+1} (n+1 \geq 4)$. Let $P = MN$ be a self–associate maximal parabolic subgroup of G over k such that the unique simple root in N is the root α_1 in Bourbaki's numbering* [**7**]. *Let π be an irreducible admissible generic representation of $M(k)$.*

Then $C_\psi(s,\pi)$ *is stable, that is, if* ν *is a character of* $K^\times$*, realized as a character* $\tilde\nu$ *of* $M(k)$ *by*

$$\tilde\nu(m) = \nu(\det(\mathrm{Ad}_{\mathfrak n}(m))),$$

then

$$C_\psi(s,\pi_1\otimes\tilde\nu) = C_\psi(s,\pi_2\otimes\tilde\nu)$$

for any two such representations π_1 *and* π_2 *with the same central characters and all sufficiently highly ramified* ν*. Here* $\mathfrak n$ *is the Lie algebra of* $N(k)$*.*

Note that this covers the local quasisplit classical groups that are under consideration here if we take $G = G_{n+1}$. The splitting field K is then k itself except in the ${}^2A_n$ and ${}^2D_n$ cases, where it is the associated quadratic extension E as in Section 1. According to the tables in Section 4 of [**38**], the Levi subgroups M in Theorem 4.1 are of the form $M \simeq H_1 \times G_n$. To use the stability result in the application to functoriality we need the following elementary lemma.

Lemma 4.1. *Let* $m \in M(k)$ *and write* $m = a \times m'$ *with* $a \in H_1(k)$ *and* $m' \in G_n(k)$*. Then* $\det(\mathrm{Ad}_{\mathfrak n}(m')) = 1$*, i.e.*

$$\det(\mathrm{Ad}_{\mathfrak n}(m)) = \det(\mathrm{Ad}_{\mathfrak n}(a)).$$

Proof: An elementary matrix calculation shows that in the symplectic and special orthogonal cases we have $\det(\mathrm{Ad}_{\mathfrak n}(m')) = \det(m') = 1$. In the unitary case, $\det(\mathrm{Ad}_{\mathfrak n}(m')) = N_{E/k}\det(m') = 1$. □

Thus we see that in Theorem 4.1 the twisting character $\tilde\nu$ of $M(k)$ factors to a character of the GL_1 factor in M. We will call a character $\tilde\nu$ of $H_1(k)$, *suitable* if it arises from a character ν of $K^\times$ via a composition of the embedding $\mathrm{GL}_1 \hookrightarrow M$ followed by the character $m \mapsto \nu(\det(\mathrm{Ad}_{\mathfrak n}(m)))$ of M. Now Theorem 4.1 has the following corollary.

Corollary 4.1.1. *Let* G_n *be a quasisplit classical group over* k *as in Section 1, so that it satisfies the hypotheses of Theorem 4.1. Let* π_1 *and* π_2 *be two irreducible admissible representations of* $G_n(k)$ *having the same central character. Then for every sufficiently highly ramified suitable character* $\tilde\nu$ *of* $H_1(k)$ *we have*

$$\gamma(s,\pi_1\times\tilde\nu,\psi) = \gamma(s,\pi_2\times\tilde\nu,\psi).$$

Proof: Let G_{n+1} be the quasisplit connected reductive algebraic group over k of rank one larger such such that the parabolic subgroup P in Theorem 4.1 has Levi subgroup $M \simeq H_1 \times G_n$. Then $\mathbf{1}\otimes\pi_1$ and $\mathbf{1}\otimes\pi_2$ determine irreducible admissible representations of $M(k)$ with the same central character. By Theorem 4.1 we know that for every sufficiently highly ramified character ν of $K^\times$, determining a character of $M(k)$ by $\tilde\nu(m) = \nu(\det(\mathrm{Ad}_{\mathfrak n}(m))$, we have $C_\psi(s,(\mathbf{1}\otimes\pi_1)\otimes\tilde\nu) = C_\psi(s,(\mathbf{1}\otimes\pi_2)\otimes\tilde\nu)$. By our lemma $\tilde\nu$ factors to only the H_1 variable in M and $(\mathbf{1}\otimes\pi_i)\otimes\tilde\nu = \tilde\nu\otimes\pi_i$ as representations of $M(k)$. Then the statement of the corollary follows from the definition of $\gamma(s,\pi_i\times\tilde\nu,\psi)$ given in [**38**]. □

Remark 4.1. Theorem 4.1 and its corollary cover the possible local situations that arise in our global problem *except* for the case of unitary groups at a place that splits in the global quadratic extension. At these places, locally the hypotheses of Theorem 4.1 are not satisfied since the parabolic subgroup P in question is no longer maximal. In this case, $G_n \simeq \mathrm{GL}_N$, $H_1 \simeq \mathrm{GL}_1 \times \mathrm{GL}_1$, and, remembering the implicit L-homomorphism $\iota \otimes \iota'$, both sides are a pair of local $\mathrm{GL}_N \times \mathrm{GL}_1$ γ-factors. In this case the stability result is due to Jacquet and Shalika [**22**]. However, as we shall see, in our application we will not need stability in this situation since the local Langlands correspondence is known for GL_N.

4.2. Stability and parametrization. Let π be an irreducible admissible representation of one of our local G_n. Assume that π is ramified, so we may not know how to parametrize π by an admissible homomorphism of the Weil-Deligne group W_k' into ${}^LG_n$. We wish to replace π by a second representation π' for which we have an arithmetic Langlands parameter and for which we still have a modicum of control over its L- and ε-factors.

We replace π with an induced representation having the same central character. To this end, let T_n be the maximal torus of G_n, take a character λ of T_n, and let $I(\lambda)$ be the associated induced representation. By appropriate choice of λ we can guarantee that π and $I(\lambda)$ have the same central character and that $I(\lambda)$ is irreducible. Let $\phi_\lambda : W_k \to {}^LT$ be the Langlands parameter for λ so that the composition $\phi_\lambda : W_k \to {}^LT_n \to {}^LG_n$ is the arithmetic Langlands parameter for $I(\lambda)$. We can take for π' any of the so constructed $I(\lambda)$. We fix one. By the corollary above, for sufficiently highly ramified suitable $\tilde{\nu}$, depending on π and our choice of $\pi' = I(\lambda)$, we have

$$\gamma(s, \pi \times \tilde{\nu}, \psi) = \gamma(s, \pi' \times \tilde{\nu}, \psi).$$

Once we have the stable γ–factor is expressed in terms of a principal series representations that we can arithmetically parametrize, then we can express the analytic γ–factor as one from arithmetic, the Artin γ–factor associated to the Galois representation $\iota \circ \phi_\lambda$.

