# DIVIDING LINES IN UNSTABLE THEORIES

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The aim of this paper is to define various properties of formulas in first order theories, and prove the appropriate implications between these properties. Most definitions are taken from [3], but the definitions themselves and many of the proofs are due to Shelah (see [4, II]). We give citations at the beginning of proofs taken from other sources.

Recall that a theory is stable if no formula has the so-called "order property", and a theory is simple if no formula has the "tree property". We first define these properties, along with a few more complicated properties of the same type. We fix some theory $T$ and a sufficiently saturated $\mathbb{M} \models T$. If $\varphi$ is a sentence with parameters from $\mathbb{M}$, we write $\models \varphi$ if $\mathbb{M} \models \varphi$.

## 1. A Chain of Properties

**Definition 1.1.** *A formula $\varphi(x,y)$ has the **order property**, OP, if there are tuples $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ such that $\models \varphi(a_i,b_j)$ if and only if $i<j$.*

*For $n \geq 3$, a formula $\varphi(x,y)$, with $l(x)=l(y)$, has the $n$-**strong order property**, $\mathrm{SOP}_n$, if*

$$\models \neg\exists x_1,\dots,x_n(\varphi(x_1,x_2)\wedge\varphi(x_2,x_3)\wedge\ldots\wedge\varphi(x_n,x_1)),$$

*and there are tuples $(a_i)_{i<\omega}$ such that $\models \varphi(a_i,a_j)$ for all $i<j<\omega$.*

*A formula $\varphi(x,y)$, with $l(x)=l(y)$, has the **strong order property**, SOP, if for all $n\geq 3$*

$$\models \neg\exists x_1,\dots,x_n(\varphi(x_1,x_2)\wedge\varphi(x_2,x_3)\wedge\ldots\wedge\varphi(x_n,x_1)),$$

*and there are tuples $(a_i)_{i<\omega}$ such that $\models \varphi(a_i,a_j)$ for all $i<j<\omega$.*

*A formula $\varphi(x,y)$ has the **strict order property**, sOP, if there are tuples $(a_i)_{i<\omega}$ such that*

$$\models \exists x(\neg\varphi(x,a_i)\wedge\varphi(x,a_j)) \quad\Leftrightarrow\quad i<j.$$

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Consider the definition of $\mathrm{SOP}_n$ and its natural extension to $n = 1$ or $n = 2$. For $n = 2$ we have the order property. Moreover, any theory with an infinite model would satisfy the definition with $n = 1$ via the formula $x \neq y$. Therefore we will redefine $\mathrm{SOP}_2$ and $\mathrm{SOP}_1$ in the same vein as the next class of properties, which are defined using trees as index sets.

Before defining these properties, we specify some notation concerning trees.

**Definition 1.2.** *Let $A$ be a set and define*

$$A^{<\omega} = \bigcup_{n\in\omega} A^n.$$

*If $(a_0, \dots, a_n), (b_0, \dots, b_m) \in A^{<\omega}$, define*

$$(a_0, \dots, a_n)\hat{\ }(b_0, \dots, b_m) := (a_0, \dots, a_n, b_0, \dots, b_m) \in A^{<\omega}.$$

*If $\mu, \eta \in A^{<\omega}$, we say $\mu \prec \eta$ if there is some $\gamma \in A^{<\omega}$ such that $\eta = \mu\hat{\ }\gamma$. For $a \in A$ we identify $a$ and $(a) \in A^{<\omega}$. If $n \in \omega$, we also define $(a)^n = (\underbrace{a, a, \dots, a}_{n\ times}) \in A^{<\omega}$. Two elements $\mu, \eta \in A^{<\omega}$ are **incomparable** if $\mu \not\prec \eta$ and $\eta \not\prec \mu$.*

The next class of properties on formulas are defined using tuples indexed by trees.

**Definition 1.3.** *A formula $\varphi(x, y)$ has the **tree property**,* TP, *if there are tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ and some $k \geq 2$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all $\eta \in \omega^{<\omega}$, $\{\varphi(x, a_{\eta\hat{\ }n}) : n < \omega\}$ is $k$-inconsistent.*

*A formula $\varphi(x, y)$ has the **tree property 1**,* $\mathrm{TP}_1$, *if there are tuples $(a_\eta)_{\eta\in\omega^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in \omega^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all incomparable $\mu, \eta \in \omega^{<\omega}$, $\{\varphi(x, a_\mu), \varphi(x, a_\eta)\}$ is inconsistent.*

*A formula $\varphi(x, y)$ has* $\mathrm{SOP}_1$ *if there are tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all $\mu, \eta \in 2^{<\omega}$, if $\mu\hat{\ }0 \prec \eta$ then $\{\varphi(x, a_{\mu\hat{\ }1}), \varphi(x, a_\eta)\}$ is inconsistent.*

*A formula $\varphi(x, y)$ has* $\mathrm{SOP}_2$ *if there are tuples $(a_\eta)_{\eta\in 2^{<\omega}}$ and some $k \in \mathbb{Z}^+$ such that for all $\sigma \in 2^\omega$, $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent; but for all incomparable $\mu, \eta \in 2^{<\omega}$, $\{\varphi(x, a_\mu), \varphi(x, a_\eta)\}$ is inconsistent.*

The goal of this section is to prove the following chain of implications (when Q $\Rightarrow$ R is written with no other information, we read this as "if $T$ has Q then $T$ has R").

