Examples Class 3

1) T complete, ctble language.

a) $\mathcal{M} \models T$ is atomic. WTS $\mathcal{M}$ is $\aleph_0$-hom.

Fix $\bar{a}, \bar{b} \in M^n$ with $\bar{a} \equiv_\emptyset \bar{b}$. Fix $c \in M$. $tp(\bar{a}, c)$ is isolated by some $\varphi(\bar{y}, x)$. $\mathcal{M} \models \exists x \varphi(\bar{a}, x)$. So $\mathcal{M} \models \exists x \varphi(\bar{b}, x)$. Choose $d \in M$ st $\mathcal{M} \models \varphi(\bar{b}, d)$. Claim $(\bar{a}, c) \equiv_\emptyset (\bar{b}, d)$

$\varphi(\bar{y}, x) \in tp(\bar{b}, d)$ ie $tp(\bar{b}, d) \in [\varphi(\bar{y}, x)] = \{tp(\bar{a}, c)\}$.

b) $\mathcal{M}, \mathcal{N} \models T$ ctble, $\aleph_0$-hom., realize the same types over $\emptyset$.

$M = \{a_n : n \geq 1\}$, $N = \{b_n : n \geq 1\}$. $f_0 = \emptyset$ is partial elementary since $\mathcal{M} \equiv \mathcal{N}$.

Suppose we have f_n. Add a_{n+1} and b_{n+1}. Enumerate $\mathrm{dom}(f_n) = \bar{c}$ and $\mathrm{Im}(f_n) = \bar{d}$ (so $\bar{c} \equiv_\emptyset \bar{d}$).

$\left[(\bar{c}, a_{n+1}) \equiv_\emptyset (\bar{d}, b) \text{ want for some } b \in N \right]$.

$tp(\bar{c}, a_{n+1})$ is realized in $\mathcal{M}$. So $\exists$ $\bar{d}', b'$ in N st

$$(\bar{c}, a_{n+1}) \equiv_\emptyset (\bar{d}', b')$$

We know $\bar{d}' \equiv_\emptyset \bar{c} \equiv_\emptyset \bar{d}$. So $\bar{d}' \equiv_\emptyset \bar{d}$

Since $\mathcal{N}$ is $\aleph_0$-hom. $\exists$ $b \in N$ st $(\bar{d}', b') \equiv_\emptyset (\bar{d}, b)$

So $(\bar{c}, a_{n+1}) \equiv_\emptyset (\bar{d}, b)$.

Similarly find $a \in M$, st $(\underline{\bar{c}, a_{n+1}}, a) \equiv_\emptyset (\underline{\bar{d}, b}, b_{n+1})$

Let $f_{n+1} = f_n \cup \{(a_{n+1}, b), (a, b_{n+1})\}$.

<u>Ex</u>: Suppose $1 < I(T, \aleph_0) < 2^{\aleph_0}$. Then T has a prime model $\mathcal{M}$ and a ctble saturated model $\mathcal{N}$. $\mathcal{M} \not\cong \mathcal{N}$. Both are $\aleph_0$-hom.

c) $\mathcal{M}, \mathcal{N} \models T$ are prime. $\mathcal{M}, \mathcal{N}$ are ctble + atomic.

$\mathcal{M}, \mathcal{N}$ are $\aleph_0$-hom. by (a). $\mathcal{M}, \mathcal{N}$ realize the same types. So $\mathcal{M} \cong \mathcal{N}$ by (b).

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4) $\mathcal{M}, \mathcal{N}$ saturated, $\mathcal{M} \equiv \mathcal{N}$, $|M| = |N| = \kappa$. Then $\mathcal{M} \cong \mathcal{N}$.

Pf: Enumerate $M = \{a_i\}_{i<\kappa}$ $N = \{b_i\}_{i<\kappa}$. ← ordinal

Build $f_0 \subseteq f_1 \subseteq \dots \subseteq f_i \subseteq \dots$ for $i < \kappa$. $|\mathrm{dom}(f_i)| < \kappa$.

$f_0 = \emptyset$. $\boxed{\alpha < \kappa}$ - limit: $f_\alpha = \bigcup_{i<\alpha} f_i$. $|\mathrm{dom}(f_\alpha)| < \kappa$.