Proposition 4.1. *With notation as above,*

$$\gamma(s, I(\lambda) \times \tilde{\nu}, \psi) = \gamma(s, (\iota \circ \phi_\lambda) \otimes \tilde{\nu}, \psi).$$

Proof: The embedding of L-groups $\iota : {}^LG_n \hookrightarrow {}^LH_N$ is defined so that ι is the map coming from the restriction of this adjoint action on ${}^L\mathfrak{n}_{n+1}$ to G_n and similarly for ι' as a representation of H_1. By our convention, $\gamma(s, I(\lambda) \times \tilde{\nu}, \psi)$ is the γ-factor associated to this representation of the L-group, i.e.,

$$\gamma(s, I(\lambda) \times \tilde{\nu}, \psi) = \gamma(s, I(\lambda) \otimes \tilde{\nu}, \iota \otimes \iota', \psi).$$

To relate this analytic γ-factor to that from the parametrization, we embed $G_n \hookrightarrow G_{n+1}$ as part of the Levi subgroup M_{n+1} of the self-associate parabolic subgroup $P_{n+1} = M_{n+1}N_{n+1} \subset G_{n+1}$ such that the unique simple root in N_{n+1} is α_1 as above. Using the product formula (or "cocycle relation") for the local coefficients from Proposition 3.2.1 of [**36**], the local coefficient $C_\psi(s, I(\lambda)\otimes\tilde{\nu})$ factors into a product over the roots appearing in the adjoint representation of ${}^LT_{n+1} \subset {}^LM_{n+1}$ on ${}^L\mathfrak{n}_{n+1}$. Sections 2 and 3 of [**23**] give a computation of the contribution of an individual root space to $C_\psi(s, I(\lambda)\otimes\tilde{\nu})$ in terms of rank one Artin factors coming from the co-roots composed with λ. The resulting expression (at $s = 0$) is found in Proposition 3.4 of [**23**]. If one takes the expression for a general s and then extracts the γ-factor from the local coefficient, one arrives at

$$\gamma(s, I(\lambda)\otimes\tilde{\nu}, \iota\otimes\iota', \psi) = \gamma(s, (\iota\circ\phi_\lambda)\otimes\tilde{\nu}, \psi).$$

This proves the proposition. □

5. The candidate lift

We now return to k denoting a number field. Let $\pi = \otimes'\pi_v$ be a globally generic cuspidal representation of $G_n(\mathbb{A}_k)$. In this section we will construct our candidate $\Pi = \otimes'\Pi_v$ for the functorial lift of π as an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A}_k)$. We will construct Π by constructing each local component, or local lift, Π_v. There will be three cases: (i) the archimedean lift, (ii) the non-archimedean unramified lift, and finally (iii) the non-archimedean ramified lift.

5.1. The archimedean lift. Let v be an archimedean place of k. By the arithmetic Langlands classification [**30, 6**], π_v is parametrized by an admissible homomorphism $\phi_v : W_{k_v} \to {}^LG_n^0$ where W_{k_v} is the Weil group of k_v. By composing with $\iota_v : {}^LG_n \hookrightarrow {}^LH_N$ we have an admissible homomorphism $\Phi_v = \iota_v\circ\phi_v : W_{k_v} \longrightarrow {}^LH_N$ and this defines an irreducible admissible representation Π_v of $H_N(k_v)$. Then Π_v is the local functorial lift of π_v. We take Π_v as our local lift of π_v. (See the local functoriality diagram in Section 1.)

The local archimedean L– and ε–factors defined via the theory of Eisenstein series we are using are the same as the Artin factors defined through the arithmetic Langlands classification [**37**]. Since the embedding $\iota_v : {}^LG_n \hookrightarrow {}^LH_N$ is the standard representation of the L–group of $G_n(k_v)$ then by the definition of the local L- and ε–factors given in [**6**] we have

$$L(s, \pi_v) = L(s, \iota_v\circ\phi_v) = L(s, \Pi_v)$$

and

$$\varepsilon(s, \pi_v, \psi_v) = \varepsilon(s, \iota_v\circ\phi_v, \psi_v) = \varepsilon(s, \Pi_v, \psi_v)$$

where in both instances the middle factor is the local Artin-Weil L– and ε–factor attached to representations of the Weil group as in [**43**].

If τ_v is an irreducible admissible representation of $H_m(k_v)$ then it is in turn parametrized by an admissible homomorphism $\phi'_v : W_{k_v} \longrightarrow {}^LH_m$. Then the tensor product homomorphism $(\iota_v\circ\phi_v)\otimes(\iota'_v\circ\phi'_v) : W_{k_v} \longrightarrow {}^LH_{mN}$ is admissible and again we have by definition

$$L(s, \pi_v\times\tau_v) = L(s, (\iota_v\circ\phi_v)\otimes(\iota'_v\circ\phi'_v)) = L(s, \Pi_v\times\tau_v)$$

and

$$\varepsilon(s,\pi_v\times\tau_v,\psi_v)=\varepsilon(s,(\iota_v\circ\phi_v)\otimes(\iota'_v\circ\phi'_v),\psi_v)=\varepsilon(s,\Pi_v\times\tau_v,\psi_v).$$

This then gives the following matching of the twisted local L- and ε-factors.

PROPOSITION 5.1. *Let v be an archimedean place of k and let π_v be an irreducible admissible generic representation of $G_n(k_v)$, Π_v its local functorial lift to $H_N(k_v)$, and τ_v an irreducible admissible generic representation of $H_m(k_v)$ with $m<N$. Then*

$$L(s,\pi_v\times\tau_v)=L(s,\Pi_v\times\tau_v)\quad\textit{and}\quad\varepsilon(s,\pi_v\times\tau_v,\psi_v)=\varepsilon(s,\Pi_v\times\tau_v,\psi_v).$$

5.2. The non-archimedean unramified lift. Now let v be a place of k which is non-archimedean and assume that π_v is an unramified representation. By the unramified arithmetic Langlands classification or the Satake classification [**6, 35**], π_v is parametrized by an unramified admissible homomorphism $\phi_v : W_{k_v}\to {}^LG_n^0$ where W_{k_v} is the Weil group of k_v. By composing with $\iota_v : {}^LG_n\hookrightarrow {}^LH_N$ we have an unramified admissible homomorphism $\Phi_v=\iota_v\circ\phi_v : W_{k_v}\longrightarrow {}^LH_N$ and this defines an irreducible admissible unramified representation Π_v of $H_N(k_v)$ [**17, 18**]. Then Π_v is again the local functorial lift of π_v and we take it as our local lift. (Again, see the local functoriality diagram in Section 1.)