**Theorem 1.4.**

$$\mathrm{sOP} \Rightarrow \mathrm{SOP} \Rightarrow \dots \Rightarrow \mathrm{SOP}_{n+1} \Rightarrow \mathrm{SOP}_n \Rightarrow \dots \Rightarrow \mathrm{SOP}_3 \Rightarrow (\mathrm{TP}_1 \Leftrightarrow \mathrm{SOP}_2) \Rightarrow \mathrm{SOP}_1 \Rightarrow \mathrm{TP} \Rightarrow \mathrm{OP}.$$

**Proposition 1.5.** sOP $\Rightarrow$ SOP.

*Proof.* Suppose $\varphi(x,y)$, with $(a_i)_{i<\omega}$, witnesses sOP. Let $l(x_1)=l(x_2)=l(y)$ and define

$$\psi(x_1,x_2) := \forall x(\varphi(x,x_1)\to\varphi(x,x_2))\wedge\exists x(\varphi(x,x_2)\wedge\neg\varphi(x,x_1)).$$

By assumption, $\models\psi(a_i,a_j)$ for all $i<j$. Suppose, towards a contradiction, that we have $n\geq 3$ and $b_1,\ldots,b_n$ such that

$$\models\psi(b_1,b_2)\wedge\ldots\wedge\psi(b_{n-1},b_n)\wedge\psi(b_n,b_1).$$

If $B_i=\psi(\mathbb{M},b_i)$ for $1\leq i\leq n$, then we have $B_1\subsetneq B_2\subsetneq\ldots\subsetneq B_n\subsetneq B_1$, which is a contradiction. Therefore $\psi(x_1,x_2)$, with $(a_i)_{i<\omega}$, witnesses SOP. □

**Proposition 1.6.** SOP $\Rightarrow$ SOP$_n$ *for all* $n\geq 3$*.*

*Proof.* Follows by definition. □

**Proposition 1.7.** *For* $n\geq 3$*,* SOP$_{n+1}\Rightarrow$ SOP$_n$*.*

*Proof.* Suppose $T$ has SOP$_{n+1}$, witnessed by $\varphi(x,y)$ and $(a_i)_{i<\omega}$. Define

$$\psi(x_1,x_2,y_1,y_2) := \varphi(x_1,x_2)\ \wedge\varphi(x_2,y_1)\ \wedge\ \varphi(x_2,y_2)\ \wedge\ \varphi(y_1,y_2).$$

If $i<j$ then $\models\psi(a_{2i},a_{2i+1},a_{2j},a_{2j+1})$. Suppose, towards a contradiction, that $(b_{1,0},b_{1,1}),\ldots,(b_{n,0},b_{n,1})$ are such that

$$\mathbb{M}\models\psi(b_{1,0},b_{1,1},b_{2,0},b_{2,1})\wedge\ldots\wedge\psi(b_{n-1,0},b_{n-1,1},b_{n,0},b_{n,1})\wedge\psi(b_{n,0},b_{n,1},b_{1,0},b_{1,1}).$$

Then we have

$$\mathbb{M}\models\varphi(b_{1,0},b_{1,1})\ \wedge\ \varphi(b_{1,1},b_{2,1})\wedge\ldots\wedge\varphi(b_{n-1,1},b_{n,1})\wedge\varphi(b_{n,1},b_{1,0}),$$

contradicting that $\varphi(x,y)$ SOP$_{n+1}$. Therefore $\psi(x_1,x_2,y_1,y_2)$, with $(a_{2i},a_{2i+1})_{i<\omega}$, witnesses SOP$_n$. □

**Proposition 1.8.** SOP$_3\Rightarrow$ SOP$_2$.

*Proof.* [2] Suppose $\varphi(x,y)$, with $(a_i)_{i<\omega}$, witnesses SOP$_3$. We have $\models\varphi(a_i,a_j)$ for all $i<j$. By compactness, we can obtain $(b_q)_{q\in\mathbb{Q}}$ such that $\models\varphi(b_q,b_r)$ for all $q<r$. Set $z=(y_1,y_2)$ and define

$$\psi(x,z) := \varphi(y_1,x)\wedge\varphi(x,y_2).$$

We define $(c_\eta)_{\eta\in 2^{<\omega}}$ inductively by $c_\emptyset=(b_0,b_1)$, and if $c_\eta=(b_q,b_r)$, with $q<r$, then

$$c_{\eta\hat{}i}=\begin{cases}(b_q,b_{\frac{1}{3}(r-q)}) & i=0\\ (b_{\frac{2}{3}(r-q)},b_r) & i=1\end{cases}$$

We claim that $\psi(x,z)$, with $(c_\eta)_{\eta\in 2^{<\omega}}$, witnesses $\mathrm{SOP}_2$. To this end, suppose $\sigma\in 2^\omega$ and $n<\omega$. There are $0=q_0<\ldots<q_n<r_n<r_{n-1}<\ldots<r_0=1$ such that for $0\leq i\leq n$,

$$c_{\sigma|_i}=(b_{q_i},b_{r_i}).$$

If $q_n<q<r_n$ then $\models\varphi(b_{q_i},b_q)\wedge\varphi(b_q,b_{r_i})$ for all $0\leq i\leq n$. Thus $\{\psi(x,c_{\sigma|_i}):0\leq i\leq n\}$ is satisfiable, and so $\{\psi(x,c_{\sigma|_n}):n<\omega\}$ is consistent by compactness.

Now suppose $\mu,\eta\in 2^{<\omega}$ are incomparable. Then, without loss of generality, we have $q<r<s<t$ such that

$$c_\mu=(b_q,b_r)\ \text{ and }\ c_\eta=(b_s,b_t).$$

If $d$ satisfies $\{\psi(x,c_\mu),\psi(x,c_\eta)\}$ then we have

$$\varphi(d,b_r)\wedge\varphi(b_r,b_s)\wedge\varphi(b_s,d),$$

contradicting that $\varphi(x,y)$ witnesses $\mathrm{SOP}_3$. Therefore $\{\psi(x,c_\mu),\psi(x,c_\eta)\}$ is inconsistent. □

**Proposition 1.9.** $\mathrm{SOP}_2\Leftrightarrow\mathrm{TP}_1$.

*Proof.* [3] Suppose $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses $\mathrm{SOP}_2$. Define $h:\omega^{<\omega}\longrightarrow 2^{<\omega}$ inductively by $h(\emptyset)=\emptyset$ and for $i<\omega$,

$$h(\eta\hat{\ }i)=h(\eta)\hat{\ }(1)^i\hat{\ }0.$$

If $\eta\prec\mu$, say $\mu=\eta\hat{\ }(n_1,\ldots,n_k)$ with $n_i\in\omega$, then $h(\mu)=h(\eta)\hat{\ }(1)^{\sum n_i}\hat{\ }0$ so $h(\eta)\prec h(\mu)$. Thus if $\sigma\in\omega^\omega$, we may define $h(\sigma):=\bigcup_{n<\omega}h(\sigma|_n)\in 2^\omega$.

