Examples Class 4

1) QE $\Rightarrow$ this condition by Thm 6.2

Assume the condition. Prove QE via Thm 6.2(ii). Fix $\mathcal{M}, \mathcal{N} \models T$.

Fix $\mathcal{A} \subseteq \mathcal{M} \cap \mathcal{N}$. WLOG $\mathcal{A}$ is finitely generated. Fix qf $\varphi(\bar{x}, y)$

and $\bar{a}$ from A. Assume $\mathcal{M} \models \exists y \varphi(\bar{a}, y)$. By Thm 16.2, $\exists$

$(|\mathcal{L}| + \aleph_0)^+$-saturated extensions $\mathcal{M}' \succeq \mathcal{M}$ and $\mathcal{N}' \succeq \mathcal{N}$.

We have $\mathcal{M}' \models \exists y \varphi(\bar{a}, y)$. By assumption $\mathcal{N}' \models \exists y \varphi(\bar{a}, y)$.

So $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

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2) $T = Th(\mathbb{Z}, <)$. Identify prime model & ctble saturated model.

Recall T has QE if we add a symbol s for the successor function.

<u>Prime model</u>: $(\mathbb{Z}, <)$

<u>Proof</u>: Fix $\mathcal{M} \models T$. Fix $a \in M$. Let $f : \mathbb{Z} \to M$ st $f(n) = s^n(a)$.

Note f is an embedding wrt $<$. Need f to be an elementary embedding.

By QE, it suffices to show that f is an embedding wrt $<$ and s.

This is clear by def. of f.

<u>Ex</u>: If $T = Th(\mathbb{Z}, +)$, then $(\mathbb{Z}, +)$ embeds in any model of T, but T has no prime model

<u>Countable Saturated model</u>: $\dots \longleftrightarrow \longleftrightarrow \longleftrightarrow \dots$ need dense $\mathbb{Z}$-chains.

$(\mathbb{Z} \times \mathbb{Q}, <)$ where $(m, q) < (n, r)$ iff $q = r$ & $m < n$ or $q < r$.

<u>Proof</u>: ctble ✓. Fix finite $A \subseteq M = \mathbb{Z} \times \mathbb{Q}$, and $p \in S_1^M(A)$.

By QE, we may assume p is a complete qf type in the language $\{<, s\}$.

If p contains $x = s^n(a)$ for some $a \in A$, $n \in \mathbb{Z}$, then p is realized in M

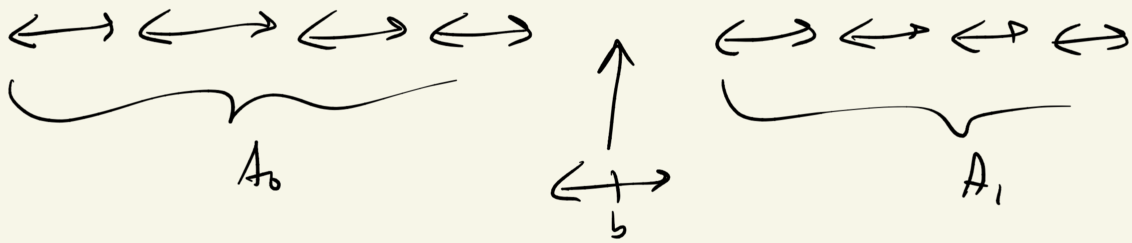
by $s^n(a)$. Assume p contains $x \neq s^n(a)$ $\forall a \in A, n \in \mathbb{Z}$.

Partition $A = A_0 \cup A_1$ st $a \in A_0$ iff p contains $x > s^n(a)$ $\forall n \in \mathbb{Z}$.

So $a \in A_1$ iff p contains $x < s^n(a)$ $\forall n \in \mathbb{Z}$.

If $a \in A_0$ and $b \in A_1$ then $s^n(a) < s^m(b)$ $\forall n, m \in \mathbb{Z}$.

Pick some $b \in M$ st the $\mathbb{Z}$-chain determined by b is between those determined by A_0 and those determined by A_1.

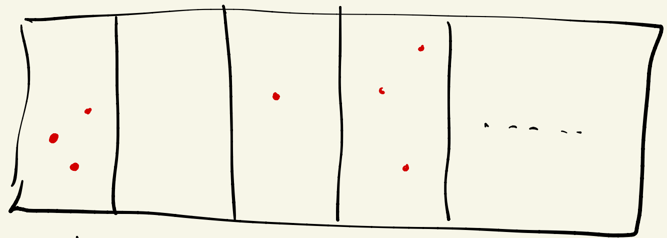


Then $b \models p$.