We will again need to know that the twisted L- and ε-factors agree for π_v and Π_v.

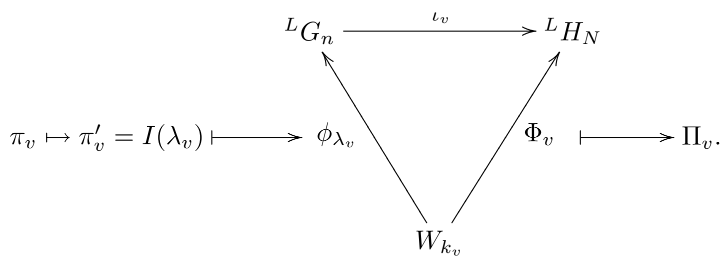
PROPOSITION 5.2. *Let v be a non-archimedean place of k and let π_v be an irreducible admissible generic unramified representation of $G_n(k_v)$. Let Π_v be its functorial local lift to $H_N(k_v)$ as above, and τ_v an irreducible admissible generic representation of $H_m(k_v)$ with $m<N$. Then*

$$L(s,\pi_v\times\tau_v)=L(s,\Pi_v\times\tau_v)\quad\textit{and}\quad\varepsilon(s,\pi_v\times\tau_v,\psi_v)=\varepsilon(s,\Pi_v\times\tau_v,\psi_v).$$

Proof: Since π_v is unramified its parameter ϕ_v factors through an unramified homomorphism into the maximal torus ${}^LT_n\hookrightarrow {}^LG_n$. The composition $\iota\circ\phi_v=\Phi_v$ then has image in a torus ${}^LT'_N\hookrightarrow {}^LH_N$, which necessarily splits, and Π_v is the corresponding unramified (isobaric) representation. Then the functoriality diagram gives that $L(s,\pi_v,\iota_v)=L(s,\Pi_v)$ and $\varepsilon(s,\pi_v,\iota_v,\psi_v)=\varepsilon(s,\Pi_v,\psi_v)$ and both can be expressed as products of one dimensional abelian Artin L-functions and ε–factors. This is the multiplicativity of the local L- and ε–factors in this case. For twisting by τ_v one appeals to the general multiplicativity of local factors from [**19, 40**] with respect to the preceding data. This is done in detail for the split groups in [**11**] and the calculation here is the same. □

5.3. The non-archimedean ramified lift. Now consider a non-archimedean place v of k where the local component π_v of π is ramified. Assume for now that we are not in the situation where G_n is a unitary group associated to a quadratic extension E/k in which the place v splits; we will return to this at the end of the section. Now we do not have the local Langlands correspondence to give us a natural local functorial lift. Instead we will use the results of Section 4.

Given π_v we choose an induced representation $\pi_v' = I(\lambda_v)$ as in Section 4.1 which has the same central character as π_v and which we do know how to parametrize. Let $\phi_{\lambda_v} : W_{k_v} \to {}^L T_n \to {}^L G_n$ be the associated parameter. By composing with $\iota_v : {}^L G_n \hookrightarrow {}^L H_N$ we have an admissible homomorphism $\Phi_v = \iota_v \circ \phi_{\lambda_v} : W_{k_v} \longrightarrow {}^L H_N$ and this defines an irreducible admissible representation Π_v of $H_N(k_v)$. We now use the local functoriality diagram in the following form:



Then Π_v is the local functorial lift of $\pi_v' = I(\lambda_v)$. We take Π_v as our local lift of π_v.

Now let $\tilde{\nu}_v$ be a sufficiently ramified suitable character of $H_1(k_v)$ as in Section 4. Then by Corollary 4.1.1 we know that

$$\gamma(s, \pi_v \times \tilde{\nu}_v, \psi_v) = \gamma(s, \pi_v' \times \tilde{\nu}_v, \psi_v)$$

and by Proposition 4.1 we have

$$\gamma(s, \pi_v' \times \tilde{\nu}_v, \psi_v) = \gamma(s, I(\lambda_v) \times \tilde{\nu}_v, \psi_v) = \gamma(s, (\iota_v \circ \phi_{\lambda_v}) \otimes \tilde{\nu}_v, \psi_v).$$

On the other hand, by the functoriality diagram above

$$\gamma(s, (\iota_v \circ \phi_{\lambda_v}) \otimes \tilde{\nu}_v, \psi_v) = \gamma(s, \Phi_v \otimes \tilde{\nu}_v, \psi_v)$$

and the work of Harris-Taylor and Henniart establishing the local Langlands conjecture for GL_n gives

$$\gamma(s, \Phi_v \otimes \tilde{\nu}_v, \psi_v) = \gamma(s, \Pi_v \times \tilde{\nu}_v, \psi_v).$$

Thus finally

$$\gamma(s, \pi_v \times \tilde{\nu}_v, \psi_v) = \gamma(s, \Pi_v \times \tilde{\nu}_v, \psi_v).$$

For sufficiently ramified $\tilde{\nu}$ the local L-functions $L(s, \pi_v \times \tilde{\nu}_v)$ and $L(s, \Pi_v \times \tilde{\nu}_v)$ both stabilize to 1 [**41, 22**] and so the stability of local γ-factors is essentially the stability of local ε–factors.

Proposition 5.3. *Let π_v be an irreducible admissible generic representation of $G_n(k_v)$ and let Π_v be the irreducible admissible representation of $H_N(k_v)$ as above. Then for sufficiently ramified suitable characters $\tilde{\nu}$ of $H_1(k_v)$ we have*

$$L(s, \pi_v \times \tilde{\nu}_v) = L(s, \Pi_v \times \tilde{\nu}_v) \quad \textit{and} \quad \varepsilon(s, \pi_v \times \tilde{\nu}_v, \psi_v) = \varepsilon(s, \Pi_v \times \tilde{\nu}_v, \psi_v).$$

There is a natural extension of this to the class of representations of $H_m(k_v)$ that we require for the application of the Converse Theorem given in the following proposition.

