Applications of Pseudofinite Model Theory

Lecture 3 (29 April 2020)

G, $\Gamma \trianglelefteq G$, G/Γ

Theorem 3.1 Any definable compactification of a pseudofinite group has an abelian connected component.

Tools

Theorem 3.2 (Peter-Weyl)

a) Any compact Hausdorff group is a projective limit of compact Lie groups

b) Any compact Lie group is a closed subgroup of $GL_n(\mathbb{C})$.

Theorem 3.3 (Jordan)

For any $n \geq 1$ $\exists$ $d \geq 1$ st any finite subgroup of $GL_n(\mathbb{C})$ contains an abelian subgroup of index $\leq d$.

Def 3.4

Let G be a group and suppose C is a compact group with a bi-invariant metric d. Then $f : G \to C$ is an ε-approximate homomorphism if $\forall$ $x, y \in G$

$$d(f(xy), f(x)f(y)) < \varepsilon.$$

Theorem 3.5 (Turing)

Let C be a compact Lie group with a bi-invariant metric d. Then $\exists$ $\delta = \delta(C, d)$ st if $\varepsilon \leq \delta$ and $f : G \to C$ is an ε-approx. hom, where G is a finite group, then $\exists$ a homomorphism $\tau : G \to C$ st $\forall$ $x \in G$,

$$d(f(x), \tau(x)) < 2\varepsilon.$$

Exercise 9 Let C be a compact Lie group with bi-inv. metric d. Given $n \geq 1$, let d_n be the product metric on C^n. Then $\delta(C^n, d_n) = \delta(C, d)$.

Fix a pseudofinite group G.

Lemma 3.6 Suppose $\tau: G \to C$ is a definable compactification, where C is a compact group with a bi-inv. metric d. Then $\forall\, \varepsilon > 0\ \exists$ a definable ε-approximate hom. $f: G \to C$ st $f(G)$ a finite $\varepsilon/2$-net in C and $\forall\, x \in G$, $d(f(x), \tau(x)) < \varepsilon/3$.

Proof

Given $\lambda \in C$, let $K_\lambda = B_{\leq \varepsilon/4}(\lambda)$ and $U_\lambda = B_{< \varepsilon/3}(\lambda)$. Let $\Lambda \subseteq C$ be a finite $\varepsilon/4$-net in C. Say $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$. $\forall\, 1 \leq i \leq n\ \exists$ a definable set $X_i \subseteq G$ st $\tau^{-1}(K_{\lambda_i}) \subseteq X_i \subseteq \tau^{-1}(U_{\lambda_i})$.

Note $G = X_1 \cup \cdots \cup X_n$. Define $f: G \to \Lambda$ st $f(x) = \lambda_i$ where i is minimal st $x \in X_i$. Then f is definable since $f(G)$ is finite and all fibers are definable.

$\forall\, x \in G$, if $f(x) = \lambda_i$ then $x \in X_i \subseteq \tau^{-1}(U_{\lambda_i})$ so $d(\tau(x), f(x)) < \frac{\varepsilon}{3}$

Since $\tau(G)$ is dense, it follows that $f(G)$ is an $\varepsilon/2$-net.

$\left[\lambda \in C,\ \exists\, x \in G \quad d(\lambda, \tau(x)) < \varepsilon/6 \quad d(\tau(x), f(x)) < \varepsilon/3\right]$

Given $x, y \in G$,

$$d(f(xy), f(x)f(y)) \leq d(f(xy), \tau(xy)) + d(\tau(x)\tau(y), f(x)\tau(y)) + d(f(x)\tau(y), f(x)f(y))$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$\square$

<u>Remark 3.7</u> If $x \in G$, $\tau(x) \in B_{\le \epsilon/4}(\lambda_1)$ then $x \in X_1$ and so $f(x) = \lambda_1$.

<u>Proof of Thm 3.1</u>

Let $\tau : G \to C$ be a definable compactification, where C is a compact group.

WLOG (Thm 3.2 (a)) we can assume C is a compact Lie group

$$\begin{tikzcd} G \arrow[r, "\cdot"] \arrow[rr, bend right] & C = \varprojlim L_i \arrow[r, "\cdot"] & L_i \end{tikzcd} \qquad C^\circ = \varprojlim L_i^\circ .$$

<u>Goal</u>: C has an abelian subgroup of finite index.

Fix a bi-inv. metric d on C. Let $I = \{ n \in \mathbb{Z}^+ : \frac{1}{n} \le \delta(C,d) \}$. Thm 3.5

Fix $n \in I$. Apply Lemma 3.6 to obtain a def. $2/5n$-approx. homomorphism

$f : G \to C$ st $f(G)$ is a finite $1/5n$-net in C

Note f has definable fibers.

So: $G \models$ "There is a $2/5n$-approx. hom with image $f(G)$."

<u>Formally</u>: Let $f(G) = \{\lambda_1, \dots, \lambda_m\}$. Let $\theta_1(x,\bar{y}), \dots, \theta_m(x,\bar{y})$ be $\mathcal{L}$-formulas st $f^{-1}(\lambda_i) = \theta_i(G,\bar{a})$ for some $\bar{a}$

For $i,j \le m$, set $S_{i,j} = \{ k \le m : d(\lambda_i \lambda_j, \lambda_k) < 2/5n \}$. Then $G \models$

$$\exists \bar{y} \Big(\forall x \bigvee_{i=1}^{m} \theta_i(x,\bar{y}) \wedge \bigwedge_{i \ne j} \neg \exists x \big(\theta_i(x,\bar{y}) \wedge \theta_j(x,\bar{y})\big) \wedge \bigwedge_{i=1}^{m} \exists x\, \theta_i(x,\bar{y}) \wedge$$

$$\bigwedge_{i,j \le m} \forall u \forall v \Big(\big(\theta_i(u,\bar{y}) \wedge \theta_j(v,\bar{y})\big) \to \bigvee_{k \in S_{i,j}} \theta_k(u \cdot v, \bar{y}) \Big).$$

So $\exists$ a finite group G_n, which yields a $2/s_n$-approx. hom.

$f_n : G_n \to C$ st $f_n(G_n)$ $(= f(G))$ is a $1/s_n$-net.

By Thm 3.5, $\exists$ a hom. $\tau_n : G_n \to C$ st $H_n = \tau(G_n)$ is a $1/n$-net in C.

[Fix $\lambda \in C$, $\exists$ $x \in G_n$ st $d(f_n(x), \lambda) < 1/s_n$. Also $d(\tau_n(x), f_n(x)) < 4/s_n$]

$\forall$ $n \in \mathbb{Z}$, we have $\tau_n : G_n \to C$ st $H_n = \tau(G_n)$ is a $1/n$-net in C.