Examples Class 2

(2) WTS For any $\varphi(\bar{x})$ $\exists$ q.f. $\psi(\bar{x})$ st $T \models \forall \bar{x}\,(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Use induction φ: The assumption provides the quantifier step.

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(6) $Th_A(\mathcal{M}) \cup q$ is consistent. Choose $\mathcal{N} \models Th_A(\mathcal{M}) \cup q$. We have $\bar{a} \in N^n$ realizing q. Let $p = tp^{\mathcal{N}}(\bar{a}/A)$. Then $q \subseteq p$ and $p \in S_n^{\mathcal{N}}(A) = S_n^{\mathcal{M}}(A)$.

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(7) Let $\mathcal{B} = \{[\varphi(\bar{x})] : \varphi \text{ an } \mathcal{L}_A\text{-formula}\}$. Show $\mathcal{B}$ is a base for a topology.

- $\mathcal{B}$ covers $S_n^{\mathcal{M}}(A)$: $[\bar{x} = \bar{x}] = S_n^{\mathcal{M}}(A)$.
- $\forall B_1, B_2 \in \mathcal{B}\ \forall p \in B_1 \cap B_2\ \exists B \in \mathcal{B}$ st $p \in B \subseteq B_1 \cap B_2$.

  Note $[\varphi(\bar{x})] \cap [\psi(\bar{x})] = [\varphi(\bar{x}) \wedge \psi(\bar{x})]$.

Fix $C \subseteq S_n^{\mathcal{M}}(A)$ clopen. $\forall p \in C\ \exists \varphi_p(\bar{x})$ st $p \in [\varphi_p(\bar{x})] \subseteq C$

C is compact. $\{[\varphi_p(\bar{x})]\}_{p \in C}$ open cover. So $\exists p_1, \dots, p_n \in C$ st

$$C = \bigcup_{i=1}^{n} [\varphi_{p_i}(\bar{x})] = \Big[\bigvee_{i=1}^{n} \varphi_{p_i}(\bar{x})\Big].$$

[closed subset of compact space is compact]

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(8) Assume $\mathcal{M}$ realizes all 1-types over $A \subseteq M$, $|A| < \kappa$.

WTS: $\forall n \geq 1$, $\mathcal{M}$ realizes all n-types over $A \subseteq M$, $|A| < \kappa$.

Induction on n. $n=1$ ✓. Fix $p \in S_{n+1}^{\mathcal{M}}(A)$, $|A| < \kappa$.

Let $\mathcal{N} \succeq \mathcal{M}$ and $\bar{b} \in N^{n+1}$ realizing p. Let $q = tp(b_1, \dots, b_n/A) \in S_n^{\mathcal{M}}(A)$

[Note: $\varphi(x_1, \dots, x_n) \in q$ iff $\varphi(x_1, \dots, x_n) \wedge x_{n+1} = x_{n+1} \in p$.] ☆

$\exists a_1, \dots, a_n \in \mathcal{M}$ st $(a_1, \dots, a_n) \models q$.

Let $r = \{\varphi(a_1, \dots, a_n, x_{n+1}) : \varphi(x_1, \dots, x_n, x_{n+1}) \in p\}$. Then $r \in S_1^{\mathcal{M}}(A \cup \{a_1, \dots, a_n\})$

E.g. $r = f\left(tp\left(b_{n+1}/A\cup\{b_1,\dots,b_n\}\right)\right)$ where f is a partial elementary map

witnessing $b_1 \dots b_n \equiv_{\mathcal{L}_A} a_1 \dots a_n$.

Note: $|A \cup \{a_1,\dots,a_n\}| < \kappa$. So let $a_{n+1} \in M$ realize r.

So $(a_1,\dots,a_{n+1}) \models p$.