Proposition 5.4. *Let v be a non-archimedean place of k. Let π_v be an irreducible admissible generic representation of $G_n(k_v)$ and let Π_v be the irreducible admissible representation of $H_N(k_v)$ as above. Let τ_v be an irreducible admissible generic representation of $H_m(k_v)$ with $m < N$ of the form $\tau_v \simeq \tau_{0,v} \otimes \tilde{\nu}_v$ with $\tau_{0,v}$ unramified and $\tilde{\nu}_v$ suitable and sufficiently ramified as above. Then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v) \quad \textit{and} \quad \varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

Proof: The proof of this proposition is by the use of multiplicativity of the local factors with respect to the H_m-variable [**40**]. Since $\tau_{0,v}$ is unramified and generic we can write it as a full induced representation from characters [**21**]

$$\tau_{0,v} \simeq \mathrm{Ind}_{B'_m(k_v)}^{H_m(k_v)}(\chi_{1,v} \otimes \cdots \otimes \chi_{m,v})$$

with each $\chi_{i,v}$ unramified. If we let $\chi_{i,v}(x) = |x|_v^{b_i}$ and let $\mu(x) = |x|_v$, then we may write τ_v as

$$\tau_v \simeq \mathrm{Ind}_{B'_m(k_v)}^{H_m(k_v)}(\tilde{\nu}_v \mu^{b_1} \otimes \cdots \otimes \tilde{\nu}_v \mu^{b_m}).$$

By the multiplicativity of the local factors [**40**] we find

$$L(s, \pi_v \times \tau_v) = \prod_{i=1}^{m} L(s + b_i, \pi_v \times \tilde{\nu}_v)$$

and

$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{m} \varepsilon(s + b_i, \pi_v \times \tilde{\nu}_v, \psi_v).$$

On the other hand, by the same results of [**19**] we also have

$$L(s, \Pi_v \times \tau_v) = \prod_{i=1}^{m} L(s + b_i, \Pi_v \times \tilde{\nu}_v)$$

and

$$\varepsilon(s, \Pi_v \times \tau_v, \psi_v) = \prod_{i=1}^{m} \varepsilon(s + b_i, \Pi_v \times \tilde{\nu}_v, \psi_v).$$

By Proposition 5.3 above we see that after factoring the L- and ε–factors for π_v and Π_v twisted by such τ_v the factors are term by term equal for $\tilde{\nu}_v$ a suitable sufficiently ramified character. This establishes the proposition. □

In the appendix (Section 7.2) we explicitly calculate the local lift of a principal series representation for the case of $G_n = \mathrm{SO}_{2n}^*$. For the other cases (at least in the unramified situation) these explicit calculations are in [**11**] and [**26, 27**].

Now let us return to the situation where G_n is a unitary group associated to a quadratic extension E/k in which the place v splits. Then $G_n(k_v) \simeq \mathrm{GL}_N(k_v)$ and $H_N(k_v) \simeq \mathrm{GL}_N(k_v) \times \mathrm{GL}_N(k_v)$. This situation is analyzed in the beginning of Section 6 of [**26**]. The L-homomorphism is simply understood in this case. If π_v is an irreducible admissible representation of $G_n(k_v)$ then, ramified or not, we take the representation Π_v of $H_N(k_v)$ to be $\pi_v \otimes \tilde{\pi}_v$. The twisting representation τ_v it a representation of $H_m(k_v) \simeq \mathrm{GL}_m(k_v) \times \mathrm{GL}_m(k_v)$ and hence of the form $\tau_v \simeq \tau_{1,v} \otimes \tau_{2,v}$. Then we have

$$\gamma(s, \pi_v \otimes \tau_v, \iota \otimes \iota', \psi_v) = \gamma(s, \pi_v \times \tau_{1,v}, \psi_v)\gamma(s, \tilde{\pi}_v \times \tau_{2,v}, \psi_v) = \gamma(s, \Pi_v \times \tau_v, \psi_v)$$

and

$$L(s, \pi_v \otimes \tau_v, \iota \otimes \iota') = L(s, \pi_v \times \tau_{1,v})L(s, \tilde{\pi}_v \times \tau_{2,v}) = L(s, \Pi_v \times \tau_v).$$

So we are in the same situation as in the unramified case, i.e., the stronger Proposition 5.2 holds in this case.

5.4. The global candidate lift. Return now to the global situation. Let $\pi \simeq \otimes' \pi_v$ be a globally generic cuspidal representation of $G_n(\mathbb{A}_k)$. Let S be a finite set of finite places such that for all non-archimedean places $v \notin S$ we have π_v and ψ_v are unramified (and if necessary the local extension E_w/k_v unramified as well). For each $v \notin S$ let Π_v be the local functorial lift of π_v as in Section 5.1 or 5.2. For the places $v \in S$ we take Π_v to be the irreducible admissible representation of $H_N(k_v)$ obtained in Section 5.3. Then the restricted tensor product $\Pi \simeq \otimes' \Pi_v$ is an irreducible admissible representation of $H_N(\mathbb{A}_k)$. It is self-dual except in the case of unitary groups, where it is self-conjugate-dual. This is our candidate lift.

For each place $v \in S$ choose a suitable sufficiently ramified character $\eta_v = \tilde{\nu}_v$ of $H_1(k_v)$ so that Proposition 5.4 is valid. Let η be any idele class character of $H_1(\mathbb{A}_k)$ which has local component η_v at those $v \in S$. Then combining Propositions 5.1 – 5.4 we obtain the following result on our candidate lift.

Proposition 5.5. *Let π be a globally generic cuspidal representation of $G_n(\mathbb{A}_k)$ and let Π be the candidate lift constructed above as a representation of $H_N(\mathbb{A}_k)$. Then for every representation $\tau \in \mathcal{T}(S;\eta) = \mathcal{T}^S(N-1) \otimes \eta$ we have*

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau) \qquad \textit{and} \qquad \varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau).$$

6. Global functoriality

6.1. Functoriality. Let us now prove Theorem 1.1. The proof is the usual one [**11, 27**], but it is short and we repeat it for completeness.