$|\mathrm{dom}(f_i)| = \begin{cases} < \aleph_0 & i < \aleph_0 \\ |i| & i \geq \aleph_0 \end{cases}$

Given f_i, let $p = tp(a_i / \mathrm{dom}(f_i)) \in S_1^{\mathcal{M}}(\mathrm{dom}(f_i))$.

let $q = f_i(p) \in S_1^{\mathcal{N}}(\mathrm{Im}(f_i))$.

Let $b \in N$ realize q. Then $f_i \cup \{(a_i, b)\}$ is partial elementary.

Go back to find $a \in M$ st $\underbrace{f_i \cup \{(a_i, b), (a, b_i)\}}_{f_{i+1}}$ is partial elementary.

Let $f = \bigcup_{i<\kappa} f_i$. Then $f: \mathcal{M} \to \mathcal{N}$ is an isomorphism

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3) $\mathcal{M}$ is κ-saturated. $A \subseteq M$, $|A| < \kappa$. Then $\mathcal{M}$ is κ-saturated as an $\mathcal{L}_A$-structure.

$B \subseteq M$, $|B| < \kappa$. $p \in S_1^{\mathrm{Th}_A(\mathcal{M})}(B)$

$\Downarrow$

$\hat{p} \in S_1^{\mathrm{Th}(\mathcal{M})}(A \cup B)$ $|A \cup B| < \kappa$.

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2) $k \geq 1$. $\mathcal{L} = \{<, U_1, \dots, U_k, c_0, c_1, c_2, \dots\}$

T says DLO + "$U_1, \dots, U_k$ partition into dense sets" + "$c_0 < c_1 < c_2 < \dots$"
+ "$c_n \in U_1 \ \forall n$"

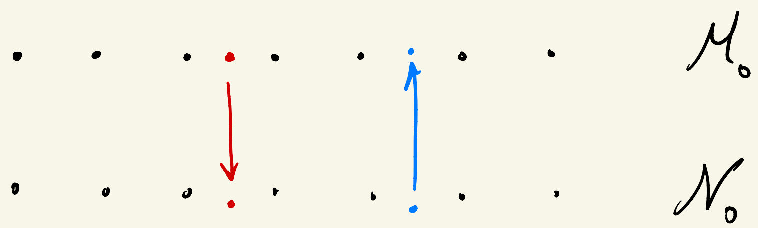
<u>T is complete</u>: Fix $\mathcal{M}, \mathcal{N} \models T$ ctble. WTS $\mathcal{M} \equiv \mathcal{N}$.

Fix $n \geq 1$. Let $\mathcal{M}_0, \mathcal{N}_0$ be reducts to $\{<, U_1, \dots, U_k, c_0, c_1, \dots, c_n\}$

We show $\mathcal{M}_0 \equiv \mathcal{N}_0$. In fact, $\mathcal{M}_0 \cong \mathcal{N}_0$.

Start with $\bar{c}^{\mathcal{M}_0} \to \bar{c}^{\mathcal{N}_0}$.

Back + Forth



<u>$I(T, \aleph_0) = k+2$.</u>

<u>Pf</u>: For $1 \leq t \leq k$, we have $\mathcal{M}_t$ st $\sup c_n$ exists + satisfies U_t.

$\mathcal{M}_{k+1}$: (c_n) is bounded above, no sup in model

$\mathcal{M}_{k+2}$: (c_n) is unbounded.

For any $\mathcal{M} \models T$ ctble $\exists\ 1 \leq t \leq k+2$ st $\mathcal{M} \cong \mathcal{M}_t$.

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5. T, $\kappa \geq |\mathcal{L}| + \aleph_0$.

Assume T is κ-stable. ($|S_1(M)| = \kappa$ for any $\mathcal{M} \models T$, $|M| = \kappa$).

Fix $n \geq 1$, $\mathcal{M} \models T$, $A \subseteq M$, $|A| \leq \kappa$. WTS $|S_n(A)| \leq \kappa$.

<u>Note</u>: $\exists\ \mathcal{N} \preceq \mathcal{M}$ st $A \subseteq N$ + $|N| \leq \kappa$. $\underline{|S_n(A)| \leq |S_n(N)|}$

WLOG $A = M$, $|M| = \kappa$.