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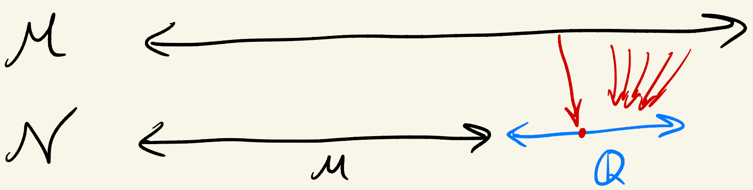
(1) Fix $\kappa > \aleph_0$.

Find $\mathcal{M} \models DLO$ of size κ st every $a \in M$ has uncountably many things above it.

$\mathcal{M} = \mathbb{Q} \cdot \kappa \models DLO$

$\mathcal{N} = \mathcal{M} + \mathbb{Q}$



$I(DLO, \aleph_0) = 1$

$I(DLO, \kappa) = 2^\kappa \quad \kappa > \aleph_0$

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(4) DLO has QE. Fix finite A linear order and $\mathcal{M}, \mathcal{N} \models DLO \cup D(A)$

Say $A = \{a_1 < a_2 < \dots < a_n\}$ ($\mathcal{M}, \mathcal{N}$ agree on order of A)

Let $f_0 : A \to A$ be identity.

Enumerate $M \setminus A = \{b_n : n \geq 0\}$, $N \setminus A = \{c_n : n \geq 0\}$

Extend f_0 to isomorphism $\hat{f}$ from $\mathcal{M} \to \mathcal{N}$ via back and forth.

f is an $\mathcal{L}_A$-isomorphism. So $\mathcal{M} \cong_A \mathcal{N}$. $DLO \cup D(A)$ complete (Vaught).

③ TFDAG

<u>Method #1</u>: Thm 6.2(iii) Fix f.g. $\mathcal{A}$ and $\mathcal{M}, \mathcal{N} \models \mathrm{TFDAG} \cup D(\mathcal{A})$ with $|M| = |N| = \kappa > \aleph_0$. $\mathcal{M}, \mathcal{N}$ are $\mathbb{Q}$-vector spaces of dim κ.

Let V_1, V_2 be the subspaces of $\mathcal{M}$ and $\mathcal{N}$ generated by A.

Then $\dim(V_1) = \dim(V_2) < \aleph_0$. There is isomorphism $f_0 : V_1 \to V_2$ fixing A.

This extends to isomorphism $f : \mathcal{M} \to \mathcal{N}$ since $\dim(\mathcal{M}/V_1) = \dim(\mathcal{N}/V_2) = \kappa$.

ADDED AFTER CLASS

<u>Method #2</u> Theorem 6.2(ii). Fix $\mathcal{M}, \mathcal{N} \models \mathrm{TFDAG}$ and $\mathcal{A} \subseteq \mathcal{M}, \mathcal{N}$. Fix a quantifier-free formula $\varphi(\bar{x}, y)$ and $\bar{a} \in A^n$, $b \in M$ st $\mathcal{M} \models \varphi(\bar{a}, b)$. WTS: $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

WLOG $\varphi(\bar{x}, y)$ is $\bigwedge_{i=1}^{m} \psi_i(\bar{x}, y)$ where each ψ_i is atomic or negated atomic.

So ψ_i is $\alpha_i y + \sum_{t=1}^{n} \beta_{i,t} x_t =_i \sum_{t=1}^{n} \gamma_{i,t} x_t$ where $=_i$ denotes either $=$ or $\neq$, and

and $\alpha_i, \beta_{i,t}, \gamma_{i,t} \in \mathbb{N}$, $\alpha_i \neq 0$. Note: $\mathrm{TFDAG} \cup D(\mathcal{A}) \models \exists! y \left(\alpha_i y + \sum_{i=1}^{n} \beta_{i,t} a_t = \sum_{i=1}^{n} \gamma_{i,t} a_t\right)$.

<u>Case 1</u>: Some $=_i$ is $=$ (WLOG $i=1$). Let $c \in N$ be the unique solution to $\psi_1(\bar{a}, y)$.