By assumption, $\{\varphi(x,a_{h(\sigma)|_n}):n<\omega\}$ is consistent. If $\eta,\mu\in\omega^{<\omega}$ are incomparable then, without loss of generality, there are $\gamma,\eta_0,\mu_0\in\omega^{<\omega}$ and $i<j$ such that $\eta=\gamma\hat{\ }i\hat{\ }\eta_0$ and $\mu=\gamma\hat{\ }j\hat{\ }\mu_0$. It follows that there are $\eta_1,\mu_1\in 2^{<\omega}$ such that $h(\eta)=h(\gamma)\hat{\ }(1)^i\hat{\ }0\hat{\ }\eta_1$ and $h(\mu)=h(\gamma)\hat{\ }(1)^j\hat{\ }0\hat{\ }\mu_1$. Therefore h($\eta$) and $h(\mu)$ are incomparable, and so $\{\varphi(x,a_{h(\eta)}),\varphi(x,a_{h(\mu)})\}$ is inconsistent. In conclusion $\varphi(x,y)$ with $(a_{h(\eta)})_{\eta\in\omega^{<\omega}}$, witnesses $\mathrm{TP}_1$.

Conversely, if $\varphi(x,y)$, with $(a_\eta)_{\eta\in\omega^{<\omega}}$, witnesses $\mathrm{TP}_1$, then clearly $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses $\mathrm{SOP}_2$. □

**Proposition 1.10.** $\mathrm{SOP}_2\Rightarrow\mathrm{SOP}_1$.

*Proof.* Suppose $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses $\mathrm{SOP}_2$. For all $\mu,\eta\in 2^{<\omega}$, if $\mu\hat{\ }0\prec\eta$ then $\mu\hat{\ }1$ and $\eta$ are incomparable, and so $\{\varphi(x,a_{\mu\hat{\ }1}),\varphi(x,a_\eta)\}$ is inconsistent. Thus $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses $\mathrm{SOP}_1$. □

**Proposition 1.11.** $\mathrm{SOP}_1\Rightarrow\mathrm{TP}$.

*Proof.* [2] Suppose $\varphi(x,y)$, with $(a_\eta)_{\eta\in 2^{<\omega}}$, witnesses $\mathrm{SOP}_1$. Define $h:\omega^{<\omega}\longrightarrow 2^{<\omega}$ inductively such that $h(\emptyset)=\emptyset$ and for $i<\omega$,

$$h(\eta\hat{\ }i)=h(\eta)\hat{\ }(0)^i\hat{\ }1.$$

For $\eta\in\omega^{<\omega}$, set $b_\eta=a_{h(\eta)}$. As in the proof of Proposition 1.9, $\mu\prec\eta$ implies $h(\mu)\prec h(\eta)$. For $\sigma\in\omega^\omega$, define $h(\sigma)=\bigcup_{n<\omega}h(\sigma|_n)$. Then $\{\varphi(x,b_{\sigma|_n}):n<\omega\}\subseteq\{\varphi(x,a_{h(\sigma)|_n}):n<\omega\}$, so $\{\varphi(x,b_{\sigma|_n}):n<\omega\}$ is consistent.

Now fix $\eta\in\omega^{<\omega}$ and suppose $i<j$. Then $h(\eta)\hat{\ }(0)^i\prec h(\eta\hat{\ }j)$ and $h(\eta\hat{\ }i)=h(\eta)\hat{\ }(0)^i\hat{\ }1$, so

$$\{\varphi(x,a_{h(\eta\hat{\ }i)}),\varphi(x,a_{h(\eta\hat{\ }j)})\}$$

is inconsistent by assumption. Therefore $\{\varphi(x,b_{\eta\hat{\ }i}),\varphi(x,b_{\eta\hat{\ }j})\}$ is inconsistent, and so $\{\varphi(x,b_{\eta\hat{\ }n}):n<\omega\}$ is 2-inconsistent. Thus $\varphi(x,y)$, with $(b_\eta)_{\eta\in\omega^{<\omega}}$, witnesses TP. □

The only remaining implication in the statement of Theorem 1.4 is TP $\Rightarrow$ OP. This argument is a bit more technical than the previous one, and we break it into two steps, the proofs of which are taken from [4].

**Lemma 1.12.** *Suppose $\varphi(x,y)$ witnesses* TP *with respect to $k\geq 2$. Then there is an infinite set $A$ such that $|S_\varphi(A)|>|A|$.*

*Proof.* [4, II] Let $\kappa$ be an infinite cardinal such that $\kappa^\omega>\max\{2^\omega,\kappa\}$. By compactness we may assume that we have $(a_\eta)_{\eta\in\kappa^{<\omega}}$ such that for all $\sigma\in\kappa^\omega$,

$$\pi_\sigma=\{\varphi(x,a_{\sigma|_n}):n<\omega\}$$

is consistent; and for all $\eta\in\kappa^{<\omega}$, $\{\varphi(x,a_{\eta\hat{\ }i}):i<\kappa\}$ is $k$-inconsistent. Given $\sigma\in\kappa^\omega$, construct $F_\sigma\subseteq\kappa^\omega$ such that

(*i*) $\sigma\in F_\sigma$;

(*ii*) $\bigcup_{\tau\in F_\sigma}\pi_\tau$ is consistent.

(*iii*) for all $\rho\in\kappa^\omega\backslash F_\sigma$, $\pi_\rho\cup\bigcup_{\tau\in F_\sigma}\pi_\tau$ is inconsistent.

