In [1, Theorem 8.8], it is shown that if $A \subseteq \mathbb{N}$ is definable in a stable expansion of $(\mathbb{Z}, +, 0)$ then $A$ has upper Banach density 0. Question 8.6 of [1] asks if one can strengthen this conclusion to: $A$ does not contain arbitrarily large finite arithmetic progressions. This question was later posed at the 2017 AIM meeting, Nonstandard Methods in Combinatorial Number Theory, and a proof was suggested by the second author using his results on “substable groups” from [5]. In subsequent conversations, we adapted the method to the “local case” (i.e., when only assuming $x + y \in A$ is a stable formula), and also reformulated the result as a characterization, in terms of arithmetic progressions, of syndetic subsets $A \subseteq \mathbb{Z}$ such that $x + y \in A$ is stable. In particular, we proved the following theorem.

**Theorem 1.** Suppose $A \subseteq \mathbb{Z}$ is stable. The following are equivalent.

- (i) $A$ is syndetic.
- (ii) $A$ is piecewise syndetic.
- (iii) $\text{BD}(A) > 0$.
- (iv) $A$ contains arbitrarily large finite arithmetic progressions.

Before getting into the proof, we recall the definitions and make some remarks.

**Definition 2.** Fix $A \subseteq \mathbb{Z}$.

1. $A$ is **syndetic** if $\mathbb{Z} = F + A$ for some finite set $F \subseteq \mathbb{Z}$.
2. $A$ is **piecewise syndetic** if $A \cup X$ is syndetic for some non-sydentic set $X$.
3. $A$ is **stable** if, for some $k \geq 1$, there does not exist $a_1, \ldots, a_k, b_1, \ldots, a_k \in \mathbb{Z}$ such that $a_i + b_j \in A$ if and only if $i \leq j$.
4. The **upper Banach density of** $A$ is

   $$\text{BD}(A) = \lim_{n \to \infty} \sup_{m \in \mathbb{Z}} \frac{|A \cap \{m+1, \ldots, m+n\}|}{n}.$$

**Remark 3.** Note that $A \subseteq \mathbb{Z}$ is stable if and only if the formula $x + y \in A$ is stable with respect to the structure $(\mathbb{Z}, +, A)$. In model theory, syndetic sets are called generic and piecewise syndetic sets are called weakly generic. It is easy to show that $A \subseteq \mathbb{Z}$ is piecewise syndetic if and only if there is a finite set $F \subseteq \mathbb{Z}$ such that $A + F$ contains arbitrarily large intervals. In particular, if $A \subseteq \mathbb{Z}$ is piecewise syndetic then $\text{BD}(A) > 0$.

The proof of Theorem 1 uses arguments similar to [5], which deals with “substable groups” (i.e. undefinable subgroups of stable groups). We adapt these arguments to the local case of a stable formula.

**Proof of Theorem 1.** First, note that $(i) \Rightarrow (ii)$ is trivial, $(ii) \Rightarrow (iii)$ follows from Remark 3, and $(iii) \Rightarrow (iv)$ is Szemerédi’s Theorem [4] for uniform density (see also Remark 4 below). We show $(iv) \Rightarrow (i)$.
Suppose $A$ contains arbitrarily large finite arithmetic progressions. We work in the theory $T = \text{Th}(\mathbb{Z}, +, A)$, and let $\varphi(x, y) := x + y \in A$, which is stable. Let $G^*$ be a monster model of $T$. By assumption, the type $\pi(x, y) := \{x + ky \in A : k \in \mathbb{Z}\}$ (in variables $x$ and $y$) is finitely satisfiable in $(\mathbb{Z}, +, A)$. So we may fix $a, b \in G^*$ such that $\pi(a, b)$ holds. So $\Gamma := b\mathbb{Z}$ is an infinite cyclic subgroup of $G^*$ contained in the translate $B = A(G^*) - a$ of $A(G^*)$.

Let $X$ be the (nonempty) set of types in $S_{\varphi}(G^*)$, which are finitely satisfiable in $\Gamma$. Then $X$ is closed set in $S_{\varphi}(G^*)$ and hence a compact Hausdorff space. By stability of $\varphi(x, y)$, we may let $S$ be the nonempty finite set of types in $X$ of maximal Cantor-Bendixson rank.

Fix $p \in S$ and note that $B \in p$ since $p$ is finitely satisfiable in $\Gamma$. Let $H = \text{Stab}_\varphi(p)$. Since $\varphi$ is stable, $p$ is definable, and so $H$ is definable. Any translate of $p$ by an element of $\Gamma$ is still finitely satisfiable in $\Gamma$ (and of maximal rank), and so $H \cap \Gamma$ has finite index in $\Gamma$ since $S$ is finite. In particular, $H$ is nontrivial. Since $H = \text{Stab}_\varphi(p)$, we have $B - h \in p$ for all $h \in H$, and so:

(i) for any finite $K \subseteq H$ there is some $g \in G^*$ such that $g + K \subseteq B$.

For a contradiction, suppose $H$ is not covered by finitely many translates of $B$. We inductively build sequences $(g_i)_{i=0}^\infty$ from $G^*$ and $(h_i)_{i=0}^\infty$ from $H$ such that $g_i + h_j \in B$ if and only if $i > j$, which contradicts stability of $x + y \in B$. Fix $n \in \mathbb{N}$ and suppose we have constructed $(g_i)_{i<n}$ and $(h_i)_{i<n}$ as desired. By (i), there is $g_n \in G^*$ such that $g_n + h_i \in B$ for all $i < n$. By our assumption, $H$ is not contained in $\bigcup_{i \leq n} g_i + B$, and so there is $h_n \in H$ such that $g_i + h_n \notin B$ for all $i \leq n$.

Altogether, we have built a nontrivial definable subgroup $H$ of $G^*$ covered by finitely many translates of $B$, and hence covered by finitely many translates of $A(G^*)$. This fact can be expressed as a first-order formula (without extra parameters), and so finitely many translates of $A$ cover a nontrivial subgroup of $\mathbb{Z}$. Since any nontrivial subgroup of $\mathbb{Z}$ is syndetic, it follows that $A$ is syndetic. \qed

\textbf{Remark 4.} In the previous proof, one can use more local stability to replace the appeal to Szemerédi’s Theorem with Van der Waerden’s Theorem, which is much easier to prove. In particular, fix $A \subseteq \mathbb{Z}$ and assume $A$ is stable and $BD(A) > 0$. Then there is a sequence $(I_n)_{n=0}^\infty$ of intervals in $\mathbb{Z}$ such that $|I_n| \to \infty$ and $\lim_{n \to \infty} |A \cap I_n|/|I_n| = BD(A)$. Let $\mu$ be the left-invariant finitely-additive probability measure on $\mathbb{Z}$ given by $\lim_{U} \mu_U$, where $U$ is a nonprincipal ultrafilter on $\mathbb{N}$ and $\mu_U$ is the normalized counting measure on $I_n$ (i.e. $\mu_U(X) = |X \cap I_n|/|I_n|$). Then $\mu(A) = BD(A) > 0$. By stability of $A$, it follows that $A$ is syndetic (see [2, Theorem 2.3]). So $A$ contains arbitrarily large finite arithmetic progressions by Van der Waerden’s Theorem.

\textbf{References}