For $i > 1$, $\mathcal{N} \models \psi_i(\bar{a}, c)$ iff $\mathcal{N} \models \sum_{t=1}^{n} \frac{\beta_{i,t} - \gamma_{i,t}}{\alpha_i} a_t =_i \sum_{t=1}^{n} \frac{\beta_{1,t} - \gamma_{1,t}}{\alpha_1} a_t$

$$\text{iff } \mathcal{N} \models \underbrace{\sum_{t=1}^{n} (\alpha_1 \beta_{i,t} + \alpha_i \gamma_{1,t}) a_t =_i \sum_{t=1}^{n} (\alpha_i \beta_{1,t} + \alpha_1 \gamma_{i,t}) a_t}_{\text{call this } \sigma_i \text{ (q.f. } \mathcal{L}_A\text{-sentence)}}$$

iff $\mathcal{M} \models \sigma_i$ iff $\mathcal{M} \models \exists y (\psi_1(\bar{a}, y) \wedge \psi_i(\bar{a}, y))$. ✓

<u>Case 2</u>: All $=_i$ are $\neq$. Then $\neg\varphi(\bar{a}, y)$ has $\leq m$ solutions in $\mathcal{N}$.

Since $\mathcal{N}$ is infinite, $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

⑤ $Th(\mathbb{Z},<)$

a) $\theta(x,y)$ is $x<y \wedge \neg\exists z(x<z<y)$ (i.e. "$y=x+1$")

Claim $\theta(x,y)$ is not equivalent (mod $Th(\mathbb{Z},<)$) to a q.f. $\psi(x,y)$

Pf: By inspection of all q.f. formulas in x,y.

OR: Suppose there is such $\psi(x,y)$. Let $\mathcal{M}=(\mathbb{Z},<)$. Let $\mathcal{N}=(2\mathbb{Z},<)$.

$\mathcal{M}\equiv\mathcal{N}$, $\mathcal{N}\subseteq\mathcal{M}$. $\mathcal{N}\models\theta(0,2)\Rightarrow\mathcal{N}\models\psi(0,2)$

$\Rightarrow\mathcal{M}\models\psi(0,2)\Rightarrow\mathcal{M}\models\theta(0,2)$. ↯ //

b) Let $s:\mathbb{Z}\to\mathbb{Z}$ st $s(x)=x+1$. $Th(\mathbb{Z},<,s)=T$

$$T\models\forall x\forall y\,(y=s(x)\leftrightarrow\theta(x,y)).$$

WTS: T has QE. Use Thm 6.2(ii).

Fix $\mathcal{M},\mathcal{N}\models T$. Fix $A\subseteq\mathcal{M}\cap\mathcal{N}$. Fix q.f. $\varphi(\bar{x},y)$ and $\bar{a}\in A^n$, $b\in\mathcal{M}$ [$\bar{x}=(x_1,\dots,x_n)$]

st $\mathcal{M}\models\varphi(\bar{a},b)$. Show $\mathcal{N}\models\exists y\varphi(\bar{a},y)$.

WLOG $\varphi(\bar{x},y)$ is $\bigwedge_{i=1}^{m}\psi_i(\bar{x},y)$ where ψ_i is atomic/negated atomic

WLOG Assume y is used in each ψ_i. So ψ_i has the form

$s^\alpha(y)=s^\beta(x_t)$, $s^\alpha(y)<s^\beta(x_t)$, $s^\alpha(y)>s^\beta(x_t)$, or negation of one of these, where $\alpha,\beta\in\mathbb{N}$ and $1\le t\le n$.

