# NORMAL BOHR NEIGHBORHOODS

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Abstract. We address an erroneous claim in [1] about the behavior of Bohr neighborhoods.

Let $G$ be a group. For convenience, we call an arbitrary *subset* $X$ of $G$ **normal** (in $G$) if $gXg^{-1} = X$ for all $g \in G$.

Let $\mathbb{T}^r$ denote the $r$-dimensional torus, with additive group structure from $(\mathbb{R}/\mathbb{Z})^r$. We equip $\mathbb{T}^r$ with the product of the arclength metric (normalized to one).

Given an integer $r \geq 1$, and a real number $\delta > 0$, we say that $B \subseteq G$ is a $(\delta, r)$**-Bohr neighborhood in** $G$ if there is a homomorphism $\tau\colon G \to \mathbb{T}^r$ such that $B = \tau^{-1}(U)$, where $U \subseteq \mathbb{T}^r$ is the open identity neighborhood of radius $\delta$. In this case, $U$ is normal in $\mathbb{T}^r$, and thus $B$ is normal in $G$ since, given $g \in G$, we have $gBg^{-1} = \tau^{-1}(\tau(g)U\tau(g)^{-1}) = \tau^{-1}(U) = B$. In [1, Proposition 4.9($a$)], we further claimed that if $H$ is a normal subgroup of $G$, then a Bohr neighborhood in $H$ is still normal in $G$. However this is false, and need not even hold when the Bohr neighborhood in question is actually a subgroup of $H$. In particular, this is due to the fact that normality of subgroups can fail to be transitive. The following example provides some further details.

**Example 1.** First, we note that if $H$ is a group and $B \leq H$ is a normal subgroup of finite index $n \geq 1$, with $H/B$ abelian, then $B$ is a $(\delta, r)$-Bohr neighborhood in $H$ with $r, \delta^{-1} \leq O_n(1)$. To see this, we first apply the classification of finite abelian groups to write $H/B \cong (\mathbb{Z}/p_1\mathbb{Z}) \times \ldots \times (Z/p_r\mathbb{Z})$ for some prime powers $p_1, \ldots, p_r$. Note that $r \leq \log_2 n$. Then via this isomorphism we can embed $H/B$ into $\mathbb{T}^r$ using the standard embedding of a cyclic group $\mathbb{Z}/p\mathbb{Z}$ into the unit circle via $p^{\text{th}}$ roots of unity. Finally, we precompose the embedding of $H/B$ to $\mathbb{T}^r$ with the canonical projection from $H$ to $H/B$ to obtain a homomorphism $\tau\colon H \to \mathbb{T}^r$. If $\delta = \min\{1/p_1, \ldots, 1/p_r\}$, and $U \subseteq \mathbb{T}^r$ is the open identity neighborhood of radius $\delta$, then $U \cap \tau(H) = \{\mathbf{0}\}$, and thus $B = \tau^{-1}(U)$. Note also that $\delta \geq 1/n$.

By the above paragraph, it follows that whenever we have a group $G$, a normal subgroup $H$ of $G$, and a normal subgroup $B$ of $H$ such that $B$ is *not* normal in $G$ and $H/B$ is finite abelian, then $B$ is a counterexample to the erroneous claim made in [1, Proposition 4.9($a$)]. For a specific example of this situation, let $G$ be the alternating group $A_4$, let $H$ be the Klein-4 subgroup $\{\text{id}, (12)(34), (13)(24), (14)(23)\}$ of $G$, and let $B$ the the two element subgroup $\{\text{id}, (12)(34)\}$.

At this point, it is worth emphasizing that this false claim from [1] is not used anywhere in the proofs of the main results (or anywhere else in the paper at all). The main result from [1] is a structural theorem for a $k$-NIP set $A$ in a finite group $G$. Roughly speaking, we prove that $A$ can be approximated by a union of translates of a Bohr neighborhood $B$ of bounded complexity in a normal subgroup $H \leq G$

*Date*: March 8, 2023.

of bounded index. The proof deals with left translates of $B$, and so the purpose of the (false) claim was only to provide an aesthetic observation that one need not worry about left versus right translates of $B$. Nevertheless, here we show how one can in fact assume $B$ is normal in $G$ in the previous structure theorem for $A$. The following is the key proposition that makes this work.

