C. LIST OF TOOLS

UPDATED: 13 MAY

In these notes I will collect the results used during lectures as "black boxes", along with discussion and citations.

Tools from Lecture 3 (29 April)

Theorem 3.1 (Peter-Weyl 1927 [12])**.**

(a) *Any compact Hausdorff group is topologically isomorphic to an inverse limit of compact Lie groups.*

(b) *Any compact Lie group is topologically isomorphic to a closed subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *for some* $n \geq 1$*.*

These two results are consequences of the *Peter-Weyl Theorem(s)*, which are more extensive results in harmonic and functional analysis concerning unitary representations of compact groups. Any textbook on compact groups or Lie groups will likely cover these theorems, and there are also multiple online lecture notes (e.g. Tao has an entry in his blog). A precise reference is [6, Corollary 2.43] for part (a) and [6, Corollary 2.40] for part (b) (via [6, Definition 2.41]).

Theorem 3.2 (Jordan 1878 [9])**.** *For any* $n \geq 1$ *there is some* $d \geq 1$ *such that any finite subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *contains an abelian subgroup of index at most* d*.*

This result is sometimes called *Jordan's Lemma* or the *Jordan-Schur Theorem* (due to a generalization by Schur). Once again, this result is easy to find in textbooks. For further discussion, and an exposition of Jordan's original proof, see Breuillard's online notes [2].

Theorem 3.3 (Turing 1938 [17])**.** *Let* C *be a compact Lie group with a bi-invariant metric* d*. Then there is a real number* $\delta = \delta(C, d)$ *such that, for any* $\epsilon \leq \delta$ *and any* ϵ*-approximate homomorphism* $f\colon G \to C$*, with* G *a finite group, there is a homomorphism* $\tau\colon G \to C$ *satisfying* $d(\tau(x), f(x)) < 2\epsilon$ *for all* $x \in G$*.*

In this case, the attribution of this result might be debatable, since Turing's paper focuses on connected Lie groups and does not formulate ϵ-approximate homomorphisms in this precise way. This exact result is proved by Alekseev, Glebskiĭ, and Gordon in [1, Theorem 5.13] using "elementary" representation theory for compact Lie groups. These authors characterize the result as a modification of a similar theorem of Kazhdan [10]. In general, Section 5 of [1] is a good modern reference for results on finite approximability of compact Lie groups. For example, [1, Theorem 5.11] is a "standard formulation" of the result that if C is a compact Lie group then the following are equivalent:

(i) There is a pseudofinite structure G expanding a group and a definable compactification $\tau\colon G \to C$.

(ii) For all $n > 0$, there is a finite group $H_n \leq C$ which is a $\frac{1}{n}$-net in C.

We proved $(i) \Rightarrow (ii)$ during lecture, and $(ii) \Rightarrow (i)$ almost follows from Exercise 10. In particular, given (ii), we can construct a surjective homomorphism $\tau\colon H \to C$, where $H = \prod_{\mathcal{U}} H_n$ for some non-principal ultrafilter $\mathcal{U}$ on $\mathbb{Z}^+$. While τ may not be definable with respect to the group language on H, it is not too difficult to show that it is definable in the (pseudofinite) expansion of H by all internal subsets.

Tools from Lecture 5 (4 May)

Theorem 5.1 (Vapnik & Chervonenkis 1971 [18]). *Suppose X is a finite set and $\mathcal{S} \subseteq \mathcal{P}(X)$ is such that $\mathrm{VC}(\mathcal{S}) = d$. Then for any $\epsilon > 0$ there is some $n \geq 1$ such that*

$$\left|\left\{\bar{a} \in X^n : \left|\mathrm{Av}_{\bar{a}}(S) - \tfrac{|S|}{|X|}\right| \geq \epsilon \text{ for some } S \in \mathcal{S}\right\}\right| \leq \frac{O(n^d)}{e^{\epsilon^2 n/32}}|X|^n.$$

This result is also called the *VC-Theorem*, and gives a "uniform law of large numbers" for set systems of bounded VC-dimension. It was proved by Vapnik and Chervonenkis for applications to learning theory. The above formulation follows [14, Theorem 6.6], which also gives a detailed proof using basic facts from probability. As discussed in lecture, the idea of the proof is to apply the weak law of large numbers, which says that $\mathrm{Av}_{\bar{a}}(S)$ approximates $|S|/|X|$ for a fixed S. One then obtains uniform control over all $S \in \mathcal{S}$ under the assumption of bounded VC-dimension. In particular, if $\mathrm{VC}(\mathcal{S}) = d$ then the *Sauer-Shelah Lemma*, implies that for any $\bar{a} \in X^n$, at most $O(n^d)$ subsets of $\{a_1, \ldots, a_n\}$ can be cut out by elements of $\mathcal{S}$. The Sauer-Shelah Lemma is implicit in the work of Vapnik and Chervonenkis, but was rediscovered by Sauer, Shelah, and others in various contexts. See [14, Lemma 6.4] for a proof of this result (which is more elementary compared to the VC Theorem).