Let $T_\sigma=\{\tau|_n:n<\omega,\ \tau\in F_\sigma\}$. Then $T_\sigma$ is a tree. Suppose, towards a contradiction, that there is $\eta\in T_\sigma$ and distinct $i_1,\dots,i_k\in\kappa$ such that $\eta\hat{\ }i_j\in T_\sigma$ for all $j$. Then there are $\tau_1,\dots,\tau_k\in F_\sigma$ such that $\eta\hat{\ }i_j\prec\tau_j$, which is a contradiction since $\{\varphi(x,a|_{\eta\hat{\ }i_j}):1\leq j\leq k\}$ is inconsistent. It follows that $T_\sigma$ can be embedded into $k^\omega$. In particular, $|F_\sigma|\leq 2^\omega$. Since $\kappa^\omega>2^\omega$, there is $F\subseteq\kappa^\omega$ such that $|F|=\kappa^\omega$ and $F_\sigma\neq F_\tau$ for all distinct $\sigma,\tau\in F$.

Let $A=(a_\eta)_{\eta\in\kappa^{<\omega}}$ and, for $\sigma\in F$, let $p_\sigma\in S_\varphi(A)$ be a complete $\varphi$-type containing $\bigcup_{\tau\in F_\sigma}\pi_\tau$. If $\sigma,\tau\in F$ are distinct then, without loss of generality, there is some $\rho\in F_\sigma\backslash F_\tau$. Then $\pi_\rho\subseteq p_\sigma$ and $p_\tau\cup\pi_\rho$ is inconsistent. Therefore $p_\sigma\neq p_\tau$, and so $|S_\varphi(A)|\geq\kappa^\omega>\kappa=|A|$. □

**Definition 1.13.** *Given formulas $\varphi(x,y)$, $\psi(y,x)$, a type $p$ $(\psi,\varphi)$-**splits** over a set $B$ if there are $a, b \in \mathrm{dom}(p)$ such that $\mathrm{tp}_\psi(a/B) = \mathrm{tp}_\psi(b/B)$, but $\varphi(x,a), \neg\varphi(x,b) \in p$.*

**Proposition 1.14.** TP $\Rightarrow$ OP.

*Proof.* [4, II] Suppose $\varphi(x,y)$ witnesses TP. By Lemma 1.12, there is some infinite cardinal $\kappa$, and a set $A$ of size $\kappa$, such that $|S_\varphi(A)| > \kappa$. Let $(c_i)_{i<\kappa^+}$ be realizations of $\kappa^+$-many distinct $\varphi$-types in $S_\varphi(A)$. Set $\psi(y,x) = \varphi(x,y)$. Let $A_0 = A$ and, given $A_n$ of size $\kappa$, define

$$A_{n+1} = A_n \cup \{a : a \models p,\ p \in S_\varphi(B) \cup S_\psi(B),\ B \subseteq A_n \text{ is finite}\}.$$

There are countably many finite subsets of $A_n$, and if $B$ is finite then $S_\varphi(B) \cup S_\psi(B)$ is finite, so $A_{n+1}$ still has size $\kappa$.

***Claim***: There is some $i < \kappa^+$ such that for all $n < \omega$ and for all $B \subseteq A_n$ finite, $\mathrm{tp}_\varphi(c_i/A_{n+1})$ $(\psi,\varphi)$-splits over $B$.

*Proof*: Suppose not. Then for all $i < \kappa^+$ there is a pair $(n,B)$ such that $B \subseteq A_n$ is finite and $\mathrm{tp}_\varphi(c_i/A_{n+1})$ does not $(\psi,\varphi)$-split over $B$. There are only countably many such pairs $(n,B)$. Thus, without loss of generality, there is a pair $(n,B)$ such that $B \subseteq A_n$ is finite and for all $i < \kappa^+$, $\mathrm{tp}(c_i/A_{n+1})$ does not $(\psi,\varphi)$-split over $B$. By definition, there is a finite set $C$ such that $B \subseteq C \subseteq A_{n+1}$ and all types in $S_\varphi(B) \cup S_\psi(B)$ are realized in $C$. Again, $S_\varphi(C)$ is finite, so without loss of generality we may assume $\mathrm{tp}_\varphi(c_i/C) = \mathrm{tp}_\varphi(c_j/C)$ for all $i, j < \kappa^+$.

Consider $c_0, c_1$. By assumption, there is some $a \in A_0$ such that $\models \varphi(c_1,a) \leftrightarrow \neg\varphi(c_0,a)$. Let $a' \in C$ such that $\mathrm{tp}_\psi(a'/B) = \mathrm{tp}_\psi(a/B)$. For all $i < \kappa^+$, $\mathrm{tp}_\varphi(c_i/A_{n+1})$ does not $(\psi,\varphi)$-split over $B$, so it follows that $\mathrm{tp}_\varphi(c_i/C)$ does not $(\psi,\varphi)$-split over $B$. Since $\mathrm{tp}_\psi(a/B) = \mathrm{tp}_\psi(a'/B)$, we have $\varphi(x,a) \in \mathrm{tp}_\varphi(c_i/C)$ if and only if $\varphi(x,a') \in \mathrm{tp}_\varphi(c_i/C)$. In other words, $\models \varphi(c_i,a) \leftrightarrow \varphi(c_i,a')$, for all $i < \kappa^+$. Altogether, we have

$$\models \varphi(c_0,a) \leftrightarrow \varphi(c_0,a') \leftrightarrow \varphi(c_1,a') \leftrightarrow \varphi(c_1,a) \leftrightarrow \neg\varphi(c_0,a),$$

which is a contradiction.$_{/\!/}$

By the claim, we have $i < \kappa^+$ such that for all $n < \omega$ and for all $B \subseteq A_n$ finite, $\mathrm{tp}(c_i/A_{n+1})$ $(\psi,\varphi)$-splits over $B$. Set $c = c_i$. Then $\mathrm{tp}_\varphi(c/A_1)$ $(\psi,\varphi)$-splits over $\emptyset$, so there are $a_0, b_0 \in A_1$ such that $\mathrm{tp}_\psi(a_0) = \mathrm{tp}_\psi(b_0)$ with $\varphi(x,a_0), \neg\varphi(x,b_0) \in \mathrm{tp}(c/A_1)$. Now $\{a_0, b_0\} \subseteq A_1$ so there is some $d_0 \in A_2$ realizing $\mathrm{tp}_\varphi(c/a_0,b_0)$.

