# Applications of Pseudofinite Model Theory

## Lecture 7 (8 May 2020)

**Setting**: $G$ is a sufficiently saturated pseudofinite group.

$A \subseteq G$ is definable + NIP (ie., $x \in yA$ is NIP)

$\mathcal{B}$ is the Boolean algebra generated by $\{gAh : g,h \in G\}$.

$$G_A^{00} = \bigcap_{g \in G} g\,\mathrm{Stab}^\mu(A) g^{-1} = \bigcap_{g \in G} \mathrm{Stab}^\mu(gA) \quad \left[\mu(g^{-1}xgA \triangle A) = \mu(xgA \triangle gA)\right]$$

**Goal**: $A$ is approximated by cosets of $G_A^{00}$

**Recall**: If $X \in \mathcal{B}$ then $X$ is NIP (Exc 20), so by Thm 5.7

$X$ is left generic iff $X$ is right generic iff $\mu(X) > 0$ ($X$ is generic)

**Def 7.1**: A $\mathcal{B}$-type-def set $X \subseteq G$ is wide if any $Y \in \mathcal{B}$ st $Y \supseteq X$ is generic.

**Remark 7.2** 1) $X \in \mathcal{B}$ is generic iff wide

2) $\mathcal{B}$-type-def. $X \subseteq G$ is wide iff $X = \bigcap_{i \in I} X_i$ for $I$ small and $X_i \in \mathcal{B}$ generic (Exc 5a)

$\{X_i : i \in I\}$ closed under finite intersections

3) If $\Gamma \leq G$ is $\mathcal{B}$-type-def. and bounded index, then $\Gamma$ is wide (Exc 6c)

**Def 7.3**: $\mathbb{E}_A = \{C \in G/G_A^{00} : C \cap A \text{ and } C \setminus A \text{ both wide}\}$

**Theorem 7.4** $\mathbb{E}_A$ is closed and has Haar measure $0$.

**Def 7.5**: Let $S(\mathcal{B})$ denote the set of complete $\mathcal{B}$-types, ie., maximal finitely consistent subsets of $\mathcal{B}$ (ie., ultrafilters over $\mathcal{B}$). [See Section B.2: Stone spaces].

**Exercise 21** $S(\mathcal{B})$ is a totally disconnected compact Hausdorff space with basic clopen sets $[X] = \{p \in S(\mathcal{B}) : X \in p\}$ for $X \in \mathcal{B}$.

**Note**: $G$ acts on $S(\mathcal{B})$: $gp = \{gX : X \in p\}$

Def 7.5 Fix $p \in S(\mathcal{B})$

1) $\mathrm{Stab}(p) = \{g \in G : gp = p\}$ (a subgroup of $G$)

2) $p$ is generic if every $X \in p$ is generic

Let $S^g(\mathcal{B})$ denote the set of generic types.

3) Given a $\mathcal{B}$-type-def set $X \subseteq G$, we write $p \vDash X$ if any $Y \in \mathcal{B}$ containing $X$ is in $p$ (ie., $X = \bigcap_{i \in I} X_i$ for $I$ small + $X_i \in p$).

Proposition 7.6

a) If $X \subseteq G$ is $\mathcal{B}$-type-def, then $X$ is wide iff $\exists\, p \in S^g(\mathcal{B})$ st $p \vDash X$.

b) If $p \in S^g(\mathcal{B})$ then $S^g(\mathcal{B}) = \overline{\{gp : g \in G\}}$.

Proof: a) ($\Leftarrow$) ✓

($\Rightarrow$). Assume $X$ is wide. Write $X = \bigcap_{i \in I} X_i$ where $I$ is small, $X_i \in \mathcal{B}$ is generic, and $\{X_i : i \in I\}$ is closed under finite intersections.