Tools from Lecture 8 (11 May)

Theorem (Simon 2015 [15]). *Let $\mathcal{M}^*$ be a sufficiently saturated structure in a countable language. Suppose $\phi(\bar{x};\bar{y})$ is an NIP formula and fix a complete ϕ-type $p \in S_\phi(\mathcal{M}^*)$ such that p is M-invariant for some countable $M \prec \mathcal{M}^*$. Then*

$$\{\bar{b} \in (\mathcal{M}^*)^{\bar{y}} : \phi(\bar{x};\bar{b}) \in p\} = \textstyle\bigcup_{n=0}^{\infty} Y_n$$

where each $Y_n \subseteq (\mathcal{M}^*)^{\bar{y}}$ is type-definable over M.

This result is sometimes referred to as "Borel definability of invariant ϕ-types for NIP formulas". It is an NIP analogue of "definability of ϕ-types for stable formulas" (proved by Shelah [13, Theorem II.2.2]), which says that if $\phi(\bar{x};\bar{y})$ is stable and $p \in S_\phi(\mathcal{M}^*)$ then there is some countable $M \prec \mathcal{M}^*$ such that the set $\{\bar{b} \in (\mathcal{M}^*)^{\bar{y}} : \phi(\bar{x};\bar{b}) \in p\}$ is definable over M. (In particular, this implies that p is M-invariant.)

The above theorem was first proved by Hrushovski and Pillay [7, Proposition 2.6] under the stronger assumption that $\mathrm{Th}(\mathcal{M}^*)$ is NIP (i.e., *all* partitioned formulas are NIP). However, some ingredients of the proof are difficult to adapt to the "local setting" of a single formula. Simon's proof uses a result of Bourgain, Fremlin, and Talagrand [4], which characterizes sequential compactness of countable sets in the Banach space $\mathcal{C}(X,\mathbb{R})$ (where X is Polish) using what is essentially a "continuous logic" version of NIP. So the above theorem deepens the connection between (continuous) model theory and functional analysis. Another remarkable example of this connection is Grothendieck's [5] characterization of relatively weakly compact sets in Banach spaces by means of a continuous version of stability, which was used by Ben Yaacov [3] to give a short proof of definability of types for stable formulas.

Theorem 8.5 (Simon 2017 [16])**.** *Suppose $\mathbb{G}$ is a second countable compact Hausdorff group and $W \subseteq \mathbb{G}$ is an NIP set such that W and $\mathbb{G}\backslash W$ are both F_σ and pointwise large. Then ∂W is Haar null.*

The motivation for this result was to fix errors in the proof of what is called *generic compact domination* for groups with "finitely satisfiable generics" definable in NIP theories (see [8, Corollary 5.10]). The notion of "compact domination" was first developed by Hrushovski, Peterzil, and Pillay [8] for their proof of the *Pillay conjectures* for definably compact groups definable in *o*-minimal theories (which are a special case of NIP groups with fsg). The proof of Theorem 8.5 combines topological methods with the VC Theorem (Theorem 5.1).

Theorem 8.11 (Matoušek 2004 [11])**.** *Let X be a finite set and suppose $\mathcal{S} \subseteq \mathcal{P}(X)$ is such that $\mathrm{VC}(\mathcal{S}) = d$. Fix $p \geq q \geq 2^{d+1}$ and suppose that among any p sets in $\mathcal{S}$, there are q with nontrivial intersection (i.e., $\mathcal{S}$ has the (p, q)-**property**). Then there is a set $F \subseteq X$, with $|F| \leq O_{p,q}(1)$, such that $S \cap F \neq \emptyset$ for all $S \in \mathcal{S}$.*