Suppose $n > 0$ and we are given $(a_i, b_i, d_i)_{i<n}$ such that for all $i < n$,

*(i)* $\mathrm{tp}_\psi(a_i/\{d_j : j < i\}) = \mathrm{tp}_\psi(b_i/\{d_j : j < i\})$;
*(ii)* $d_i \in A_{2i+2}$ realizes $\mathrm{tp}_\varphi(c/\{a_j, b_j : j \leq i\})$;
*(iii)* $\models \varphi(c,a_i) \wedge \neg\varphi(c,b_i)$.

Then $\mathrm{tp}_\varphi(c/A_{2n+1})$ $(\psi,\varphi)$-splits over $\{d_i : i < n\} \subseteq A_{2n}$ so there are $a_n, b_n \in A_{2n+1}$ such that $\mathrm{tp}_\psi(a_n/\{d_i : i < n\}) = \mathrm{tp}_\psi(b_n/\{d_i : i < n\})$ and $\varphi(x, a_n), \neg\varphi(x, b_n) \in \mathrm{tp}_\varphi(c/A_{2n+1})$. But $\mathrm{tp}_\varphi(c/\{a_i, b_i : i \leq n\})$ is realized by some $d_n \in A_{2n+2}$. This process generates $(a_n, b_n, d_n)_{n<\omega}$ such that for all $n < \omega$,

(*i*) $\mathrm{tp}_\psi(a_n/\{d_i : i < n\}) = \mathrm{tp}_\psi(b_n/\{d_i : i < n\})$;

(*ii*) $d_n \in A_{2n+2}$ realizes $\mathrm{tp}_\varphi(c/\{a_i, b_i : i \leq n\})$;

(*iii*) $\models \varphi(c, a_n) \wedge \neg\varphi(c, b_n)$.

Note first that for all $j \leq i$, we have $\models \varphi(d_i, a_j) \wedge \neg\varphi(d_i, b_j)$. Moreover, for all $i < j$,

$$\models \varphi(d_i, a_j) \leftrightarrow \psi(a_j, d_i) \leftrightarrow \psi(b_j, d_i) \leftrightarrow \varphi(d_i, b_j).$$

Therefore we have

$$\models \varphi(d_i, a_j) \leftrightarrow \varphi(d_i, b_j) \quad \Leftrightarrow \quad i < j.$$

Altogether, if $z = (y_1, y_2)$ and $\theta(x, z) := \varphi(x, y_1) \leftrightarrow \varphi(x, y_2)$, then $\theta(x, z)$, with $(d_i)_{i<\omega}$ and $(a_i, b_i)_{i<\omega}$, witnesses OP. □

This completes the proof of Theorem 1.4.

## 2. Further Properties

We now define two more properties, which do not fit exactly into the chain in Theorem 1.4.

**Definition 2.1.** *A formula $\varphi(x, y)$ has the* ***independence property****, IP, if there are $(a_i)_{i<\omega}$ and $(c_\sigma)_{\sigma\in 2^\omega}$ such that $\models \varphi(a_i, c_\sigma)$ if and only if $\sigma(i) = 1$.*

*A formula $\varphi(x, y)$ has the* ***tree property 2****, $\mathrm{TP}_2$, if there are $(a_{i,j})_{i,j<\omega}$ such that for any $\sigma \in \omega^\omega$, $\{\varphi(x, a_{n,\sigma(n)}) : n < \omega\}$ is consistent; but for all $j < k < \omega$, $\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k})\}$ is inconsistent.*

**Proposition 2.2.** IP $\Rightarrow$ OP.

*Proof.* Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$ and $(c_\sigma)_{\sigma\in 2^\omega}$, witnesses IP. Given $i < \omega$, let $\sigma_i : \omega \longrightarrow \omega$ such that $\sigma_i(j) = 0$ if and only if $i \leq j$. Then we have

$$\models \varphi(a_i, c_{\sigma_j}) \quad \Leftrightarrow \quad \sigma_j(i) = 1 \quad \Leftrightarrow \quad i < j.$$

So $\varphi(x, y)$, with $(a_i)_{i<\omega}$ and $(c_{\sigma_i})_{i<\omega}$, witnesses OP. □

**Proposition 2.3.** $\mathrm{TP}_2 \Rightarrow$ TP.

*Proof.* [1] Suppose $\varphi(x,y)$, with $(a_{i,j})_{i,j<\omega}$, witnesses $\mathrm{TP}_2$. Fix an injection $f:\omega\times\omega\longrightarrow\omega$. Set $b_\emptyset=a_{0,0}$, and for $i<\omega$, set $b_{(j)}=a_{1,j}$. Suppose $0<n<\omega$ and for all $\eta\in\omega^n$ we have $j<\omega$ such that $b_\eta=a_{n,j}$. Let $(b_{\eta_i})_{i<\omega}$ be an enumeration of $\omega^n$ and for $j<\omega$ define $b_{\eta_i\hat{\ }j}=a_{n+1,f(i,j)}$.