We begin with our globally generic cuspidal representation of $G_n(\mathbb{A}_k)$. Decompose $\pi \simeq \otimes' \pi_v$ into its local components and let S be a non-empty set of non-archimedean places such that for all non-archimedean places $v \notin S$ we have that π_v and ψ_v (and if necessary E_w/k_v) are unramified. Let $\Pi \simeq \otimes' \Pi_v$ be the irreducible admissible representation of $H_N(\mathbb{A}_k)$ constructed in Section 5 as our candidate lift. By construction Π is self-dual or self-conjugate-dual and is the local functorial lift of π at all places $v \notin S$. Choose η, an idele class character of $H_1(\mathbb{A}_k)$, such that its local components η_v are suitable and sufficiently ramified at those $v \in S$ so that Proposition 5.5 is valid. Furthermore, since we have taken S non-empty, we may choose η so that for at least one place $v_0 \in S$ we have that η_{v_0} is sufficiently ramified so that Theorem 3.1 is also valid. Fix this character.

We are now ready to apply the Converse Theorem to Π. Consider $\tau \in \mathcal{T}(S;\eta)$. By Proposition 5.5 we have that

$$L(s,\pi\times\tau)=L(s,\Pi\times\tau) \qquad \text{and} \qquad \varepsilon(s,\pi\times\tau)=\varepsilon(s,\Pi\times\tau).$$

On the other hand, by Theorem 3.1 we know that each $L(s,\pi\times\tau)$ and hence $L(s,\Pi\times\tau)$ is nice. Thus Π satisfies the hypotheses of the Converse Theorem, Theorem 2.1. Hence there is an automorphic representation $\Pi'\simeq\otimes'\Pi'_v$ of $H_N(\mathbb{A}_k)$ such that $\Pi'_v\simeq\Pi_v$ for all $v\notin S$. But for $v\notin S$, by construction Π_v is the local functorial lift of π_v. Hence Π' is a functorial lift of π as required in the statement of Theorem 1.1. The lift is then uniquely determined, independent of the choice of S and η, by the classification theorem of Jacquet and Shalika [**20**]. □

6.2. The image of functoriality. In this section we would like to record the image of functoriality. Assuming the existence of functoriality, the global image has been analyzed in the papers of Ginzburg, Rallis, and Soudry using their method of descent [**16, 42**]. From their method of descent of automorphic representations from $H_N(\mathbb{A}_k)$ to the classical groups $G_n(\mathbb{A}_k)$ and its local analogue, Ginzburg, Rallis, and Soudry were able to characterize the image of functoriality from generic representations.

There is a central character condition that must be satisfied by the lift. For each classical group we associate a quadratic idele class character of $\mathbb{A}_k^\times$ as follows. If G_n is of type B_n, C_n, D_n, or ${}^2A_n$ we simply take χ_{G_n} to be the trivial character $\mathbf{1}$. If G_n is of type ${}^2D_n$, so a quasisplit even special orthogonal group SO^*_{2n}, then the two-dimensional anisotropic kernel of the associated orthogonal space is given by the norm form of a quadratic extension E/k; in this case we set $\chi_{G_n}=\eta_{E/k}$ the quadratic character coming from class field theory.

The arithmetic part of their characterization relies on a certain L-function $L(s,\Pi_i,R)$ for a H_{N_i} having a pole at $s=1$. The corresponding representation R of the L-group depends on the G_n from which we are lifting. If the dual group ${}^LG_n$ is of orthogonal type, then $R=\mathrm{Sym}^2$, if it is of symplectic type then $R=\Lambda^2$, and in the unitary case it is either the Asai representation $R=\mathrm{Asai}_{E/k}$ for $G_n=\mathrm{U}_{2n+1}$ or the twist by the quadratic character $\eta_{E/k}$ of the associated quadratic extension $R=\mathrm{Asai}_{E/k}\otimes\eta_{E/k}$ for $G_n=\mathrm{U}_{2n}$. For the definition of the Asai representation, if it is not familiar, see [**26, 27**].

The image of the lifting then has the following characterization [**42**].

THEOREM 6.1. *Let π be a globally generic cuspidal representation of $G_n(\mathbb{A}_k)$. Then any functorial lift of π to an automorphic representation Π of $H_N(\mathbb{A}_k)$ is self-dual (respectively self-conjugate-dual in the unitary case) with central character $\omega_\Pi=\chi_{G_n}$ (resp. $\omega_\Pi|_{\mathbb{A}^\times}=\chi_{G_n}$) and is of the form*

$$\Pi=\mathrm{Ind}(\Pi_1\otimes\cdots\otimes\Pi_{\mathrm{d}})=\Pi_1\boxplus\cdots\boxplus\Pi_{\mathrm{d}},$$

where each Π_i is a unitary self-dual (resp. self-conjugate-dual) cuspidal representation of $H_{N_i}(\mathbb{A}_k)$ such that the partial L-function $L^T(s,\Pi_i,R)$, with any sufficiently large finite set of places T containing all archimedean places, has a pole at $s=1$ and $\Pi_i\not\simeq\Pi_j$ for $i\neq j$. Moreover, any such Π is the functorial lift of some π.

Let us make a few elementary observations on the distinguishing characterizations of the various lifts. We summarize the image characterization data in the following table:

G_n	R	χ_{G_n}
SO_{2n+1}	Λ^2	$\mathbf{1}$
SO_{2n}	Sym^2	$\mathbf{1}$
SO_{2n}^*	Sym^2	$\eta_{E/k}$
Sp_{2n}	Sym^2	$\mathbf{1}$
U_{2n}	$\mathrm{Asai}_{E/k} \otimes \eta_{E/k}$	$\mathbf{1}$
U_{2n+1}	$\mathrm{Asai}_{E/k}$	$\mathbf{1}$

We first consider lifts from orthogonal and symplectic groups. Suppose for simplicity that Π is a self-dual cuspidal representation of $\mathrm{GL}_N(\mathbb{A}_k)$. If $N = 2n+1$ is odd, then Π can only be a lift from Sp_{2n} and will be iff $L^T(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$ and ω_Π is trivial. If $N = 2n$ is even, then Π can only be a lift from an orthogonal group. If $L^T(s, \Pi, \Lambda^2)$ has a pole at $s = 1$ and ω_Π is trivial then it is a lift from the split SO_{2n+1}. On the other hand, if it is a lift from an even orthogonal group, then necessarily $L^T(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$. Since the central character ω_Π is necessarily quadratic, this character will distinguish between the various even orthogonal groups. If ω_Π is trivial, then Π is a lift from the split SO_{2n} while if $\omega_\Pi = \eta_{E/k}$ for some quadratic extension E/k, then Π is a lift from the quasisplit SO_{2n}^* associated to this extension. (If Π is isobaric, one applies the same conditions to the summands.)

