Examples Class 1

4) $\mathcal{L} = \{+, 0\}$. Let T be the theory of nontrivial torsion-free divisible abelian groups.

TF: $\forall x\, (nx = 0 \to x = 0)$ $\quad$ nx is $x + \dots + x$ (n times)

D: $\forall x\, \exists y\, (x = ny)$

<u>Claim</u>: T is κ-categorical $\forall\ \kappa > \aleph_0$

<u>Proof</u>: Models of T "are" vector spaces over $\mathbb{Q}$.

NTTFDAG G, $a \in G$, $\lambda = \frac{m}{n} \in \mathbb{Q}$. Define λa to be the unique solution of $ma = nx$ in G

$\mathcal{M}, \mathcal{N} \models T$ are isomorphic iff $\dim(\mathcal{M}) = \dim(\mathcal{N})$ ☆

If $\mathcal{M} \models T$ is uncountable then $\dim(\mathcal{M}) = |\mathcal{N}|$. $\square$

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7) $\mathcal{N}$, $A \subseteq N$. Build substructure generated by A "by hand"

<u>Remark</u>: If $\mathcal{M} \subseteq \mathcal{N}$ and $A \subseteq M$, then for any $\mathcal{L}$-term $t(x_1, \dots, x_n)$ and $\bar{a} \in A^n$, $t^{\mathcal{M}}(\bar{a}) = \underline{t^{\mathcal{N}}(\bar{a})}$ (see Claim in proof of Thm 2.2)

Define $M = \{ t^{\mathcal{N}}(\bar{a}) : \text{any } \mathcal{L}\text{-term } t, \text{ any } \bar{a} \text{ from } A \}$. Define $\mathcal{M}$

- constant symbol c : $\underline{c^{\mathcal{M}} = c^{\mathcal{N}}} \in M$
- n-ary function symbol f: Given $b_1, \dots, b_n \in M$, define $f^{\mathcal{M}}(b_1, \dots, b_n)$.
  Fix terms $t_1, \dots, t_n$ and tuples $\bar{a}_1, \dots, \bar{a}_n$ from A st $b_i = t_i^{\mathcal{N}}(\bar{a}_i)$
  Let $\underline{f^{\mathcal{M}}(b_1, \dots, b_n) = f^{\mathcal{N}}(t_1^{\mathcal{N}}(\bar{a}_1), \dots, t_n^{\mathcal{N}}(\bar{a}_n))} = t^{\mathcal{N}}(\bar{a}_1, \dots, \bar{a}_n) \in M$
  where t is $f(t_1, \dots, t_n)$.
- n-ary relation symbol R : $\underline{R^{\mathcal{M}} := R^{\mathcal{N}} \cap M^n}$

By construction $\mathcal{M} \subseteq \mathcal{N}$. $A \subseteq M$. It's the smallest by the Remark.

8) Chain $(\mathcal{M}_i)_{i<\alpha}$. Define $\mathcal{N} = \bigcup_{i<\alpha} \mathcal{M}_i$. Define $N = \bigcup_{i<\alpha} M_i$.

- constant c : $\forall\ i<j<\alpha$, $c^{\mathcal{M}_i} = c^{\mathcal{M}_j}$ since $\mathcal{M}_i \subseteq \mathcal{M}_j$.

  Let $c^{\mathcal{N}} = c^{\mathcal{M}_i}$.

- n-ary f: Given $a_1, \ldots, a_n \in N$. Choose $i<\alpha$ st $a_1, \ldots, a_n \in M_i$

  $\forall\ j>i$, $f^{\mathcal{M}_i}(\bar{a}) = f^{\mathcal{M}_j}(\bar{a})$ since $\mathcal{M}_i \subseteq \mathcal{M}_j$.

  Define $f^{\mathcal{N}}(\bar{a}) = f^{\mathcal{M}_i}(\bar{a})$ for some/any i st $a_1, \ldots, a_n \in M_i$.

- n-ary relation R : $\forall\ i<j<\alpha$ $\underline{R^{\mathcal{M}_i}} = \underline{R^{\mathcal{M}_j}} \cap M_i^n$ since $\mathcal{M}_i \subseteq \mathcal{M}_j$.

  Define $R^{\mathcal{N}} = \bigcup_{i<\alpha} R^{\mathcal{M}_i}$. Then $\forall\ i$, $R^{\mathcal{N}} \cap M_i^n = R^{\mathcal{M}_i}$.

a) Clear from construction that $\mathcal{M}_i \subseteq \mathcal{N}$ $\forall$ $i<\alpha$.

b) Assume $(\mathcal{M}_i)_{i<\alpha}$ is elementary. Fix $\varphi(x_1, \ldots, x_n)$ $\mathcal{L}$-formula

☆WTS: $\forall\ i<\alpha$ $\forall\ \bar{a} \in M_i^n$, $\mathcal{M}_i \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(\bar{a})$.