We claim that $\varphi(x,y)$, with $(b_\eta)_{\eta\in\omega^{<\omega}}$, witnesses TP with respect to 2. If $\sigma\in\omega^\omega$ then for all $n<\omega$, $b_{\sigma|_n}=a_{n,j}$ for some $j<\omega$. So if $\tau:\omega\longrightarrow\omega$ is such that $\tau(n)=j$, we have that

$$\{\varphi(x,b_{\sigma|_n}):n<\omega\}=\{\varphi(x,a_{n,\tau(n)}):n<\omega\}$$

is consistent. Furthermore suppose $\eta\in\omega^{<\omega}$ and $j<k<\omega$. If $|\eta|=n$, then $b_{\eta\hat{\ }j}=a_{n+1,f(i,j)}$ and $b_{\eta\hat{\ }k}=a_{n+1,f(i,k)}$, where $\eta=\eta_i$ in the enumeration of $\omega^n$. Since $f$ is injective, it follows that $f(i,j)\neq f(i,k)$, and so

$$\{\varphi(x,b_{\eta\hat{\ }j}),\varphi(x,b_{\eta\hat{\ }k})\}=\{\varphi(x,a_{n+1,f(i,j)}),\varphi(x,a_{n+1,f(i,k)})\}$$

is inconsistent by assumption. □

**Proposition 2.4.** $\mathrm{TP}_2\Rightarrow\mathrm{IP}$.

*Proof.* [1] Suppose $\varphi(x,y)$, with $(a_{i,j})_{i,j<\omega}$, witnesses $\mathrm{TP}_2$. Let $\sigma\in 2^\omega$. By assumption,

$$\{\varphi(x,a_{i,1}):\sigma(i)=1\}\cup\{\varphi(x,a_{i,0}):\sigma(i)=0\}$$

is consistent, say satisfied by some $b_\sigma$. Furthermore, $\{\varphi(x,a_{i,0}),\varphi(x,a_{i,1})\}$ is inconsistent for all $i<\omega$, and so it follows that $b_\sigma$ satisfies

$$\{\varphi(x,a_{i,1}):\sigma(i)=1\}\cup\{\neg\varphi(x,a_{i,1}):\sigma(i)=0\}.$$

Therefore $\varphi(x,y)$, with $(a_{i,1})_{i<\omega}$ and $(b_\sigma)_{\sigma\in 2^\omega}$, witnesses IP. □

Altogether, we have shown the following:

**Theorem 2.5.**

$$\begin{array}{l}
\mathrm{sOP}\Rightarrow\mathrm{SOP}\Rightarrow\ldots\Rightarrow\mathrm{SOP}_{n+1}\Rightarrow\mathrm{SOP}_n\Rightarrow\ldots\Rightarrow\mathrm{SOP}_3\Rightarrow(\mathrm{TP}_1\Leftrightarrow\mathrm{SOP}_2)\Rightarrow\mathrm{SOP}_1\Rightarrow \begin{array}{ccc}\mathrm{TP} & \Rightarrow & \mathrm{OP}\\ \Uparrow & & \Uparrow\\ \mathrm{TP}_2 & \Rightarrow & \mathrm{IP}\end{array}
\end{array}$$

**Remark 2.6.** In [4], the following equivalences are proved,

$$\mathrm{OP}\Leftrightarrow(\mathrm{IP}\ \text{ or }\ \mathrm{sOP})\ \text{ and }\ \mathrm{TP}\Leftrightarrow(\mathrm{TP}_1\ \text{ or }\ \mathrm{TP}_2).$$

We detail these proofs in the last section.

Recall again that a theory $T$ is stable if and only if $T$ does not have OP; and $T$ is simple if and only if $T$ does not have TP.

## 3. Alternate Definitions

In the literature, it is easy to find sources with slightly different definitions of the properties discussed above. While this can sometimes make a nominal difference when considering the property with respect to the formula, it usually does not make any difference when considering the property with respect to a theory.

**Theorem 3.1.** *Let $n \geq 3$. Then $T$ has* $\mathrm{SOP}_n$ *if and only if there is a formula $\varphi(x, y)$, with $l(x) = l(y)$, such that for all $k \leq n$,*

$$\models \neg\exists x_1, \ldots, x_k(\varphi(x_1, x_2) \wedge \ldots \wedge \varphi(x_{k-1}, x_k) \wedge \varphi(x_k, x_1)),$$

*and there are $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$.*

*Proof.* Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses $\mathrm{SOP}_n$. Then for all $k < n$, there are $\varphi_k(x, y)$ and $(a_i^k)_{k<\omega}$ witnessing $\mathrm{SOP}_k$ if $k \geq 3$ and OP (respectively an infinite model) if $k = 2$ (resp. $k = 1$). Define

$$\psi(x_1, \ldots, x_n, y_1, \ldots, y_n) := \varphi(x_n, y_n) \wedge \bigwedge_{k<n} \varphi_k(x_k, y_k).$$

Clearly, for all $k \leq n$, we have

$$\models \neg\exists \bar{x}_1, \ldots, \bar{x}_k(\psi(\bar{x}_1, \bar{x}_2) \wedge \ldots \wedge \psi(\bar{x}_{k-1}, \bar{x}_k) \wedge \psi(\bar{x}_k, \bar{x}_1)),$$

Moreover, if $\bar{a}_i = (a_i^1, \ldots, a_i^{n-1}, a_i^n)$, then $\models \psi(\bar{a}_i, \bar{a}_{i+1})$ for all $i < \omega$.

Conversely, suppose we have $\varphi(x, y)$, with $l(x) = l(y)$ and $(a_i)_{i<\omega}$ such that $\models \varphi(a_i, a_{i+1})$ for all $i < \omega$ and for all $k \leq n$,

$$\models \neg\exists x_1, \ldots, x_k(\varphi(x_1, x_2) \wedge \ldots \wedge \varphi(x_{k-1}, x_k) \wedge \varphi(x_k, x_1)).$$

□

**Theorem 3.2.** *$T$ has* sOP *if and only if there is a formula $\psi(x, y)$, with $l(x) = l(y)$, defining a partial order (reflexive, antisymmetric, transitive) with infinite chains.*

*Proof.* Suppose $\varphi(x, y)$, with $(a_i)_{i<\omega}$, witnesses that $T$ has sOP. Define the formula,

$$\psi(y_1, y_2) := y_1 = y_2 \vee \Big(\forall x(\varphi(x, y_1) \to \varphi(x, y_2)) \wedge \exists x(\neg\varphi(x, y_1) \wedge \varphi(x, y_2))\Big).$$

In other words, for all $b, c \in \mathbb{M}$,

$$\models \psi(b,c) \quad \Leftrightarrow \quad b = c \text{ or } \varphi(\mathbb{M}, b) \subsetneq \varphi(\mathbb{M}, c).$$

Therefore $\psi(y_1, y_2)$ defines a partial order. By assumption we have $\varphi(\mathbb{M}, a_i) \subsetneq \varphi(\mathbb{M}, a_j)$ for all $i < j$, so $(a_i)_{i<\omega}$ is an infinite chain with respect to $\psi(y_1, y_2)$.