We next consider lifts from unitary groups and we begin with a cuspidal representation Π of $\mathrm{GL}_N(\mathbb{A}_k)$ for some number field k. If Π is to be a lift from a unitary group U_N, then we must have a quadratic sub-field $k_0 \subset k$ with non-trivial Galois automorphism σ such that both $\omega_\Pi|_{\mathbb{A}_{k_0}^\times} = \mathbf{1}$ and $\Pi \simeq \widetilde{\Pi}^\sigma$. Then we can realize $\mathrm{GL}_N(\mathbb{A}_k) = H_N(\mathbb{A}_{k_0})$ with $H_N = \mathrm{Res}_{k/k_0}\mathrm{GL}_N$. If $N = 2n+1$ is odd, then for Π to be a transfer from $\mathrm{U}_N(\mathbb{A}_{k_0})$ we would need $L^T(s, \Pi, \mathrm{Asai}_{k/k_0})$ to have a pole at $s = 1$ and if $N = 2n$ is even then for Π to be a transfer from $\mathrm{U}_{2n}(\mathbb{A}_{k_0})$ we would need $L^T(s, \Pi, \mathrm{Asai}_{k/k_0} \otimes \eta_{k/k_0})$ to have a pole at $s = 1$.

From these descriptions, it is clear that there is no intersection between lifts from different orthogonal groups nor between them and symplectic groups since the Λ^2 and Sym^2 L-functions can never share poles. On the other hand, there seems to be much room for overlap in the images from orthogonal/symplectic and unitary groups as well as potential overlap in the images from different unitary groups. It would be interesting to understand these.

7. Appendix: Quasisplit orthogonal groups

In this section we present some explicit computation for the case of $G_n = \mathrm{SO}^*_{2n}$. While these are not necessary for what preceded, they can be quite helpful in understanding this case.

7.1. The L-homomorphism. Let us start with some L-group generalities. Let k be a local or global field and let $\Gamma_k = \mathrm{Gal}(\overline{k}/k)$. Let G be a connected reductive group over k. Let B be a Borel subgroup and $T \subset B$ a Cartan subgroup. Let $\Psi_0(G) = (X, \Delta, X^\vee, \Delta^\vee)$ be the based root datum for G associated to (B,T).

Since $\mathrm{Out}(G) \simeq \mathrm{Aut}(\Psi_0(G))$ we have the short exact sequence of automorphisms

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\Psi_0(G)) \to 1.$$

We fix a splitting of this sequence as follows. For each $\alpha \in \Delta$ we fix $x_\alpha \in G_\alpha$. Then

$$\mathrm{Aut}(\Psi_0(G)) \simeq \mathrm{Aut}(G, B, T, \{x_\alpha\})$$

realizes $\mathrm{Aut}(\Psi_0(G))$ as a subgroup of $\mathrm{Aut}(G)$.

The cocycle

$$[\sigma \mapsto f(f^\sigma)^{-1}] \in H^1(\Gamma_k, \mathrm{Aut}(G))$$

then lands in $\mathrm{Aut}(\Psi_0(G))$ and becomes a homomorphism

$$\mu_G : \Gamma_k \to \mathrm{Aut}(\Psi_0(G)).$$

We then have the dual action

$$\mu_G^\vee : \mathrm{Gal}(\overline{k}/k) \to \mathrm{Aut}(\Psi_0(G)^\vee) = \mathrm{Aut}(\Psi_0(\widehat{G})).$$

In the case of SO^*_{2n} which splits over E, with $(E:k)=2$, let $\mathrm{Gal}(E/k) = \{1, \sigma\}$. As before, we realize $\mathrm{Aut}(\Psi_0(\widehat{G})) \simeq \mathrm{Aut}(\widehat{G}, \widehat{B}, \widehat{T}, \{x_{\alpha^\vee}\})$. If $\tau \in \sigma\mathrm{Gal}(\overline{k}/E)$ then $\mu_G^\vee(\tau)$ must send T to itself, each $x_{\alpha_i^\vee}$ to itself for $1 \le i \le n-2$, and interchange $x_{\alpha_{n-1}^\vee}$ and $x_{\alpha_n^\vee}$. In particular, it must send $e_n^\vee$ to $-e_n^\vee$. So the element $\mu_G^\vee(\tau)$ can be represented by an element $[\hat{w}]$ representing a coset of T in the normalizer of T in $\mathrm{O}_{2n}(\mathbb{C})$. In fact

$$[\hat{w}] = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} T.$$

Let

$$\hat{w} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \hat{t}$$

be an element of this coset that fixes the splitting $\{x_{\alpha^\vee}\}$. Then $\hat{t} \in Z(\widehat{G}) = Z(\mathrm{O}_{2n}(\mathbb{C})) = \{\pm 1\}$. Thus

$$\hat{w} = \pm \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

are the only possibilities for an element in $\mathrm{O}_{2n}(\mathbb{C})$ representing $\mu_G^\vee(\tau)$ by conjugation. The choice of $\pm$ is irrelevant. So we set

$$\hat{w} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Then our embedding $\iota : \mathrm{SO}_{2n}(\mathbb{C}) \rtimes \Gamma_k \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \Gamma_k$ must send

$$(1, \tau) \mapsto (\hat{w}, \tau) \quad \text{if} \quad \tau \in \sigma \mathrm{Gal}(\overline{k}/E)$$

while

$$(1, \tau) \mapsto (1, \tau) \quad \text{it} \quad \tau \in \mathrm{Gal}(\overline{k}/E).$$

This follows from the fact that $\mu_G^\vee(\tau)(g) = \hat{w} g \hat{w}^{-1}$ and

$$(1, \tau)(g, 1) = (\tau(g), \tau) = (\mu_G^\vee(\tau)(g), \tau) = (\hat{w} g \hat{w}^{-1}, \tau).$$

In particular

$$(1, \tau)(g, 1)(1, \tau)^{-1} = (\hat{w} g \hat{w}^{-1}, 1).$$

Note that by the matrix representation given for $\hat{w}$ we are clearly fixing

$$\widehat{T} = \mathrm{GL}_1(\mathbb{C})^{n-1} \times (\mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}))/\mathbb{C}^\times$$

the latter being the L-group of SO_2^*. Moreover note that $\hat{w}$ basically represents only one sign change ($e_n^\vee \mapsto -e_n^\vee$) and thus cannot be in the Weyl group of $\widehat{T}$ in

$\mathrm{SO}_{2n}(\mathbb{C})$. It represents the outer automorphism (the graph automorphism) of SO_{2n}^* and $(\hat{w},\tau)$ gives the embedding of $\tau\in\sigma\mathrm{Gal}(\overline{k}/E)$ in $\mathrm{GL}_{2n}(\mathbb{C})\times\Gamma_k$.