<u>Claim</u>: $\forall\ \mathcal{M} \models T$, $|M| = \kappa$, we have $|S_n(M)| = \kappa$.

<u>Pf</u>: Induction on n. $n=1$ ✓ $|S_1(M)| = \kappa$

Fix $\underline{N \succeq M}$ st $|N| = \kappa$ and N realizes all types in $S_1(M)$.

Given $p \in S_{n+1}(M)$, let $p_0 = \{ \varphi(x_{n+1}) : \varphi(x_{n+1}) \wedge \bar{x} = \bar{x} \in p \}$

So $p_0 \in S_1(M)$. Choose $a_p \in N$ st $a_p \models p$.

Let $q_p \in S_n(N)$ be any extension of $p(x_1, \dots, x_n, a_p)$

Show $p \mapsto (a_p, q_p)$ is injective. So $|S_{n+1}(M)| \leq |N \times S_n(N)| \leq \kappa$.

Suppose $(a_{p_1}, q_{p_1}) = (a_{p_2}, q_{p_2})$

$$\begin{aligned} \varphi(x_1, \dots, x_{n+1}) \in p_1 \quad &\text{iff} \quad \varphi(x_1, \dots, x_n, a_{p_1}) \in q_{p_1} \\ &\text{iff} \quad \varphi(x_1, \dots, x_n, a_{p_2}) \in q_{p_2} \\ &\text{iff} \quad \varphi(x_1, \dots, x_{n+1}) \in p_2. \end{aligned}$$

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⑥ Compactness.

$$\Gamma(\bar{x}_n, \bar{y}_n)_{n \geq 1} = \{ \varphi(\bar{x}_i, \bar{y}_j) : i \leq j \} \cup \{ \neg\varphi(\bar{x}_i, \bar{y}_j) : i > j \}.$$

$\Gamma \cup T$ is fin. satisfiable.

So $\exists$ $M \models T$ and $(\bar{a}_n)_{n \geq 1}$ $(\bar{b}_n)_{n \geq 1}$ from M

st $M \models \varphi(\bar{a}_i, \bar{b}_j)$ iff $i \leq j$.

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(7) $p \in S_n(M)$ $\quad \varphi(\bar{x}, \bar{y})$, $\psi(\bar{x}, \bar{z})$

If $\chi(\bar{y})$ is a φ-def. for p and $\Theta(\bar{z})$ is a ψ-def. for p.

Then $\neg\chi(\bar{y})$ is a $(\neg\varphi)$-def. for p and

$$\chi(\bar{y}) \wedge \Theta(\bar{z}) \text{ is a } (\varphi \wedge \psi)\text{-def. for } p.$$

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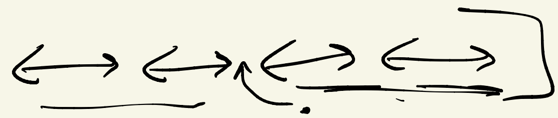
(8) $T = Th(\mathbb{Z}, <)$. $\quad p \in S_n(\mathbb{Z})$ $\quad$ Then p is definable.

$M \succeq \mathbb{Z}$ 

Fix $x_1, \ldots, x_n$. Atomic formulas $\varphi(\bar{x}, \bar{y})$ are

1. $x_i = s^\alpha(y)$ or 2. $x_i < s^\alpha(y)$ $\quad \alpha \in \mathbb{Z}$

[e.g. $x_i > s^\alpha(y) \leftrightarrow \neg(x_i < s^\alpha(y) \vee x_i = s^\alpha(y))$]

Case 1: $x_i = a \in p$ for some $a \in \mathbb{Z}$.

$a = s^\alpha(y)$ def. for 1.

$a < s^\alpha(y)$ def. for 2.

Case 2: $x_i \neq a \in p$ for all $a \in \mathbb{Z}$.

For 1., use $y \neq y$ then $x_i = s^\alpha(b) \in p$ iff $b \neq b$

For 2. case 2a. $x_i > a \in p$ $\forall a \in \mathbb{Z}$. Use $y \neq y$.

case 2b. $x_i < a \in p$ $\forall a \in \mathbb{Z}$. Use $y = y$.
