A NOTE ON STABILITY IN A MODEL

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This note contains a combinatorial result about bipartite graphs (Proposition 2), which is motivated by certain definitions in model theory. Given an $\mathcal{L}$-structure M, an $\mathcal{L}$-formula $\varphi(x,y)$ is called **stable in** M if there do not exist sequences $(a_i)_{i\in I}$ in M^x and $(b_i)_{i\in I}$ in M^y, indexed by an infinite linear order I, such that $M\models\varphi(a_i,b_j)$ if and only if $i\leq j$. Note that it suffices to assume I is either ω or ω^* (where the latter denotes the reverse order on ω). This variation of the order property seems to have first appeared in an early paper of Pillay [4], and was later popularized by a paper of Ben Yaacov [1] on the connection between stability and work of Grothendieck [2] in functional analysis.

Following the definition of stable in a model, it is natural to ask for an analogous definition of NIP in a model. In this case, one does not want to look for the independence property to be fully realized by sequences in M, since this would require M to have size at least continuum. Thus, in [3], Khanaki and Pillay say that $\varphi(x,y)$ is **NIP in** M if there does not exist an infinite set $A\subseteq M^x$ such that the set system $\{\varphi(x,b):b\in M^y\}$ shatters all finite subsets of A. Equivalently, $\varphi(x,y)$ is NIP in M if and only if there do not exist sequences $(a_i)_{i\in\omega}$ in M^x and $(b_X)_{X\subseteq\omega}$ in N^y, where $N\succeq M$, such that $N\models\varphi(a_i,b_X)$ if and only if $i\in X$ (see [3, Remark 1.2] for further discussion).

Now, since the definition of NIP in a model is not directly analogous to the definition of stable in a model, it becomes necessary to prove that if $\varphi(x,y)$ is stable in M then it is also NIP in M. This can be done by showing that stability in a model has a formulation analogous to NIP in model. Specifically, the following is [3, Lemma 2.6].

Fact 1. *A formula $\phi(x,y)$ is stable in M if and only if there is no infinite linear order I, and sequences $(a_i)_{i\in I}$ in M^x and $(b_i)_{i\in I}$ in N^y, with $N\succeq M$, such that $N\models\phi(a_i,b_j)$ if and only if $i\leq j$.*

In [3], this is obtained as a quick corollary of Pillay's account in [5] of the Grothendieck approach to stability and, in particular, definability of types. Given this short but rather high-powered proof, Pillay asked for a direct combinatorial proof, which we will present here. Since this is a combinatorial argument, we will phrase our result (Proposition 2) in terms of bipartite graphs, and align things with an equivalent formulation of the righthand side of Fact 1 involving only the initial model M. For aesthetic reasons, Proposition 2 will also correspond to switching the positions of M and N in Fact 1. But this does not matter since $\varphi(x,y)$ is stable in M if and only if $\varphi^*(y,x)$ is stable in M, where $\varphi^*(y,x)$ is identical to $\varphi(x,y)$ but with the roles of x and y switched.

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A relation $R \subseteq U \times V$ is **stable** if there is no infinite linear order I, and sequences $(a_i)_{i\in I}$ from U and $(b_i)_{i\in I}$ from V, such that $R(a_i, b_j)$ holds if and only if $i \leq j$. (So $\varphi(x,y)$ is stable in M if and only if φ is stable as a relation on $M^x \times M^y$.)

Proposition 2. *Given $R \subseteq U \times V$, the following are equivalent.*

(*i*) *R is stable.*

(*ii*) *There is no infinite linear order I and sequences $(a_i^X)_{i\in X\subset_{\text{fin}} I}$ in U and $(b_i)_{i\in I}$ in V, such that given $i,j \in X \subset_{\text{fin}} I$, $R(a_i^X, b_j)$ holds if and only if $i \leq j$.*

Proof. $(ii) \Rightarrow (i)$ is trivial. We prove the contrapositive of $(i) \Rightarrow (ii)$. Suppose there is some infinite linear order I as in (ii). We may assume I is either ω or ω^*. Suppose first that I is ω. So we have $(a_i^k)_{i\leq k\in\omega}$ in U and $(b_i)_{i\in\omega}$ in V such that, for all $i,j \leq k \in \omega$, $R(a_i^k, b_j)$ holds if and only if $i \leq j$. Call $t \in \omega$ *good* if for all $k \geq t$, the set $\{u \in \omega : R(a_t^k, b_u)\}$ is finite.