<u>Induction on φ</u> : φ is atomic, use $\mathcal{M}_i \subseteq \mathcal{N}$. $\wedge, \neg$ clear.

Assume the result for $\varphi(x_1, \ldots, x_n, y)$. Consider $\exists y \varphi(\bar{x}, y)$.

Fix $i<\alpha$. Fix $\bar{a} \in M_i^n$. Suppose $\mathcal{M}_i \models \exists y \varphi(\bar{a}, y)$. Then $\exists\ b \in M_i$ st $\mathcal{M}_i \models \varphi(\bar{a}, b)$. Then $\exists\ b \in N$ st $\mathcal{N} \models \varphi(\bar{a}, b)$ by induction.

So $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

Suppose $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$. Fix $b \in N$ st $\mathcal{N} \models \varphi(\bar{a}, b)$. Fix $j \geq i$ st $b \in M_j$. Then $\mathcal{M}_j \models \varphi(\bar{a}, b)$ by Induction. So $\mathcal{M}_j \models \exists y \varphi(\bar{a}, y)$.

Since $\mathcal{M}_i \preceq \mathcal{M}_j$, we have $\mathcal{M}_i \models \exists y \varphi(\bar{a}, y)$. $\square$.

⑨ $\mathcal{M}$, $A \subseteq M$. Want $\mathcal{N} \preceq \mathcal{M}$ st $A \subseteq N$ & $|N| \leq |A| + |\mathcal{L}| + \aleph_0$

Attempt #1 Let $\mathcal{N}$ be the substructure generated by A. $|N| \leq |A| + |\mathcal{L}| + \aleph_0$ (by #7)

Only know $\mathcal{N} \subseteq \mathcal{M}$.

Ex: Let $\mathcal{M}$ be infinite set in $\mathcal{L} = \phi$. Let $A \subset M$ be finite.

Then $\mathcal{N} = A$. $\mathcal{M} \not\equiv \mathcal{N}$.

Attempt #2 Apply DLST to $Th_A(\mathcal{M})$ to get $\mathcal{N} \models Th_A(\mathcal{M})$ with $|N| \leq |A| + |\mathcal{L}| + \aleph_0$. There may be no elementary embedding of $\mathcal{N}$ into $\mathcal{M}$ fixing A.

Ex: $\mathcal{M} = (\mathbb{Q}, +, 0)$ $A = \{0\}$ $\mathcal{N} = (\mathbb{Q}^2, +, 0)$

$\mathcal{N}$ does not embed into $\mathcal{M}$.

Proof of 9: $\mathcal{L}_0 = \mathcal{L}_A$. Expand $\mathcal{M}$ to an $\mathcal{L}_0$-structure.

Given $\mathcal{L}_k$ and expansion of $\mathcal{M}$ to $\mathcal{L}_k$-structure.

Let $\mathcal{L}_{k+1} = \mathcal{L}_k \cup \left\{ c_{\varphi, \bar{a}} : \begin{array}{l} \varphi(x_1, \dots, x_n, y) \text{ is an } \mathcal{L}_k\text{-formula} \\ \bar{a} \in A^n \text{ st } \mathcal{M} \models \exists y \varphi(\bar{a}, y) \end{array} \right\}$

Expand $\mathcal{M}$ to $\mathcal{L}_{k+1}$-structure st $\mathcal{M} \models \varphi(\bar{a}, c^{\mathcal{M}}_{\varphi, \bar{a}})$.

Let $\mathcal{L}^* = \bigcup \mathcal{L}_k$. Let $\mathcal{N}$ be the $\mathcal{L}^*$-substructure of $\mathcal{M}$ generated by A.

$A \subseteq N$. $|N| \underset{\#7}{\leq} |A| + |\mathcal{L}^*| + \aleph_0 \leq |A| + |\mathcal{L}| + \aleph_0$.

Fix $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $b_1, \dots, b_n \in N$

WTS: $\mathcal{M} \models \varphi(\bar{b})$ iff $\mathcal{N} \models \varphi(\bar{b})$.

Induction on φ: atomic is clear since $\mathcal{N} \subseteq \mathcal{M}$. $\wedge, \neg$ easy.

Assume the result for $\varphi(x_1, \dots, x_n, y)$. Consider $\exists y \varphi(\bar{x}, y)$.

Fix $b_1, \dots, b_n \in N$. $\mathcal{N} \models \exists y \varphi(\bar{b}, y) \implies \mathcal{M} \models \exists y \varphi(\bar{b}, y)$ by induction.

Suppose $\mathcal{M} \models \exists y \varphi(\bar{b}, y)$. Choose $\mathcal{L}^*$-terms $t_1, \dots, t_n$ st $b_i = t_i^{\mathcal{M}}(\bar{a}_i)$

for some tuple $\bar{a}_i$ from A. Let $\psi(\bar{v}_1, \dots, \bar{v}_n, y)$ be

$\varphi(t_1(\bar{v}_1), \dots, t_n(\bar{v}_n), y) \leftarrow \mathcal{L}^*$-formula.