7.2. Computation of the local lift and its central character: $\mathbf{SO}_{2n}^*$. In the local lifts of Section 5, both the unramified lift and the ramified lift relied on lifting a principal series representation of G_n to a representation of GL_{2n}. We will analyze this in a bit more detail here, computing the local lift in the quasisplit case and the central character of the local lift in both cases.

Let k be a non-archimedean local field. Let $T'\subset\mathrm{GL}_{2n}$ be the standard maximal split torus. Let

$$\varphi: W_k\to {}^LT' \qquad \text{and} \qquad \chi: T'(k)\to\mathbb{C}^\times.$$

Write

$$\varphi((x,w))=(\varphi_0(x),w) \qquad \text{with} \qquad \varphi_0\in H^1(k,\widehat{T}').$$

Let $\rho^\vee\in X_*(T')=X^*(\widehat{T}')$ be such that for $x\in\overline{k}^\times$ we have $\rho^\vee(x)=\mathrm{diag}(x,\dots,x)\in\mathrm{GL}_{2n}(\overline{k})$. Then $\chi(\rho^\vee(x))=\omega_\pi(x)$ if $x\in k^\times$ and $\pi=\mathrm{Ind}(\chi)$.

Suppose we are in the case of a split SO_{2n} and $\varphi=\iota\circ\phi'$ with $\iota:\mathrm{SO}_{2n}(\mathbb{C})\hookrightarrow\mathrm{GL}_{2n}(\mathbb{C})$ and $\phi': W_k\to {}^LT$. Then

$$\chi(\rho^\vee(x))=\det(\varphi(x))=\det(\iota(\phi'(x)))=1.$$

Now let us look at the non-split case SO_{2n}^* associated to the quadratic extension E/k as above. In this case

$$T(k)=(k^\times)^{n-1}\times E^1 \qquad \text{or} \qquad T=\mathbb{G}_m^{n-1}\times\mathrm{SO}_2^*.$$

Thus

$${}^LT=\mathrm{GL}_1(\mathbb{C})^{n-1}\times(\mathrm{GL}_1(\mathbb{C})\times\mathrm{GL}_1(\mathbb{C}))/\mathbb{C}^\times.$$

and

$$\varphi=\iota\circ\phi': W_k\to {}^LT\to {}^L\widetilde{T}'$$

where $\widetilde{T}'(k)\simeq(k^\times)^{n-2}\times E^\times\times(k^\times)^{n-2}$ is a torus of $\mathrm{GL}_{2n}(k)$ with $E^\times$ embedded in $\mathrm{GL}_2(k)$ as in Langlands-Labesse [**33**]. Let $\mu=(\mu_1,\dots,\mu_{n-1},\chi_n)$, with $\mu_i\in\widehat{k^\times}$ and $\chi_n\in\widehat{E^1}$, be a character of $T(k)$. Then, by Hilbert's Theorem 90, $E^\times/k^\times\simeq E^1$ through the map $x\mapsto x/x^\sigma$. Thus we can extend χ_n to a character $\widetilde{\chi}_n$ of $E^\times$. We can then consider the character

$$\widetilde{\mu}=(\mu_1,\dots,\mu_{n-1},\widetilde{\chi}_n,\mu_{n-1}^{-1},\dots,\mu_1^{-1})$$

of $\widetilde{T}'(k)$.

To get a principal series on $\mathrm{GL}_{2n}(k)$, $\widetilde{\chi}_n$ must factor through the norm map, so write $\widetilde{\chi}_n=\mu_n\circ N_{E/k}$ with $\mu_n\in\widehat{k^\times}$. Since $\widetilde{\chi}_n$ is trivial on restriction to $k^\times$, then $\mu_n^2=1$. Since endoscopy then gives the Weil representation of $\mathrm{GL}_2(k)$ defined by $\mathrm{Ind}_{W_E}^{W_k}\widetilde{\chi}_n$, it gives the principal series representation $I(\mu_n\eta_{E/k},\mu_n)=$

$I(\mu_n\eta_{E/k}, \mu_n^{-1})$, where $\eta_{E/k}$ is the quadratic character of $k^\times$ associated to the quadratic extension E/k by local class field theory. So if the principal series representation $I(\mu)$ of $\mathrm{SO}_{2n}^*(k)$ transfers to a principal series representation Π_v of $\mathrm{GL}_{2n}(k)$, it will be induced from the character

$$(\mu_1, \ldots, \mu_{n-1}, \mu_n\eta_{E/k}, \mu_n^{-1}, \mu_{n-1}^{-1}, \ldots, \mu_1^{-1})$$

of $T'(k)$. Its central character is then simply $\eta_{E/k}$.

Even if $\widetilde{\chi}_n$ does not factor through the norm, the lift Π_v will be the representation of $\mathrm{GL}_{2n}(k)$ induced from

$$(\mu_1, \ldots, \mu_{n-1}, \pi(\mathrm{Ind}_{W_E}^{W_k}\widetilde{\chi}_n), \mu_{n-1}^{-1}, \ldots, \mu_1^{-1})$$

and its central character is still $\eta_{E/k}$. In fact, the central character of $\pi(\mathrm{Ind}_{W_E}^{W_k}\widetilde{\chi}_n)$ is the restriction of $\eta_{E/k}\widetilde{\chi}_n$ to $k^\times$, which is simply $\eta_{E/k}$. Note that now the transfer is tempered, but not necessarily a principal series.

References

[1] J. Arthur, Unipotent automorphic representations: global motivation. *Automorphic forms, Shimura varieties, and L-functions*, Vol. I (Ann Arbor, MI, 1988), 1–75, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

[2] J. Arthur, The problem of classifying automorphic representations of classical groups. *Advances in mathematical sciences: CRM's 25 years* (Montreal, PQ, 1994), 1–12, *CRM Proc. Lecture Notes*, **11**, Amer. Math. Soc., Providence, RI, 1997.