Case 1: There is some good $t \in \omega$.

We construct a strictly increasing sequence $(k_i)_{i\in\omega}$ such that $k_0 = t$ and, for all $i < j \in \omega$, $\neg R(a_t^{k_i}, b_{k_j})$. Let $k_0 = t$ and suppose we have constructed $k_0, \ldots, k_i$ as above. Since t is good, the set $\{u \in \omega : R(a_t^{k_j}, b_u)$ for some $j \leq i\}$ is finite. So we may choose some $k_{i+1} > k_i$ such that $\neg R(a_t^{k_j}, b_{k_{i+1}})$ holds for all $j \leq i$. This finishes the construction.

Now set $c_i = a_t^{k_i}$ and $d_i = b_{k_i}$. If $i \geq j$ then $R(c_i, d_j)$ holds since $k_i \geq k_j \geq t$. If $i < j$ then $\neg R(c_i, d_j)$ holds since $k_i < k_j$. Altogether, R is not stable.

Case 2: There is no good $t \in \omega$.

We construct $(k_t)_{t\in\omega}$ in ω and $(I_t)_{t\in\omega}$ in $\mathcal{P}(\omega)$ such that, for all $t \in \omega$, I_t is infinite, $i_t := \min I_t \leq k_t$, and $I_{t+1} = \{u \in I_t : u > k_t,\ R(a_{i_t}^{k_t}, b_u)\}$. Let $I_0 = \omega$. Fix $t \geq 0$ and suppose have constructed I_s for all $s \leq t$, and k_s for all $s < t$. We find I_{t+1} and k_t. Since i_t is not good, and I_t is infinite, there is some $k_t \geq i_t$ such that the set $I_{t+1} := \{u \in I_t : u > k_t,\ R(a_{i_t}^{k_t}, b_u)\}$ is infinite. This finishes the construction.

Now set $c_t = a_{i_t}^{k_t}$ and $d_t = b_{i_t}$. Fix $s,t < \omega$. If $s = t$ then $R(c_s, d_t)$ holds since $i_t \leq k_t$. If $s > t$ then $\neg R(c_s, d_t)$ holds since $i_t < i_s \leq k_s$. Suppose $s < t$. Then $i_t = \min I_t \subseteq I_{s+1}$, and so $i_t \in I_{s+1}$. So $R(a_{i_s}^{k_s}, b_{i_t})$ holds, i.e., $R(c_s, d_t)$ holds. Altogether, R is not stable.

Finally, suppose I is ω^*. So we have $(a_i^k)_{i\leq k\in\omega}$ in U and $(b_i)_{i\in\omega}$ in V such that, for all $i,j \leq k \in \omega$, $R(a_i^k, b_j)$ holds if and only if $i \geq j$. For $i \in \omega$, let $c_i = b_{i+1}$. Then, for all $i,j \leq k \in \omega$, we have $\neg R(a_i^k, c_j)$ if and only if $i \leq j$. So $\neg R$ is not stable by the above argument, which implies that R is not stable. ☐

References

[1] I. Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symb. Log. **20** (2014), no. 4, 491–496. MR 3294276

[2] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), 168–186. MR 0047313

[3] K. Khanaki and A. Pillay, *Remarks on the NIP in a model*, MLQ Math. Log. Q. **64** (2018), no. 6, 429–434. MR 3898100

[4] A. Pillay, *Dimension theory and homogeneity for elementary extensions of a model*, J. Symbolic Logic **47** (1982), no. 1, 147–160. MR 644760

[5] ———, *Generic stability and Grothendieck*, South Amer. J. Log. **2** (2016), no. 2, 437–442. MR 3671044