Conversely, suppose we have $\psi(x, y)$ defining a partial order with infinite chains. Let $(a_i)_{i<\omega}$ be an infinite chain, i.e., $\models \psi(a_i, a_j)$ and $a_i \neq a_j$ for all $i < j$. We claim that $\psi(x, y)$, with $(a_i)_{i<\omega}$ witnesses sOP. Indeed, if $i < j$ then we have $\models \neg\psi(a_j, a_i) \wedge \psi(a_j, a_j)$. On the other hand, if $c \in \mathbb{M}$ such that $\models \neg\psi(c, a_i) \wedge \psi(c, a_j)$ then $i < j$, since otherwise we would have $\models \psi(c, a_j) \wedge \psi(a_j, a_i)$, and so $\models \psi(c, a_i)$ by transitivity. □

## 4. Equivalence Theorems

**Definition 4.1.** *A formula $\varphi(x, y)$ is **unstable** if there is some infinite set $A$ such that $|S_\varphi(A)| > |A|$.*

Recall that $T$ is stable if and only if no formula is unstable.

**Lemma 4.2.** *A formula $\varphi(x, y)$ is unstable if and only if it has* OP.

*Proof.* [4, II] Suppose $\varphi(x, y)$ is unstable. As in the proof of Proposition 1.14, there are $(a_i, b_i, d_i)_{i<\omega}$ such that

$$\models \varphi(d_i, a_j) \leftrightarrow \varphi(d_i, b_j) \text{ for all } i < j, \quad \text{and} \quad \models \varphi(d_i, a_j) \wedge \neg\varphi(d_i, b_j) \text{ for all } j \leq i.$$

Let $[\omega] = \{(i, j) : i < j < \omega\}$ and define $f : [\omega] \longrightarrow \{0, 1\}$ such that $f(i, j) = 0$ if and only if $\models \varphi(d_i, a_j)$. By Ramsey's Theorem, there is an infinite subset $I \subseteq \omega$ such that $f$ is constant on $\{(i, j) \in I^2 : i < j\}$. By renaming, we may assume $f$ is constant on $[\omega]$. If $f \equiv 0$ then we have $\models \varphi(d_i, b_j)$ if and only if $i < j$, so $\varphi(x, y)$ has OP. If $f \equiv 1$ then we have $\models \neg\varphi(d_i, a_j)$ if and only if $i < j$. Define

$$\Delta = T \cup \{\varphi(x_i, y_j) : i < j < \omega\} \cup \{\neg\varphi(x_i, y_j) : j \leq i < \omega\}.$$

If $\Delta_0 \subseteq \Delta$ is finite then let $n$ be maximal such that $x_n$ or $y_n$ occurs as a variable in $\Delta_0$. For $i \leq n$, interpret $x_i$ as $d_{n-i}$ and $y_j$ as $a_{n-j}$, which satisfies $\Delta_0$. Therefore $\Delta$ is satisfied by compactness and so $\varphi(x, y)$ has OP.

Suppose $\varphi(x, y)$ has OP. By compactness we may assume OP is witnessed by $(a_q)_{q\in\mathbb{Q}}$ and $(b_q)_{q\in\mathbb{Q}}$. Note that for all $q < r$ we have $\models \varphi(a_q, b_r) \wedge \neg\varphi(a_q, b_q)$, so if $A = \{b_q : q < \omega\}$ then $A$ is countably infinite. Given $t \in \mathbb{R}\backslash\mathbb{Q}$, define the $\varphi$-type $p_t = \{\varphi(x, b_q) : q > t\} \cup \{\neg\varphi(x, b_q) : q < t\}$. By assumption and compactness, each $p_t$ is consistent. If $s < t$ are irrational and $q \in \mathbb{Q}$ with $s < q < t$ then $\varphi(x, b_q) \in p_s$ and $\neg\varphi(x, b_q) \in p_t$. Therefore $|S_\varphi(A)| > |A|$ and so $\varphi(x, y)$ is unstable. □

**Theorem 4.3.** *A formula $\varphi(x,y)$ is unstable if and only if $\theta(y,x) := \varphi(x,y)$ has* IP *or, for some $n<\omega$ and $\eta \in 2^n$*

$$\psi_\eta(x, y_0, \ldots, y_{n-1}) := \bigwedge_{\eta(i)=1} \varphi(x, y_i) \ \wedge \bigwedge_{\eta(i)=0} \neg\varphi(x, y_i)$$

*has* sOP.

*Proof.* [4, II] First, if $\varphi(x,y)$ has IP then it is unstable by Proposition 2.2 and Lemma 4.2. On the other hand suppose there is some $n<\omega$ and $\eta \in 2^n$ such that $\psi_\eta(x,\bar{y})$ has sOP, witnessed by $(a_i)_{i<\omega}$. If $b_i$ is such that $\models \neg\psi_\eta(b_i,a_i) \wedge \psi_\eta(b_i,a_{i+1})$, then $\models \psi_\eta(b_i,a_j)$ if and only if $i<j$, so $\psi_\eta(x,y)$ is unstable by Lemma 4.2. Let $A$ be infinite such that $|S_{\psi_\eta}(A)| > |A|$. Given $p \in S_{\psi_\eta}(A)$, let $a_p \models p$ and define

$$\hat{p} = \{\varphi(x,a) : a \in A,\ \models \varphi(a_p,a)\} \cup \{\neg\varphi(x,a) : a \in A,\ \models \neg\varphi(a_p,a)\}.$$

Clearly, each $\hat{p}$ is a consistent $\varphi$-type. Furthermore, if $p,q \in S_{\psi_n}(A)$ and $\hat{p} = \hat{q}$, then $p=q$. Therefore $|S_\varphi(A)| \geq |S_{\psi_n}(A)| > |A|$, and so $\varphi(x,y)$ is unstable.