☆ WLOG: s^{-1} is in the language. Note: $u=s^{-1}(v)$ iff $s(u)=v$

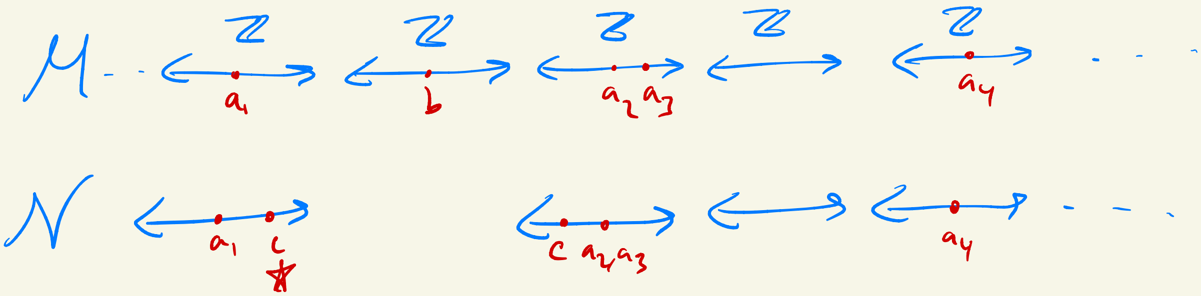
WLOG: $b\notin A$. Otherwise $b\in\mathcal{N}$. $\mathcal{N}\models\varphi(\bar{a},b)$.

So: 1) No ψ_i is $y=s^\alpha(x_t)$

2) No ψ_i,ψ_j are $y<s^\alpha(x_t)$ and $y>s^\beta(x_t)$

Note: $T\models\forall x\forall y\left(\left(y<s^\alpha(x)\wedge y>s^\beta(x)\right)\to\bigwedge_{\alpha<\gamma<\beta}y=s^\gamma(x_t)\right)$

Models:



There is a partition $\{1,\dots,n\} = I \cup J$ and $\alpha > 0$, $\beta < 0$

st $\overset{\text{wlog}}{\ }$ $\varphi(\bar{x}, y)$ says $\displaystyle\bigwedge_{i \in I} y > s^\alpha(x_i) \wedge \bigwedge_{i \in J} y < s^\beta(x_i)$

If $I \neq \emptyset$ take $c = s\left(\max\{s^\alpha(a_i) : i \in I\}\right) \in A$

If $I = \emptyset$ take $c = s^{-1}\left(\min\{s^\beta(a_i) : 1 \le i \le n\}\right) \in A$.

So $\mathcal{N} \models \varphi(\bar{a}, c)$.

Fact: Let T be an $\mathcal{L}$-theory. TFAE.

1) T has QE.

2) Thm 6.2(ii) but with $\mathcal{M}, \mathcal{N}$ $|\mathcal{L}|^+$-saturated. $(|\mathcal{L}| + \aleph_0)^+$

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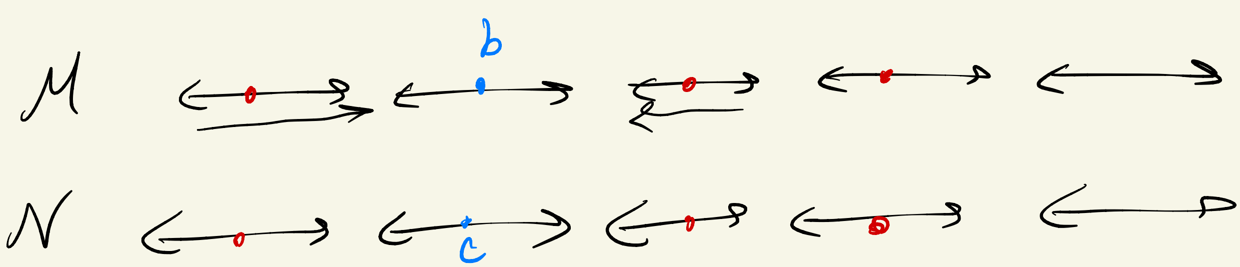
QE test involving saturated models.



$\aleph_1$-saturated

Notation: $\kappa^+ = \min\{\text{cardinals } \lambda : \lambda > \kappa\}$

$(\aleph_0)^+ = \aleph_1$.