<u>General principle</u>: If T has QE in a language $\mathcal{L}' \supseteq \mathcal{L}$ and every symbol in $\mathcal{L}'$ is first-order definable using $\mathcal{L}$, then any $p \in S_n^M(A)$ is uniquely determined by its restriction to quantifier-free $\mathcal{L}'$-formulas.

3) $\mathcal{L} = \{E\}$. T says "E is an equivalence relation with infinitely many classes which are all infinite."

<u>ctble models of T</u>:



$\aleph_0$-many classes each of size $\aleph_0$.

So T is $\aleph_0$-categorical, hence complete by Vaught's Test.

<u>Claim</u>: T has QE.

<u>Pf</u>: If A is a finite $\mathcal{L}$-structure, then $T \cup D(A)$ is $\aleph_0$-categorical (assuming it is consistent). Apply Thm 6.2.

<u>Claim</u>: Fix $\kappa \geq \aleph_0$. Then T is κ-stable.

<u>Pf</u>: Fix $\mathcal{M} \models T$. $|M| = \kappa$. Fix $p \in S_1(M)$.

<u>Case 1</u>: p says $x = a$ for some $a \in M$. (κ choices)

<u>Case 2</u>: p says $x \neq a$ $\forall a \in M$, and $E(x,b)$ for some $b \in M$. (at most κ choices)

<u>Case 3</u>: p says $\neg E(x,a)$ $\forall a \in M$. (1 choice)

So $|S_1(M)| = \kappa$.

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4) T complete theory. Show T is stable iff no formula $\varphi(x,\bar{y})$ has OP wrt T (with $|x| = 1$).

<u>Proof</u>: ($\Rightarrow$). ✓. Assume no formula $\varphi(x,\bar{y})$ has OP wrt T.
Fix $\kappa \geq (|\mathcal{L}| + \aleph_0)$ st $\kappa^{|\mathcal{L}|+\aleph_0} = \kappa$. We show T is κ-stable.
Fix $\mathcal{M} \models T$, $|M| = \kappa$. WTS $|S_1(M)| = \kappa$. By FTS3 $\Rightarrow$ FTS2, any $p \in S_1(M)$ is definable wrt any formula $\varphi(x,\bar{y})$. By FTS2 $\Rightarrow$ FTS1, we have $|S_1(M)| \leq \kappa$. $\square$

5) a) $Th(\mathbb{N},+)$. Let $\varphi(x,y)$ be $\exists z(y=x+z)$

$\mathbb{N}\models\varphi(m,n)$ iff $m\leq n$.

b) $Th(\mathbb{N},\cdot)$ Let $\varphi(x,y)$ be $\exists z(y=x\cdot z)$

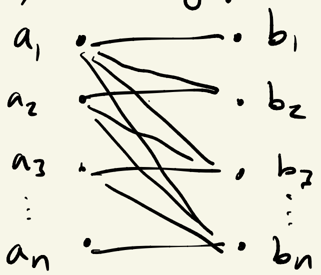
$\mathbb{N}\models\varphi(m,n)$ iff $m \mid n$ (or $m=n=0$)

Let a_i be the i^{th} prime. Let $b_i=a_0 a_1\cdot\ldots\cdot a_i$

Then $\mathbb{N}\models\varphi(a_i,b_j)$ iff $a_i\mid b_j$ iff $i\leq j$.

c) $T=RG$. We show $E(x,y)$ has OP. Fix $\mathcal{M}\models RG$.

Any finite graph embeds in $\mathcal{M}$ as an induced subgraph (Corollary 7.7)



$\mathcal{M}\models E(a_i,b_j)$ iff $i\leq j$.

So $E(x,y)$ has OP wrt T by ES3 #6.

d) $T=Th(\mathbb{Z},+,A)$ $A=\{n^2: n\geq 0\}$.

$$\underbrace{\exists y_1\exists y_2\exists y_3\exists y_4\left(x=y_1+y_2+y_3+y_4\wedge\bigwedge_{i=1}^{4}A(y_i)\right)}_{\theta(x)}$$

Lagrange: $\mathbb{Z}\models\theta(n)$ iff $n\geq 0$

Let $\varphi(x,y)$ be $\theta(y-x)$. Then $\mathbb{Z}\models\varphi(m,n)$ iff $m\leq n$.

Ex: $Th(\mathbb{Z},+,\{2^n: n\geq 0\})$ is stable.

$2^{\mathbb{Z}}$

$\{A\in 2^{\mathbb{Z}}: (\mathbb{Z},+,A)$ is unstable$\}$

contain a co-meager set.