Conversely, suppose $\varphi(x,y)$ is unstable. By Lemma 4.2, there are $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ witnessing that $\varphi(x,y)$ has OP. By replacing $(a_i,b_i)_{i<\omega}$ with a realization of $EM((a_i,b_i)_{i<\omega})$, we may assume $(a_i,b_i)_{i<\omega}$ is indiscernible. Suppose that for all $n<\omega$ and $\mu \in 2^n$ we have

$$\models \exists x \left( \bigwedge_{\mu(i)=1} \varphi(x,b_i) \ \wedge \bigwedge_{\mu(i)=0} \neg\varphi(x,b_i) \right).$$

Then for any $\sigma \in 2^\omega$, we have a solution $c_\sigma$ to $\{\varphi(x,b_i) : \sigma(i)=0\} \cup \{\neg\varphi(x,b_i) : \sigma(i)=1\}$ by compactness. Setting $\theta(y,x) = \varphi(x,y)$, it follows that $\theta(y,x)$, with $(b_i)_{i<\omega}$ and $(c_\eta)_{\eta\in 2^n}$, witnesses IP. Therefore we may assume that there is some $n<\omega$ and $\mu \in 2^n$ such that

$$\models \neg\exists x \left( \bigwedge_{\mu(i)=1} \varphi(x,b_i) \ \wedge \bigwedge_{\mu(i)=0} \neg\varphi(x,b_i) \right).$$

Let $X_0 = \{i : \mu(i) = 1\}$ and set $m = |X_0|$. Note that $0<m<n$. For some $N<\omega$, we construct sets $X_0, \ldots, X_N$ satisfying the following properties:

- $(i)$ $X_N = \{n-m, n-m+1, \ldots, n-1\}$;
- $(ii)$ for all $k \leq N$, $|X_k| = m$ and $X_k \subseteq \{0,\ldots,n-1\}$;
- $(iii)$ for all $k<N$ there is some $l \in X_k$ such that $X_{k+1} = (X_k \backslash \{l\}) \cup \{l+1\}$ (note that altogether this implies $l \in X_k \backslash X_{k+1}$ and $l+1 \in X_{k+1} \backslash X_k$).

This can be done in the following way. Let $X_0 = \{l_1,\ldots,l_m\}$ with $l_1 < \ldots < l_m$. Then $l_i \leq n-1+m-i$ for all $i$. The next set in the sequence is obtained from the current one by choosing $i$ maximal with $l_i < n-1+m-i$ and replacing $l_i$ with $l_i+1$. Eventually we find $l_i = n-1+m-i$ for all $i$.

We have

$$\models \neg\exists x\left(\bigwedge_{i\in X_0}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_0,\ i<n}\neg\varphi(x,b_i)\right)\ \text{ and }\ \models \exists x\left(\bigwedge_{i\in X_N}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_N, i<n}\neg\varphi(x,b_i)\right),$$

where the second statement is witnessed with $x = a_{n-m-1}$. Therefore there is some $k < N$ such that

$$\models \neg\exists x\left(\bigwedge_{i\in X_k}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_k,\ i<n}\neg\varphi(x,b_i)\right)\ \text{ and }\ \models \exists x\left(\bigwedge_{i\in X_{k+1}}\varphi(x,b_i)\ \wedge \bigwedge_{i\notin X_{k+1}, i<n}\neg\varphi(x,b_i)\right),$$

Let $l \in X_k$ be such that $X_{k+1} = (X_k\backslash\{l\}) \cup \{l + 1\}$. Set

$$\psi(x, y, y_0, \dots, y_{l-1}, y_{l+2}, \dots, y_{n-1}) := \varphi(x, y)\ \wedge \bigwedge_{i\in X_k\backslash\{l\}}\varphi(x,y_i)\ \wedge \bigwedge_{i\notin X_{k+1}\cup\{l\},\ i<n}\neg\varphi(x,y_i).$$

For $r < \omega$, let $\bar{b}_r = (b_0, \dots, b_{l-1}, b_{l+2+r}, \dots, b_{n-1+r})$. Then we have $\models \exists x(\psi(x, b_{l+1}, \bar{b}_0) \wedge \neg\varphi(x, b_l))$. Fixing $r < \omega$, for all $i, j < \omega$ with $l \leq i < j < l + 2 + r$, we have by indiscernibility

$$\models \exists x(\psi(x, b_j, \bar{b}_r) \wedge \neg\varphi(x, b_i)).$$

But $\models \neg\exists x(\psi(x, b_l, \bar{b}_0) \wedge \neg\varphi(x, b_{l+1}))$ so, similarly, for $r < \omega$ and $l \leq i < j < l + 2 + r$, we have

$$\models \neg\exists x(\psi(x, b_i, \bar{b}_r) \wedge \neg\varphi(x, b_j)).$$

It follows that for all $r < \omega$ and $l \leq i < j < l + 2 + r$,

$$\models \exists x(\psi(x, b_j, \bar{b}_r) \wedge \neg\psi(x, b_i, \bar{b}_r))\ \text{ and }\ \models \neg\exists x(\psi(x, b_i, \bar{b}_r) \wedge \neg\psi(x, b_j, \bar{b}_r)).$$

For $r < \omega$ and $i < r$, let $\bar{a}^r_i = (b_{l+i}, \bar{b}_r)$. Then for all $r < \omega$ we have

$$\models \exists x(\neg(\psi(x, \bar{a}^r_i) \wedge \varphi(x, \bar{a}^r_j) \quad \Leftrightarrow \quad i < j.$$

By compactness, $\psi(x, y)$ has sOP. Clearly, $\psi$ is of the desired form $\psi_\eta$, for some $\eta \in 2^{<\omega}$. □

**Corollary 4.4.** OP $\Leftrightarrow$ (IP *or* sOP). □

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