[3] J. Arthur, The principle of functoriality. *Bull. Amer. Math. Soc.* **40** (2002), 39–53.

[4] J. Arthur, Automorphic representations of GSp(4). *Contributions to automorphic forms, geometry, and number theory*, 65–81, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[5] M. Asgari and F. Shahidi, Generic transfer for general spin groups. *Duke Math. J.* **132** (2006), no. 1, 137–190.

[6] A. Borel, Automorphic L-functions. *Proc. Sympos. Pure Math.* **33**, part 2, (1979), 27–61.

[7] N. Bourbaki, Lie Groups and Lie Algebras, Chapters 4–6. Elements of Mathematics, Springer-Verlag, Berlin, 2002.

[8] W. Casselman and F. Shahidi, On reducibility of standard modules for generic representations. *Ann. Scient. Éc. Norm. Sup.* **31** (1998), 561–589.

[9] J.W. Cogdell, Dual groups and Langlands Functoriality. *An Introduction to the Langlands Program* (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston, 2003, 251–268.

[10] J.W. Cogdell, H.H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, On lifting from classical groups to GL_N. *Publ. Math. IHES* **93** (2001), 5–30.

[11] J.W. Cogdell, H.H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, Functoriality for the classical groups. *Publ. Math. IHES* **99** (2004), 163–233.

[12] J.W. Cogdell and I.I. Piatetski-Shapiro, Converse Theorems for GL_n. *Publ. Math. IHES* **79** (1994), 157–214.

[13] J.W. Cogdell and I.I. Piatetski-Shapiro, Converse Theorems for GL_n, II. *J. reine angew. Math.* **507** (1999), 165–188.

[14] J.W. Cogdell, I.I. Piatetski-Shapiro, and F. Shahidi, Stability of γ-factors for quasisplit groups. *J. Inst. Math. Jussieu* **7** (2008), 27-66.

[15] S. Gelbart and F. Shahidi, Boundedness of automorphic L-functions in vertical strips. *J. Amer. Math. Soc.* **14** (2001), 79–107.

[16] D. Ginzburg, S. Rallis, and D. Soudry, Generic automorphic forms on $\mathrm{SO}(2n+1)$: functorial lift to $\mathrm{GL}(2n)$, endoscopy, and base change. *Internat. Math. Res. Notices* **2001**, no. 14, 729–764.

[17] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Annals of Math. Studies No. 151, Princeton University Press, Princeton, 2001.

[18] G. Henniart, Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. *Invent. math.* **139** (2000), 439–455.
[19] H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, Rankin-Selberg convolutions. *Amer. J. Math.* **105** (1983), 367–464.
[20] H. Jacquet and J. Shalika, On Euler products and the classification of automorphic representations. *Amer. J. Math.* **103** (1981), 499–558 and 777–815.
[21] H. Jacquet and J. Shalika, The Whittaker models for induced representations. *Pacific J. Math.* **109** (1983). 107–120.
[22] H. Jacquet and J. Shalika, A lemma on highly ramified ϵ–factors. *Math. Ann.* **271** (1985), 319–332.
[23] D. Keys and F. Shahidi, Artin L-functions and normalization of intertwining operators. *Ann. Sci. École Norm. Sup.* (4) **21** (1988), no. 1, 67–89.
[24] H. Kim, Langlands–Shahidi method and poles of automorphic L-functions, II. *Israel J. Math.* **117** (2000), 261–284.
[25] H. Kim, On local L-functions and normalized intertwining operators. *Canad. J. Math.*, **57** (2005), 535–597.
[26] H. Kim and M. Krishnamurthy, Base change lift for odd unitary groups. *Functional analysis VIII*, 116–125, Various Publ. Ser. (Aarhus), **47**, Aarhus Univ., Aarhus, 2004.
[27] H. Kim and M. Krishnamurthy, Stable base change lift from unitary groups to GL_n. *IMRP Int. Math. Res. Pap.* **2005**, no. 1, 1–52.
[28] H. Kim and F. Shahidi, Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. *Ann. of Math.* **155** (2002), 837–893.
[29] M. Krishnamurthy, he Asai transfer to GL_4 via the Langlands-Shahidi method. *Int. Math. Res. Not.* **2003**, no. 41, 2221–2254.
[30] R.P. Langlands, On the classification of irreducible representations of real algebraic groups. *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Math. Surveys Monographs, 31, Amer. Math. Soc., Providence, RI, 1989, 101–170
[31] R.P. Langlands, Automorphic representations, Shimura varieties, and motives. Ein Märchen. *Proc. Sympos. Pure Math.* **33**, Part 2, (1979), 205–246.
[32] R.P. Langlands, Where stands functoriality today. *Proc. Sympos. Pure Math.* **61** (1997), 457–471.
[33] J-P. Labesse snd R.P. Langlands, L-indistinguishability for SL(2). *Canad. J. Math.* **31** (1979), no. 4, 726–785.
[34] G. Muić, A proof of Casselman-Shahidi's conjecture for quasisplit classical groups. *Canad. Math. Bull.* **44** (2001), 298-312.
[35] I. Satake, Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields. *Publ. Math. IHES* **18** (1963), 5–69.
[36] F. Shahidi, On certain L-functions. *Amer. J. Math.* **103** (1981), 297–355.
[37] F. Shahidi, Local coefficients as Artin factors for real groups *Duke Math. J.* **52** (1985), 973–1007.
[38] F. Shahidi, On the Ramanujan conjecture and finiteness of poles for certain L-functions. *Ann. of Math.* **127** (1988), 547–584.
[39] F. Shahidi, A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. *Ann. of Math.* **132** (1990), 273–330.
[40] F. Shahidi, On multiplicativity of local factors. *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday*, Part II (Ramat Aviv, 1989), Israel Math. Conf. Proc., 3, Weizmann, Jerusalem, 1990, 279–289.
[41] F. Shahidi, Twists of a general class of L-functions by highly ramified characters. *Canad. Math. Bull.* **43** (2000), 380–384.
[42] D. Soudry, On Langlands functoriality from classical groups to GL_n. *Formes Automorphes (I), Asterisque* **298** (2005), 335–390.
[43] J. Tate, Number theoretic background. *Proc. Sympos. Pure Math.* **33**, Part 2, (1979), 3–26.

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