**Proposition 2.** *Let $G$ be a group, and let $H$ be normal subgroup of $G$ of finite index $m$. Suppose $B$ is a $(\delta, r)$-Bohr neighborhood in $H$, and let $B' = \bigcap_{g\in G} gBg^{-1}$. Then $B'$ is a $(\delta, rm)$-Bohr neighborhood in $H$.*

*Proof.* Let $g_1, \ldots, g_m \in G$ be coset representatives for $H$ in $G$. We first observe that $B' = \bigcap_{i=1}^m g_iBg_i^{-1}$. In particular, given $g \in G$, we can write $g = g_ih$ for some $1 \leq i \leq m$ and $h \in H$, whence $gBg^{-1} = g_ihBh^{-1}g_i^{-1} = g_iBg_i^{-1}$ (recall that $B$ is normal in $H$).

Now fix $\tau\colon H \to \mathbb{T}^r$ such that $B = \tau^{-1}(U)$, where $U \subseteq \mathbb{T}^r$ is the open identity neighborhood of radius $\delta$. Since $H$ is normal in $G$, it follows that for any $g \in G$, the map $\tau_g\colon H \to \mathbb{T}^r$ such that $\tau_g(x) = \tau(g^{-1}xg)$ is a well-defined homomorphism. Moreover, $\tau_g^{-1}(U) = gBg^{-1}$. Identify $\mathbb{T}^{rm}$ as $(\mathbb{T}^r)^m$, and define $\hat{\tau}\colon H \to \mathbb{T}^{rm}$ such that

$$\hat{\tau}(x) = (\tau_{g_1}(x), \ldots, \tau_{g_m}(x)).$$

Then $\hat{\tau}^{-1}(U^m) = \bigcap_{i=1}^m \tau_{g_i}^{-1}(U) = \bigcap_{i=1}^m g_iBg_i^{-1} = B'$. Since $U^m$ is the open identity neighborhood of radius $\delta$ in $\mathbb{T}^{rm}$, we conclude that $B'$ is a $(\delta, rm)$-Bohr neighborhood in $H$. □

Now we prove a modified version of the key lemma from [1].

**Lemma 3.** *For any $k \geq 1$ and $\epsilon >$, and any function $\gamma\colon (\mathbb{Z}^+)^2 \times (0,1] \to \mathbb{R}^+$, there is $n = n(k, \epsilon, \gamma)$ such that the following holds. Suppose $G$ is a finite group and $A \subseteq G$ is $k$-NIP. Then there are*

- *a normal subgroup $H \leq G$ of index $m \leq n$,*
- *a $(\delta, r)$-Bohr neighborhood $B$ in $H$, which is normal in $G$, and with $r \leq n$ and $\frac{1}{n} \leq \delta \leq 1$, and*
- *a set $Z \subseteq G$, with $|Z| < \epsilon|G|$,*

*such that for any $g \in G\backslash Z$, either $|gB \cap A| < \gamma(m, r, \delta)|B|$ or $|gB\backslash A| < \gamma(m, r, \delta)|B|$.*

*Proof.* If one removes the requirement that $B$ is normal in $G$, then the statement follows immediately from [1, Lemma 5.6] (that result also includes an "approximate Bohr set" $Y$ containing $B$, which we do not need to mention here). So, with $k$, $\epsilon$, and $\gamma$ fixed, define $\gamma_0\colon (\mathbb{Z}^+)^2 \times (0,1] \to \mathbb{R}^+$ such that $\gamma_0(x, y, z) = z^{xy}\gamma(x, xy, z)$, and let $n_0 = n(k, \epsilon, \gamma_0)$ be given by [1, Lemma 5.6]. Now fix a group $G$ and a $k$-NIP set $A \subseteq G$. Then we have a normal subgroup $H \leq G$ of index $m \leq n_0$, a $(\delta, r_0)$-Bohr neighborhood $B_0$ in $H$, with $r_0 \leq n_0$ and $\frac{1}{n_0} \leq \delta \leq 1$, and a set $Z \subseteq G$, with $|Z| < \epsilon|G|$, such that for any $g \in G\backslash Z$, either $|gB_0 \cap A| < \gamma_0(m, r_0, \delta)|B_0|$ or $|gB_0\backslash A| < \gamma_0(m, r_0, \delta)|B_0|$.

Set $r = r_0m$, $n = n_0^2$, and $B = \bigcap_{g\in G} gB_0g^{-1}$. By Proposition 2, $B$ is a $(\delta, r)$-Bohr neighborhood in $H$, which is normal in $G$, and we have $r \leq n$. So to finish the proof, we fix $g \in G\backslash Z$ and show that either $|gB \cap A| < \gamma(m, r, \delta)|B|$ or $|gB\backslash A| < \gamma(m, r, \delta)|B|$. Since $B \subseteq B_0$ we have $|gB \cap A| \leq |gB_0 \cap A|$ and $|gB\backslash A| \leq |gB_0\backslash A|$. So it suffices to show that $\gamma_0(m, r_0, \delta)|B_0| \leq \gamma(m, r, \delta)|B|$. By [1, Proposition 4.5], we have $|B| \geq \delta^r |H| \geq \delta^r |B_0|$. Therefore

