Applications of Pseudofinite Model Theory

Lecture 9 (13 May 2020)

Setting: G saturated pseudofinite, $A \subseteq G$ def. + NIP, $\mathcal{B} = \langle \{gAh : g,h \in G\} \rangle$

$$\mathbb{G}_A = G/G_A^{00} \qquad \pi : G \to \mathbb{G}_A$$

$$\mathbb{E}_A = \{C \in \mathbb{G}_A : C \cap A, C \setminus A \text{ wide}\} \qquad \sigma : S(\mathcal{B}) \to \mathbb{G}_A.$$

Update: We don't need Lemma 8.10

Lemma 9.1 If $p \in S^g(\mathcal{B})$, $p \models G_A^{00}$, and $X \in \mathcal{B}$ then $\mathbb{D}_X^p$ is pointwise large.

Proof Fix open $U \subseteq \mathbb{G}_A$ st $\exists\, aG_A^{00} \in \mathbb{D}_X^p \cap U$. WTS $\eta(\mathbb{D}_X^p \cap U) > 0$.

Let $K = \sigma^{-1}(\{aG_A^{00}\})$ and $V = \sigma^{-1}(U)$. So K is closed, V open, $K \subseteq V$.

By Exercise 21(d), $\exists\, Y \in \mathcal{B}$ st $K \subseteq [Y] \subseteq V$. We have $X \in ap$. Also $Y \in ap$ since $ap \in K$. So $X \cap Y \in ap$. So $X \cap Y$ is generic.

Now $\eta(\mathbb{D}_X^p \cap \mathbb{D}_Y^p) = \eta(\mathbb{D}_{X\cap Y}^p) = \eta_p(X \cap Y) > 0$ (by Prop 8.9)

ETS $\mathbb{D}_Y^p \subseteq U$. Fix $gG_A^{00} \in \mathbb{D}_Y^p$. Then $Y \in gp$.

So $gG_A^{00} = \sigma(gp) \in \sigma([Y]) = U$. $\square$

Proof of Thm 7.4 ($\mathbb{E}_A$ is closed + $\eta(\mathbb{E}_A) = 0$).

$\mathbb{E}_A$ is closed (8.1). Fix $p \in S^g(\mathcal{B})$ st $p \models G_A^{00}$ (by 7.6(a)). Then $\mathbb{E}_A \subseteq \partial \mathbb{D}_A^p$ (8.3), $\mathbb{D}_A^p$ is NIP (8.6), both $\mathbb{D}_A^p$ + $\mathbb{G}_A \setminus \mathbb{D}_A^p$ are F_σ (8.7), are ptwise large (9.1). Since $\mathbb{G}_A$ is second countable (6.6), we have $\eta(\partial \mathbb{D}_A^p) = 0$ by Thm 8.5. $\square$

Proposition 9.2 If $K \subseteq \mathbb{G}_A$ is closed then

$$\eta(K) = \inf\{\mu(X) : X \in \mathcal{B},\ \pi^{-1}(K) \subseteq X\}.$$

Proof: There is a (unique) left-inv. regular Borel prob. measure $\tilde{\mu}$ on $S(\mathcal{B})$ st $\tilde{\mu}([X]) = \mu(X)$ $\forall X \in \mathcal{B}$ (see Misc. Notes B.1, B.2; Exercise 25).

Given a Borel set $W \subseteq G_A$, set $\nu(W) = \tilde{\mu}(\sigma^{-1}(W))$.

Then ν is a left-inv. regular Borel prob. measure on G_A. So $\nu = \mu$.

If $K \subseteq G_A$ is closed then:

$$\eta(K) = \nu(K) = \tilde{\mu}(\sigma^{-1}(K)) = \inf\{\tilde{\mu}(U) : U \text{ is clopen, } \sigma^{-1}(K) \subseteq U\}$$

(Exc 25(a))

$$= \inf\{\tilde{\mu}([X]) : X \in \mathcal{B},\ \sigma^{-1}(K) \subseteq [X]\}$$

$$= \inf\{\mu(X) : X \in \mathcal{B},\ \pi^{-1}(K) \subseteq X\} \quad \text{(Exc 24(b))} \qquad \square$$

Corollary 9.3 Let $G_A^{00} = \bigcap_{n=0}^{\infty} X_n$ where X_n is definable + $X_{n+1} \subseteq X_n$.
Then $\forall \varepsilon > 0$ $\exists n \geq 0$ and $Z \in \mathcal{B}$ st $\mu(Z) < \varepsilon$ and if $g \in G \setminus Z$ then either $\mu(gX_n \cap A) = 0$ or $\mu(gX_n \setminus A) = 0$.

Proof: Fix $\varepsilon > 0$. By Thm 7.4 + Prop 9.2, $\exists Z \in \mathcal{B}$ st $\pi^{-1}(\mathbb{E}_A) \subseteq Z$ + $\mu(Z) < \varepsilon$

[Aside: Saturation $\Rightarrow$ $\forall g \in G \setminus Z$ $\exists n \geq 0$ st $\mu(gX_n \cap A) = 0$ or $\mu(gX_n \setminus A) = 0$.]

Toward a contradiction, suppose $\forall n \geq 0$ $\exists a_n \in G \setminus Z$ st $\mu(a_n X_n \cap A) > 0$ and $\mu(a_n X_n \setminus A) > 0$. Let $U = \{C \in G_A : C \subseteq Z\}$, which is open by Exc 7d.

Note $\mathbb{E}_A \subseteq U$, $\pi^{-1}(U) \subseteq Z$, and $\forall n \geq 0$, $a_n G_A^{00} \notin U$ since $a_n \notin Z$

Passing to a subsequence, assume $(a_n G_A^{00}) \to aG_A^{00} \in G_A \setminus U \subseteq G_A \setminus \mathbb{E}_A$.

Either $aG_A^{00} \cap A$ is not wide or $aG_A^{00} \setminus A$ is not wide.

By Exercise 5a, $\exists n \geq 0$ st $\mu(aX_n \cap A) = 0$ or $\mu(aX_n \setminus A) = 0$.

By Exercise 6a, $\exists i \geq n$ st $X_i^2 \subseteq X_n$. Define $V = \{C \in G_A : C \subseteq aX_i\}$, which is an open nbhd of aG_A^{00}. So $\exists m \geq i$ st $a_m G_A^{00} \in V$.

Therefore

$$a_m X_m \subseteq a X_i X_m \subseteq a X_i^2 \subseteq a X_n .$$

So $\mu(a_m X_m \cap A) = 0$ or $\mu(a_m X_m \setminus A) = 0$.

This contradicts the choice of a_m.