Let $p_0 = \{X_i : i \in I\} \cup \{Y \in \mathcal{B} : \mu(Y) = 1\}$. Then $p_0$ has the finite intersection property since if $i \in I$ and $Y_1, \ldots, Y_n$ with measure 1 then

$$\mu(X_i \cap Y_1 \cap \ldots \cap Y_n) = \mu(X_i) > 0$$

Let $p \in S(\mathcal{B})$ st $p_0 \subseteq p$. Then $p \vDash X$. If $Z \in p$ then $\mu(Z) > 0$ since if $\mu(Z) = 0$ then $G \setminus Z \in p_0$. So $p \in S^g(\mathcal{B})$.

b) Fix $p \in S^g(\mathcal{B})$. Then $gp \in S^g(\mathcal{B})$ $\forall\, g \in G$, so $\overline{\{gp : g \in G\}} \subseteq S^g(\mathcal{B})$

[$S^g(\mathcal{B})$ is closed: If $q \notin S^g(\mathcal{B})$ then $\exists$ non-generic $Y \in q$. So $[Y] \cap S^g(\mathcal{B})$].

Fix $q \in S^g(\mathcal{B})$, and let $U \subseteq S(\mathcal{B})$ be open with $q \in U$. So $\exists\, X \in \mathcal{B}$ st $q \in [X] \subseteq U$. So $X \in q$ and thus $G = FX$ for some finite $F \subseteq G$.

So $FX \in p$ and thus $gX \in p$ for some $g \in F$. So $g^{-1}p \in [X] \subseteq U$. $\square$

Theorem 7.7

a) If $p \in S^g(\mathcal{B})$ then $G_A^{00} = \mathrm{Stab}(p)$.

b) If $\Gamma \leq G$ is $\mathcal{B}$-type-def with bounded index then $G_A^{00} \subseteq \Gamma$.

Proof Fix $p \in S^g(\mathcal{B})$ and $\Gamma$ as in part (b). We show:

$$G_A^{00} \subseteq \mathrm{Stab}(p) \subseteq \Gamma.$$

Then Corollary 6.6 $\Rightarrow$ (a) and Prop 7.6(a) $\Rightarrow$ (b).

$G_A^{00} \subseteq \mathrm{Stab}(p)$ : Fix $x \notin \mathrm{Stab}(p)$. Then $x^{-1}p \neq p$. So $x^{-1}p$ and $p$ disagree on a generator of $\mathcal{B}$, i.e., $\exists\, g, h \in G$ st $xgAh \vartriangle gAh \in p$.

Then $\mu(xgAh \vartriangle gAh) > 0 \Rightarrow \mu(xgA \vartriangle gA) > 0 \Rightarrow x \notin g\,\mathrm{Stab}^{\mu}(A)g^{-1}$
$\Rightarrow x \notin G_A^{00}$.

$\mathrm{Stab}(p) \subseteq \Gamma$ : There is a unique right coset $C$ of $\Gamma$ st $p \models C$.

$$\left[\begin{array}{l} \text{Let } \Gamma = \bigcap_{i \in I} X_i.\ \forall i \in I,\ \exists\, g_i \in G \text{ st } g_i X_i \in p. \text{ So } p \models \bigcap_{i \in I} g_i X_i \\ \text{Pick } a \in \bigcap_{i \in I} g_i X_i.\ \text{Show } \bigcap_{i \in I} g_i X_i \subseteq a\Gamma. \text{ So } p \models a\Gamma. \end{array}\right] (\star)$$

If $x \in \mathrm{Stab}(p)$ then $p = xp \models xC$. So $xC \cap C \neq \emptyset \Rightarrow x \in \Gamma$. $\square$

Def 7.8 Let $\mathbb{G}_A = G/G_A^{00}$, $\pi : G \to \mathbb{G}_A$ quotient map.

Define $\sigma : S(\mathcal{B}) \to \mathbb{G}_A$ st $p \models \sigma(p)$ (well-defined by $(\star)$)

Proposition 7.9 $\sigma$ is continuous.

Proof: Suppose $K \subseteq \mathbb{G}_A$ is closed and $p \notin \sigma^{-1}(K)$. Then $\sigma(p) \subseteq G \setminus \pi^{-1}(K)$.

By Exercise 5a, $\exists\, X \in \mathcal{B}$ st $\sigma(p) \subseteq X \subseteq G \setminus \pi^{-1}(K)$. So $p \models \sigma(p) \subseteq X$.

So $p \in [X]$. Also $[X] \cap \sigma^{-1}(K) = \emptyset$ $\square$.