$x+y\in A$ is a RG

6) $T = Th(F, +, \cdot, 0, 1)$. F field. Suppose $\varphi(x,y)$ is q.f. + has OP wrt T.

There is $K \models T$ and $(a_i)_{i<\omega}$ and $(b_j)_{j<\omega}$ from K st

$K \models \varphi(a_i, b_j)$ iff $i \leq j$. Let $\overline{K}$ be alg. closure of K.

Then K is a substructure of $\overline{K}$. Since $\varphi(x,y)$ is q.f.,

$\overline{K} \models \varphi(a_i, b_j)$ iff $i \leq j$. Then $Th(\overline{K})$ is unstable

But ACF_p is stable $\forall$ p. Contradiction.

Recall: It's unknown if $Th(\mathbb{C}(t), +, \cdot, 0, 1)$ is stable ??

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9) G exp. of group. $Th(G)$ stable.

a) Assume G^0 has finite index. Prove G^0 is definable.

Pf: If $H \leq G$ is def. and finite index, then $G^0 \leq H$.

So H is a union of cosets of G^0.

So there are only finitely many such H, say $H_1, \ldots, H_n$.

So $G^0 = \bigcap_{i=1}^{n} H_i$ is definable.

b) Assume G is $(|\mathcal{L}| + \aleph_0)$-saturated and G^0 is definable.

Prove G^0 has finite index.

Proof: Recall $G^0 = \bigcap_{\varphi} G^0(\varphi)$ ($\varphi(x, \bar{y})$ is an $\mathcal{L}$-formula)

G^0_φ is definable, say by $\Theta_\varphi(x)$. (Baldwin-Saxl)

Let $\Theta(x)$ define G^0.

We show that $\exists$ $\varphi_1, \ldots, \varphi_n$ st $G^0 = \bigcap_{i=1}^{n} G^0(\varphi_i)$

If not, then $\{\neg\Theta(x)\} \cup \{\Theta_\varphi(x) : \varphi\}$

is finitely satisfiable in G, and thus realized in G by saturation.

This contradicts $G^0 = \bigcap_\varphi G^0(\varphi)$.

Each $G^0(\varphi)$ has finite index. So $G^0 = \bigcap_{i=1}^{n} G^0(\varphi_i)$ has finite index.

ADDED AFTER CLASS

7). Let $q^* = q \cup \{G \setminus Y : Y \subseteq G \text{ definable, not bi-generic}\}$.

Claim: q^* is finitely satisfiable in G.

Proof Fix $X_1, \dots, X_m \in q$ and $Y_1, \dots, Y_n \subseteq G$ definable & not bi-generic.

WTS: $X_1 \cap \dots \cap X_m \cap G \setminus Y_1 \cap \dots \cap G \setminus Y_n \neq \emptyset$. Suppose not.

Then $X_1 \cap \dots \cap X_m \subseteq Y_1 \cup \dots \cup Y_n$. Note that $Y_1 \cup \dots \cup Y_n$ is not bi-generic by Lemma 21.7. So $X_1 \cap \dots \cap X_m$ is not bi-generic. But $X_1 \cap \dots \cap X_m \in q$, contradiction. //

Let $p \in S_1(G)$ be st $q^* \subseteq p$. Then $q \subseteq p$, and if $Y \in p$ then Y is bi-generic since otherwise $G \setminus Y \in q^* \subseteq p$.

8) Fix $p \in S_1(G)$ and $g \in p$. Note: $X \in gp$ iff $g^{-1}X \in p$.

gp is finitely satisfiable: Fix $X_1, \dots, X_n \in gp$. Then

$g^{-1}(X_1 \cap \dots \cap X_n) = g^{-1}X_1 \cap \dots \cap g^{-1}X_n \in p$. So $g^{-1}(X_1 \cap \dots \cap X_n) \neq \emptyset$.

So $X_1 \cap \dots \cap X_n \neq \emptyset$.

gp is complete: Fix definable $X \subseteq G$. Then $g^{-1}X$ is definable so $g^{-1}X \in p$ or $g^{-1}(G \setminus X) = G \setminus g^{-1}X \in p$, i.e. $X \in gp$ or $G \setminus X \in gp$.

Therefore $gp \in S_1(G)$.

Now assume p is bi-generic.

gp is bi-generic: Fix $X \in gp$. Then $g^{-1}X \in p$ so $\exists\, a_1, \dots, a_n, b_1, \dots, b_n \in G$ st $G = \bigcup_{i=1}^{n} a_i (g^{-1}X) b_i$. Let $a_i' = a_i g^{-1}$. Then $G = \bigcup_{i=1}^{n} a_i' X b_i$. So X is bi-generic.