CORRECTIONS TO "AN AXIOMATIC APPROACH TO FREE AMALGAMATION"

GABRIEL CONANT

Recent work of Scott Mutchnik [10, 11] has led to renewed interest in [5] and the class of free amalgamation theories. The purpose of this note is to correct a number of errors in the paper. I would like to thank Michele Bailetti for bringing the issues in Section 1.1 and Section 3 to my attention. The issue in Section 2 arose from discussions with Alex Kruckman, who also deserves very special thanks for fixing the issue in Section 3 by providing the proof of Proposition 3.1.

1. Indiscernible oversights

1.1. **TP$_2$.** Definition 7.1 of [5] gives a nonstandard formulation of TP$_2$, which is incorrect and, in particular, too weak. However, the witness to TP$_2$ constructed in the proof of [5, Theorem 7.7] satisfies some additional indiscernibility assumptions (described in the next proposition), which suffice to overcome the discrepancy.

Proposition 1.1. *Suppose there are tuples $a, b \in \mathbb{M}$ and an array $(b^m_n)_{m,n<\omega}$ in $\mathbb{M}$ satisfying the following properties.*

(*i*) *There does not exist a tuple a_* such that $a_* b^0_n \equiv ab$ for all $n < \omega$.*
(*ii*) *For all $\sigma\colon \omega \to \omega$ there is a tuple a_* such that $a_* b^m_{\sigma(m)} \equiv ab$ for all $m < \omega$.*
(*iii*) *The sequence $(b^0_i)_{i<\omega}$ is indiscernible and, for all $m < \omega$, $b^m_{<\omega} \equiv b^0_{<\omega}$.*

Then T has TP$_2$.

Proof. Note that $b^m_n \equiv b$ for all $m, n < \omega$ by condition (*ii*). Let $p(x, b) = \mathrm{tp}(a/b)$. Then by condition (*i*), $\bigcup_{n<\omega} p(x, b^0_n)$ is inconsistent. By compactness there is a formula $\varphi(x, b) \in p$ and some finite $I \subset \omega$ such that $\{\varphi(x, b^0_n) : n \in I\}$ is inconsistent. Since $b^0_{<\omega}$ is indiscernible, it follows that $\{\varphi(x, b^0_n) : n < \omega\}$ is k-inconsistent where $k = |I|$. Therefore, for any $m < \omega$, $\{\varphi(x, b^m_n) : n < \omega\}$ is k-inconsistent since $b^m_{<\omega} \equiv b^0_{<\omega}$. With (*ii*), $\varphi(x, y)$ now satisfies the standard definition of TP$_2$. □

The incorrect definition of TP$_2$ in the paper includes only condition (*ii*) and a stronger form of (*i*) (which is implied by (*i*) and (*iii*)). In any case, it is clear in the proof of [5, Theorem 7.7] that we have built an array satisfying (*i*), (*ii*), and (*iii*). It is also worth pointing out that (*iii*) is a weak form of array indiscernibility, and thus the converse of the previous proposition holds as well. Further discussion and details can be found on MathStackExchange [2].

1.2. **SOP$_3$.** In the characterization of SOP$_3$ given by [5, Proposition 7.2], one must assume that $(b_i)_{i<\omega}$ is indiscernible in order to be able to lift indiscernibility to the pair sequence $(a_i, b_i)_{i<\omega}$ using EM-types. Alternatively, it would suffice to replace the types p and q by formulas. The only part of the paper that uses [5, Proposition 7.2] is the construction of SOP$_3$ in the proof of [5, Theorem 7.17]. In the proof,

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we have a type $r(x, y)$ over a parameter set C, as well as some $b_0 \equiv_C b_1$ such that $r(x, b_0) \cup r(x, b_1)$ is inconsistent. Then, with $z = (x_1, x_2)$, we use $p(x, z) = r(x, y_1)$ and $q(x, z) = r(x, y_2)$ in the application of [5, Proposition 7.2]. However, by compactness, there is a formula $\varphi(x, y)$ in $r(x, y)$ such that $\varphi(x, b_0) \wedge \varphi(x, b_1)$ is inconsistent, and we can replace r with φ in the rest of the proof.

2. Algebraic closure in dividing independence

Fact 7.4 in [5] states that $a \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C} b$ if and only if $\mathrm{acl}(aC) \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{\mathrm{acl}(C)} \mathrm{acl}(bC)$. The justification for this is [1, Remark 5.4(3)], which was recently discovered to be false (see [7]). **Therefore, Fact 7.4 in** [5] **is incorrect.** Fortunately, all uses of this erroneous result can be replaced by weaker statements, which are true.