$$\gamma_0(m, r_0, \delta)|B_0| \leq \delta^{-r}\gamma_0(m, r_0, \delta)|B| = \delta^{-r}\delta^{r_0 m}\gamma(m, r_0 m, \delta)|B| = \gamma(m, r, \delta)|B|,$$

as desired. □

Using the previous lemma in place of [1, Lemma 5.6], one can now follow the proof of [1, Theorem 5.7] essentially verbatim to obtain a stronger version in which the Bohr neighborhood $B$ in the statement is also normal in $G$. However, we will repeat the proof here in detail both for completeness and also to officially show that one can obtain "functional" control on the error in regular translates of $B$. (This is evident from [1, Lemma 5.6], but was not made explicit in [1, Theorem 5.7].)

**Theorem 4.** *For any $k \geq 1$ and $\epsilon > 0$, and any function $\gamma\colon (\mathbb{Z}^+)^2 \times (0, 1] \to \mathbb{R}^+$, there is $n = n(k, \epsilon, \gamma)$ such that the following holds. Suppose $G$ is a finite group and $A \subseteq G$ is $k$-NIP. Then there are*

- *a normal subgroup $H \leq G$ of index $m \leq n$,*
- *a $(\delta, r)$-Bohr neighborhood $B$ in $H$, which is normal in $G$, with $r \leq n$ and $\frac{1}{n} \leq \delta \leq 1$, and*
- *a subset $Z \subseteq G$, with $|Z| < \epsilon|G|$,*

*satisfying the following properties.*

(i) (structure) *There is a set $D \subseteq G$, which is a union of at most $m(\frac{2}{\delta})^r$ translates of $B$, such that $|(A \triangle D)\backslash Z| < \gamma(m, r, \delta)|B|$.*

(ii) (regularity) *For any $g \in G\backslash Z$, either $|gB \cap A| < \gamma(m, r, \delta)|B|$ or $|gB\backslash A| < \gamma(m, r, \delta)|B|$.*

*Proof.* Fix $k \geq 1$, $\epsilon > 0$, and $\gamma\colon (\mathbb{Z}^+)^2 \times (0, 1] \to \mathbb{R}^+$. Define $\gamma^*\colon (\mathbb{Z}^+)^2 \times (0, 1] \to \mathbb{R}^+$ such that $\gamma^*(x, y, z) = \gamma(x, y, z)x^{-1}(\frac{z}{2})^y$. Let $n = n(k, \epsilon, \gamma^*)$ be given by Lemma 3. Fix a finite group $G$ and a $k$-NIP suset $A \subseteq G$. By Lemma 3 there are

- a normal subset $H \leq G$ of index $m \leq n$,
- a $(\delta, r)$-Bohr neighborhood $B$ in $H$, which is normal in $G$, with $r \leq n$ and $\frac{1}{n} \leq \delta \leq 1$, and
- a set $Z \subseteq G$, with $|Z| < \epsilon|G|$,

such that for any $g \in G\backslash Z$, either $|gB \cap A| < \gamma^*(m, r, \delta)|B|$ or $|gB\backslash A| < \gamma^*(m, r, \delta)|B|$. Since $\gamma^*(m, r, \delta) \leq \gamma(m, r, \delta)$, this immediately yields condition $(ii)$.

For condition $(i)$, we argue as follows. First, by [1, Proposition 4.9$(b)$], there is a set $F \subseteq G\backslash Z$ such that $|F| \leq m(\frac{2}{\delta})^r$ and $G\backslash Z \subseteq FB$. Let $I = \{g \in F : |gB\backslash A| < \gamma^*(m, r, \delta)|B|\}$, and note that if $g \in F\backslash I$ then $|gB \cap A| < \gamma^*(m, r, \delta)|B|$. Let $D = IB$. Since $G \subseteq Z \subseteq FB$, we have

$$A \triangle D \subseteq Z \cup \bigcup_{g \in I}(gB\backslash A) \cup \bigcup_{g \in F\backslash I}(gB \cap A), \text{ and so}$$

$$|(A \triangle D)\backslash Z| \leq \sum_{g \in I}|gB\backslash A| + \sum_{g \in F\backslash I}|gB \cap A| < |F|\gamma^*(m, r, \delta)|B| \leq \gamma(m, r, \delta)|B|. \ \square$$

## References

[1] G. Conant, A. Pillay, and C. Terry, *Structure and regularity for subsets of groups with finite VC-dimension*, J. Eur. Math. Soc. (JEMS) **24** (2022), no. 2, 583–621.

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