$\mathcal{M} \models \exists y\, \psi(\bar{a}_1, \dots, \bar{a}_n, y)$. So if $c = c^{\mathcal{M}}_{\psi, \bar{a}}$, then $\mathcal{M} \models \psi(\bar{a}, c)$

So $\mathcal{M} \models \varphi(\bar{b}, c)$. So $\mathcal{N} \models \varphi(\bar{b}, c)$ (since $c \in N$).

So $\mathcal{N} \models \exists y\, \varphi(\bar{b}, y)$

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(6) $\mathcal{M}, \mathcal{N}$, $M \subseteq N$, $h: M \to N$ inclusion.

WTS: h is an $\mathcal{L}$-embedding (i.e. $\mathcal{M} \subseteq \mathcal{N}$) iff h preserves q.f. formulas.

($\Rightarrow$): By proof of Thm 2.2 (only needed surjectivity for quantifier step)

($\Leftarrow$): · constant symbol c : $\varphi(x)$ be $x = c$

$\mathcal{M} \models \varphi(c^{\mathcal{M}})$ So $\mathcal{N} \models \varphi(c^{\mathcal{M}})$ ie. $c^{\mathcal{M}} = c^{\mathcal{N}}$.

· n-ary f : Let $\varphi(x_1, \dots, x_n, y)$ be $y = f(x_1, \dots, x_n)$

Given $\bar{a} \in M^n$. $\mathcal{M} \models \varphi(\bar{a}, f^{\mathcal{M}}(\bar{a}))$. So $\mathcal{N} \models \varphi(\bar{a}, f^{\mathcal{M}}(\bar{a}))$

ie. $f^{\mathcal{M}}(\bar{a}) = f^{\mathcal{N}}(\bar{a})$.

· n-ary R : Let $\varphi(x_1, \dots, x_n)$ be $R(x_1, \dots, x_n)$

Given $\bar{a} \in M^n$, $\bar{a} \in R^{\mathcal{M}}$ iff $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(\bar{a})$

iff $\bar{a} \in R^{\mathcal{N}}$. So $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^n$.

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(5) $\mathcal{M} = (V, E)$ is a graph which is finitely k-colorable.

$\mathcal{L}^\circ = \{E\}$, $\mathcal{L} = \{E, C_1, \dots, C_k\}$ each C_i is a unary relation symbol.

Let σ say "$C_1, \dots, C_k$ partition the universe & yield a k-coloring" ie.

$$\forall x \bigvee_{i=1}^{k} C_i(x) \wedge \forall x \bigwedge_{i \neq j} \neg(C_i(x) \wedge C_j(x)) \wedge \forall x \forall y \left(E(x,y) \to \bigwedge_{i=1}^{k} \neg(C_i(x) \wedge C_i(y))\right)$$

Let $T = D(\mathcal{M}) \cup \{\text{graph axioms}\} \cup \{\sigma\}$

A model of T is a k-colored graph which has $\mathcal{M}$ as a subgraph.

Fix $\Sigma \subseteq T$ finite. Let A be the set of elements in V whose constant symbols occur in Σ. A is a finite subgraph of $\mathcal{M}$.

Expand A to an $\mathcal{L}_V$-structure satisfying Σ. By Compactness T has a model.

ADDED AFTER CLASS

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① Let $\Delta = \{\sigma_n : n \geq 1\}$ where σ_n says "there are at least n distinct elements."

By assumption, $T \cup \Delta$ is finitely satisfiable.

By Compactness, $T \cup \Delta$ has a model.

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② (i) $\Leftrightarrow$ (ii) by definition.

$\neg$(i) $\Rightarrow$ $\neg$(iii) If T is not complete then there is an $\mathcal{L}$-sentence φ st $T \not\models \varphi$ and $T \not\models \neg\varphi$. So there is $\mathcal{M} \models T$ st $\mathcal{M} \models \neg\varphi$ and $\mathcal{N} \models T$ st $\mathcal{N} \models \varphi$. So $\mathcal{M} \not\equiv \mathcal{N}$.

$\neg$(iii) $\Rightarrow$ $\neg$(i). Suppose $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{M} \not\equiv \mathcal{N}$. Then there is an $\mathcal{L}$-sentence φ st $\mathcal{M} \models \varphi$ and $\mathcal{N} \models \neg\varphi$. So $T \not\models \neg\varphi$ and $T \not\models \varphi$.

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③ If $\mathcal{L} = \emptyset$ then "$\mathcal{L}$-isomorphism" = "bijection."

So if $\mathcal{M}, \mathcal{N}$ are infinite and $|M| = |N|$, then $\mathcal{M} \cong \mathcal{N}$.

By Vaught's Test, the theory of infinite sets is complete.