Proposition 2.1.

(*a*) *If* $\mathrm{acl}(aC) \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{\mathrm{acl}(C)} \mathrm{acl}(bC)$ *then* $a \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C} b$.

(*b*) *If* $a \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C} b$ *then* $a \mathop{\smile\mkern-13mu\mid}^{\mathrm{a}}_{C} b$.

(*c*) *If* $\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}$ *and* $\mathop{\smile\mkern-13mu\mid}^{\mathrm{a}}$ *do not coincide then there is an algebraically closed set* C *and algebraically closed tuples* a, b *such that* $a \cap b = C$ *and* $a \not\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C} b$.

(*d*) *Fact 7.4 of* [5] *holds under the additional assumption that* T *is simple.*

Proof. Part (*a*). Suppose $\mathrm{acl}(aC) \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{\mathrm{acl}(C)} \mathrm{acl}(bC)$. Then $a \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{\mathrm{acl}(C)} b$ by monotonicity. To prove $a \mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C} b$, fix a C-indiscernible sequence $(b_i)_{i<\omega}$ with $b_0 = b$. Then $(b_i)_{i<\omega}$ is $\mathrm{acl}(C)$-indiscernible (see [3, Corollary 1.7]), hence there is some a^* such that $a^* b_i \equiv_{\mathrm{acl}(C)} ab_i$ for all $i < \omega$ (so, in particular, $a^* b_i \equiv_C ab_i$ for all $i < \omega$).

Part (*b*). This is a folklore fact, which is often cited in the literature as a corollary of the incorrect remark from [1]. See [6, Section 2] for discussion and a direct proof.

Part (*c*). Assume $\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}$ and $\mathop{\smile\mkern-13mu\mid}^{\mathrm{a}}$ do not coincide. By part (*b*), there are a', b', C' such that $a' \not\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C'} b'$ and $a' \mathop{\smile\mkern-13mu\mid}^{\mathrm{a}}_{C'} b'$. Let a enumerate $\mathrm{acl}(a'C')$, b enumerate $\mathrm{acl}(b'C')$, and $C = \mathrm{acl}(C')$. Then $a' \mathop{\smile\mkern-13mu\mid}^{\mathrm{a}}_{C'} b'$ says precisely that $a \cap b = C$. Moreover, since $a' \not\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C'} b'$, we have $a \not\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}_{C} b$ by part (*a*).

Part (*d*). Recall that if T is simple then $\mathop{\smile\mkern-13mu\mid}^{\mathrm{d}}$ coincides with $\mathop{\smile\mkern-13mu\mid}^{\mathrm{f}}$. Moreover, the analogue of [5, Fact 7.4] for $\mathop{\smile\mkern-13mu\mid}^{\mathrm{f}}$ in a simple theory is well-known (see, e.g., [3, Proposition 5.20], whose proof is left as an exercise). In fact, it is not hard to show that $\mathop{\smile\mkern-13mu\mid}^{\mathrm{f}}$ satisfies the analogue of [5, Fact 7.4] in *any* theory (see [7]). □

We now provide patches for each use of [5, Fact 7.4] in the paper.

(1) In the proof of Theorem 7.7[$(ii) \Rightarrow (iii)$], our use of Fact 7.4 was only in order to apply Proposition 2.1(*c*).
(2) In Fact 7.10, we assume T is simple. Thus Fact 7.4 holds for T by Proposition 2.1(*d*).
(3) In Lemma 7.15, we tacitly used Fact 7.4, but only in order to conclude Proposition 2.1(*b*).
(4) In the proof of Theorem 7.17, our use of Fact 7.4 was only in order to apply Proposition 2.1(*c*).
(5) In the proof of Theorem 7.22, our use of Fact 7.4 was only in order to apply Proposition 2.1(*a*).