This result is sometimes called *Matoušek's (p, q)-Theorem*, and has origins in discrete geometry, especially related to classical work of Helly on families of convex sets in $\mathbb{R}^d$. One can also view this result as a (qualitative) generalization of the existence of ϵ-nets for set systems of bounded VC-dimension, which is an immediate consequence of the VC Theorem. In particular, suppose X is a finite set and $\mathcal{S} \subseteq \mathcal{P}(X)$ is such that $\mathrm{VC}(\mathcal{S}) = d$. Fix $\epsilon > 0$ and let $\mathcal{S}_\epsilon = \{S \in \mathcal{S} : |S| \geq \epsilon|X|\}$. Then, by the VC Theorem, there is $F \subseteq X$, with $|F| \leq O_{d,\epsilon}(1)$, such that $S \cap F \neq \emptyset$ for all $S \in \mathcal{S}_\epsilon$. On the other hand, one can show that for any $q \geq 1$, there is $p = O_{q,\epsilon}(1)$ such that $\mathcal{S}_\epsilon$ has the (p, q)-property (see Exercise 22). So Theorem 8.11 yields the same conclusion (taking $q = 2^{d+1}$).

Update (13 May): The previous result is used to prove that if G is a pseudofinite expansion of a group, $A \subseteq G$ is definable and NIP, and $\mathcal{B}$ is the Boolean algebra generated by all bi-translates of A, then the restriction of the pseudofinite counting measure to $\mathcal{B}$ is the unique left-invariant finitely additive probability measure on $\mathcal{B}$. It turns out that we don't need this for the main results of the course.

References

[1] M. A. Alekseev, L. Yu. Glebskiĭ, and E. I. Gordon, *On approximations of groups, group actions and Hopf algebras*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **256** (1999), no. Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 3, 224–262, 268.

[2] E. Breuillard, *An exposition of Jordan's original proof of his theorem on finite subgroups of* $\mathrm{GL}_n(\mathbb{C})$.
∗ available: `https://www.imo.universite-paris-saclay.fr/~breuilla/Jordan.pdf`

[3] Itaï Ben Yaacov, *Model theoretic stability and definability of types, after A. Grothendieck*, Bull. Symb. Log. **20** (2014), no. 4, 491–496.

[4] J. Bourgain, D. H. Fremlin, and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. **100** (1978), no. 4, 845–886.

[5] A. Grothendieck, *Critères de compacité dans les espaces fonctionnels généraux*, Amer. J. Math. **74** (1952), 168–186.

[6] Karl H. Hofmann and Sidney A. Morris, *The structure of compact groups*, augmented ed., De Gruyter Studies in Mathematics, vol. 25, Walter de Gruyter & Co., Berlin, 2006, A primer for the student—a handbook for the expert.
∗ available from the Cambridge library via ProQuest

[7] Ehud Hrushovski and Anand Pillay, *On NIP and invariant measures*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 4, 1005–1061.

[8] Ehud Hrushovski, Anand Pillay, and Pierre Simon, *Generically stable and smooth measures in NIP theories*, Trans. Amer. Math. Soc. **365** (2013), no. 5, 2341–2366.

[9] C. Jordan, *Mémoire sur les équations différentielles lineéaires à intégrale algébrique*, J. Reine Agnew. Math. **84** (1878), 89–215.
[10] D. Kazhdan, *On ε-representations*, Israel J. Math. **43** (1982), no. 4, 315–323.
[11] Jiří Matoušek, *Bounded VC-dimension implies a fractional Helly theorem*, Discrete Comput. Geom. **31** (2004), no. 2, 251–255.
[12] F. Peter and H. Weyl, *Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe*, Math. Ann. **97** (1927), no. 1, 737–755.
[13] Saharon Shelah, *Classification theory and the number of nonisomorphic models*, second ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990.
$*$ available: `https://shelah.logic.at/v1/class/`
[14] Pierre Simon, *A guide to NIP theories*, Lecture Notes in Logic, vol. 44, Association for Symbolic Logic, Chicago, IL; Cambridge Scientific Publishers, Cambridge, 2015.
$*$ available: `http://www.normalesup.org/~simon/NIP_guide.pdf`
[15] ______, *Rosenthal compacta and NIP formulas*, Fund. Math. **231** (2015), no. 1, 81–92.
[16] ______, *VC-sets and generic compact domination*, Israel J. Math. **218** (2017), no. 1, 27–41.
[17] A. M. Turing, *Finite approximations to Lie groups*, Ann. of Math. (2) **39** (1938), no. 1, 105–111.
[18] V. N. Vapnik and A. Ja. Červonenkis, *The uniform convergence of frequencies of the appearance of events to their probabilities*, Teor. Verojatnost. i Primenen. **16** (1971), 264–279.