3. The dichotomy for modular free amalgamation theories

Theorem 7.17 of [5] states that any modular free amalgamation theory T is either simple or SOP_3. However, there is a gap in the proof of Claim 1. In order to discuss the details, let us first recall how the proof starts. Assume T is not simple. Then $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}}$ and $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}$ do not coincide by [5, Theorem 7.7]. By Proposition 2.1(c), there are algebraically closed tuples a, b such that, setting $C = a \cap b$, we have $a \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_C b$. So there is a C-indiscernible sequence $(b_i)_{i<\omega}$ such that, if $p(x, y) = \mathrm{tp}(a, b/C)$, then $\{p(x, b_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$. At this point, Claim 1 attempts to reduce to the case that $k = 2$. We do this by choosing a minimal k witnessing $a \not\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}_C b$, and then defining the sequence $b^*_i = \mathrm{acl}(b_{i(k-1)}b_{i(k-1)+1} \dots b_{i(k-1)+k-2})$. By minimality, there is some a_* such that $a_*b_i \equiv_C ab$ for all $i < k - 1$, and it is (correctly) shown that $\{p^*(x, b^*_i) : i < \omega\}$ is 2-inconsistent, where $p^*(x, y) = \mathrm{tp}(a_*, b^*_0/C)$. However, we may have lost $a_* \cap b^*_0 = C$. To fix this, the reader is told to set $C^* = a_* \cap b^*_0$ and replace $(b^*_i)_{i<\omega}$ with a C^*-indiscernible realization of its EM-type, while also moving by an automorphism to ensure we keep b^*_0 as the first term of the sequence. But in order for this to work, we would need to know that $b^*_i \equiv_{C^*} b^*_j$ for all $i < j < \omega$, and there is no reason for this to be true.

Before getting into the details of how to fix this gap, we first note that if one strengthens the modularity assumption to *disintegration* of algebraic closure, then the above proof works. Indeed, each b_i is algebraically closed, and thus disintegration would imply $b^*_i = b_{i(k-1)} \dots b_{i(k-1)+k-2}$. Thus, since $a \cap b = C$ and $a_*b_i \equiv_C ab$ for all $i < k - 1$, we do have $a_* \cap b^*_0 = C$ in the disintegrated case. The reason to point this out is that all of the examples of free amalgamation theories in [5] have disintegrated algebraic closure (but see Remark 3.2 below for further details).

We now turn to fixing the gap. In [11], Mutchnik used the general proof strategy of [5, Theorem 7.17] to show that SOP_1 and SOP_2 coincide at the level of theories. The overall scope of this result draws from many areas of neostability (e.g., [4] and [8]). However, in the special case of modular free amalgamation theories, Mutchnik's strategy in the analogue of Claim 1 suggests an elementary fix for the gap above. We also note that, while modularity of T is used in the proof of Claim 2 of [5, Theorem 7.17], it is not used in the (incorrect) proof of Claim 1. So we emphasize that the following correct proof does require modularity.

Proposition 3.1. *Let T be a modular complete theory such that $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}}$ and $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{d}}$ do not coincide. Then there is an algebraically closed set C, algebraically closed tuples a and b, and a C-indiscernible sequence $(b_i)_{i<\omega}$, with $b_0 = b$, such that $a \cap b = C$ and there is no a^* with $a^*b_0 \equiv_C a^*b_1 \equiv_C ab$.*

Proof. We first argue that it suffices to find an algebraically closed set C, a C-indiscernible sequence $(b'_i)_{i<\omega}$, and some tuple a' such that

$$(\dagger) \qquad a' \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{a}}_C b'_0 \text{ and there is no } a^* \text{ with } a^*b'_0 \equiv_C a^*b'_1 \equiv_C a'b'_0.$$

Indeed, with (†) in hand, let a enumerate $\mathrm{acl}(a'C)$ and b enumerate $\mathrm{acl}(b'_0C)$. So $a \cap b = C$. Let $b_0 = b$. For $i > 0$, choose an automorphism σ_i over C such that $\sigma_i(b'_0) = b'_i$ and set $b_i = \sigma_i(b_0)$. Let $(b^*_i)_{i<\omega}$ be a C-indiscernible realization of the EM-type of $(b_i)_{i<\omega}$ over C. Let b''_i be the subtuple of b^*_i corresponding to the location of b'_i in b_i. Since each b_i realizes $\mathrm{tp}(b_0/C)$, so does each b^*_i and, in particular, b^*_i enumerates $\mathrm{acl}(b''_i)$. Since $(b'_i)_{i<\omega}$ is C-indiscernible, we have $b''_{<\omega} \equiv_C b'_{<\omega}$. Thus after moving by an automorphism over C, we may assume $(b_i)_{i<\omega}$ is C-indiscernible. By (†), we clearly have no a^* such that $a^*b_0 \equiv_C a^*b_1 \equiv_C ab$.

Now we construct C, $(b'_i)_{i<\omega}$, and a' satisfying (†).[1] Fix a, b, and C such that $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b$ and $a \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{d}}_C b$ (recall Proposition 2.1(b)). By Proposition 2.1(a) (or (c)), we may assume C is algebraically closed. Fix a C-indiscernible sequence $(b_i)_{i<\omega}$, with $b_0 = b$, such that there is no a^* satisfying $a^*b_i \equiv_C ab$ for all $i < \omega$. Since $a \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b$, we may choose $k \geq 1$ maximal such that there is a tuple a_0 with $a_0 \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b_{<k}$ and $a_0 b_t \equiv_C ab$ for all $t < k$. Set $b'_i = b_{ik}b_{ik+1}\dots b_{ik+k-1}$. Then $(b'_i)_{i<\omega}$ is C-indiscernible, and we have $a_0 \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b'_0$. Let $\kappa = |\operatorname{acl}(b_{\leq k}C)|^+$ and, using extension for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$, construct a sequence $(a_i)_{i<\kappa}$ such that $a_i \equiv_{Cb'_0} a_0$ and $a_i \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C a_{<i}b'_0$. By base monotonicity for $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$, we have $a_i \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_{a_{<i}C} b'_0$ for all $i < \kappa$. Using left transitivity and induction, we see that $a_{<i} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b'_0$ for all $i < \kappa$, hence $a_{<\kappa} \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b'_0$.

Finally, we show that there is no $a^*_{<\kappa}$ with $a^*_{<\kappa}b'_0 \equiv_C a^*_{<\kappa}b'_1 \equiv_C a_{<\kappa}b'_0$, and so we can set $a' = a_{<\kappa}$ to obtain (†). Toward a contradiction, suppose that there is such an $a^*_{<\kappa}$. Fix some $i < \kappa$. Then $a^*_i b'_0 \equiv_C a_i b'_0 \equiv a_0 b'_0$, and so for all $t < k$, we have $a^*_i b_t \equiv_C a_0 b_t \equiv_C ab$. Also, $a^*_i b'_1 \equiv_C a_i b'_0 \equiv_C a_0 b'_0$, and so $a^*_i b_k \equiv_C a_0 b_0 \equiv_C ab$. So $a^*_i b_k \equiv_C ab$ for all $t \leq k$ and, by maximality of k, it follows that $a^*_i \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b_{\leq k}$.

Now, for each $i < \kappa$, fix a witness $e_i \in (\operatorname{acl}(a^*_i C) \cap \operatorname{acl}(b_{\leq k}C))\backslash C$ to $a^*_i \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C b_{\leq k}$. By choice of κ, there are $j < i < \kappa$ such that $e_i = e_j$, and so $a^*_i \not\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C a^*_j$. But $a^*_i a^*_j \equiv_C a_i a_j$, and $a_i \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}_C a_j$, which is a contradiction. □

Remark 3.2. As mentioned above, the previous proof is inspired by work of Mutchnik [11], who uses a strategy similar to [5, Theorem 7.17] to construct SOP_3 in any theory with SOP_1 and NSOP_2 (hence there are no such theories, so SOP_1 and SOP_2 coincide). In subsequent work, Mutchnik [10] uses similar tools to prove several new results about free amalgamation theories. Recall that all of the examples in [5] of such theories are modular and, in fact, disintegrated. Moreover, any *simple* free amalgamation theory is modular by [5, Corollary 7.13]. Thus, Question 7.19 of [5] asks if *every* free amalgamation theory is modular. In Section 4 of [10], Mutchnik provides a negative answer by showing that the theory $T_{f,c}$ of a generic binary function with a distinguished constant is a non-modular free amalgamation theory. By work of Kruckman and Ramsey (see Corollary 3.13 and Proposition 3.14 of [9]), $T_{f,c}$ is not simple, but is NSOP_3 (in fact, NSOP_1). Altogether, the modularity assumption cannot be removed from [5, Theorem 7.17]. However, Mutchnik extends [5, Theorem 7.17] to the non-modular case by proving the following results for an arbitrary free amalgamation theory T (see Section 2 of [10]).

(1) T is either NSOP_1 or SOP_3.
(2) Moreover, if T is NSOP_1 then Kim-independence coincides with algebraic independence over models.
(3) Therefore, T is simple if and only if it is NSOP_1 and modular.

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Department of Mathematics, The Ohio State University, Columbus, OH, USA
Email address: conant.38@osu.edu