Part III Model Theory, Lecture 0, 9 Oct

Review

A language is a set $\mathcal{L}$ of function symbols, relation symbols, and constant symbols. Each function/relation symbol has an arity $n \geq 1$

Convention: constant symbols are "function symbols of arity 0".

An $\mathcal{L}$-structure $\mathcal{M}$ consists of:

- a nonempty set M (the universe of $\mathcal{M}$)
- for every function symbol f of arity n, a function $f^{\mathcal{M}}: M^n \to M$
- for every relation symbol R of arity n, a subset $R^{\mathcal{M}} \subseteq M^n$
- for every constant symbol c, a element $c^{\mathcal{M}} \in M$.

Syntax: We build formulas using symbols in $\mathcal{L}$ along with

$\wedge \quad \neg \quad \forall \quad = \quad (\quad)$, and ctbly many variable symbols.

$\mathcal{L}$-term (new functions)

- constant symbols and variables are terms
- if $t_1, \dots, t_n$ are terms and f is an n-ary function symbol then $f(t_1, \dots, t_n)$ is a term

Given a structure $\mathcal{M}$ and a term t, inductively define

$$t^{\mathcal{M}}: M^r \longrightarrow M \quad \text{for appropriate } r$$

- constant symbol $c \rightsquigarrow c^{\mathcal{M}}$ $(r=0)$
- variable $x \rightsquigarrow$ identity function $(r=1)$
- $f(t_1, \dots, t_n) \rightsquigarrow f^{\mathcal{M}}(t_1^{\mathcal{M}}, \dots, t_n^{\mathcal{M}})$

$\mathcal{L}$-formulas (new relations)

- If t_1 and t_2 are terms, $(t_1 = t_2)$ is a formula
- If R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms, then $R(t_1, \dots, t_n)$ is a formula.

} atomic $\mathcal{L}$-formulas

- If φ and ψ are formulas then so are

$$\neg\varphi, \quad (\varphi \wedge \psi), \quad \text{and } \forall x \varphi \text{ for any variable } x$$

An occurrence of v variable x is free in φ if x does not occur in the scope of $\forall x$

Otherwise the occurrence is bound.

$$\forall x \neg (f(x) = y)$$

Notation: Given a formula φ, write $\varphi(x_1, \dots, x_n)$ to denote that $x_1, \dots, x_n$ are the free variables in φ.

Given a formula $\varphi(x_1, \dots, x_n)$, a structure $\mathcal{M}$, $a_1, \dots, a_n \in M$ we define "$\bar{a}$ satisfies φ in $\mathcal{M}$", written $\mathcal{M} \models \varphi(a_1, \dots, a_n)$, as follows:

- If φ is $(t_1 = t_2)$ then $\mathcal{M} \models \varphi(\bar{a})$ iff $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$
- If φ is $R(t_1, \dots, t_n)$ then $\mathcal{M} \models \varphi(\bar{a})$ iff $(t_1^{\mathcal{M}}(\bar{a}), \dots, t_n^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$.
- $\mathcal{M} \models (\varphi \wedge \psi)(\bar{a})$ iff $\mathcal{M} \models \varphi(\bar{a})$ and $\mathcal{M} \models \psi(\bar{a})$.
- $\mathcal{M} \models \neg\varphi(\bar{a})$ iff $\mathcal{M} \not\models \varphi(\bar{a})$
- Suppose φ is $\forall w \psi(x_1, \dots, x_n, w)$. Then $\mathcal{M} \models \varphi(\bar{a})$ iff for all $b \in M$, $\mathcal{M} \models \psi(\bar{a}, b)$.

Abbreviations

Global: $(\varphi \vee \psi)$ is $\neg(\neg\varphi \wedge \neg\psi)$

"or"

$(\varphi \rightarrow \psi)$ is $(\neg\varphi \vee \psi)$

$(\varphi \leftrightarrow \psi)$ is $(\varphi \to \psi) \wedge (\psi \to \varphi)$

$\exists x \varphi$ is $\neg \forall x \neg \varphi$ (universes are nonempty)

<u>Local</u>: e.g. $\mathcal{L} = \{+, \cdot, <, 0, 1\}$ (language of ordered rings)

$x + y$ is $+(x, y)$

$x < y$ is $<(x, y)$

$x \leq y$ is $(x<y) \vee (x=y)$

$x<y<z$ is $(x<y) \wedge (y<z)$

x^2 is $x \cdot x$

nx is $x + x + \ldots + x$ (n times)

An <u>$\mathcal{L}$-sentence</u> is an $\mathcal{L}$-formula with no free variables.

$\forall x \, (f(x) \neq y)$ $\qquad$ $\exists y \forall x \, (f(x) \neq y)$

If φ is a sentence and $\mathcal{M}$ is a structure, then we have the notion of $\mathcal{M} \models \varphi$, "$\mathcal{M}$ satisfies φ" or "$\mathcal{M}$ models φ"

<u>Def</u>: An <u>$\mathcal{L}$-theory</u> is a set of $\mathcal{L}$-sentences.

Given a theory T, we write $\mathcal{M} \models T$ ($\mathcal{M}$ is a <u>model</u> of T) if $\mathcal{M} \models \varphi$ for all $\varphi \in T$.

T is <u>satisfiable</u> if it has a model.

<u>Ex</u>: $T = \{\neg \exists x \, (x = x)\}$ is unsatisfiable.

$\exists x \, (x = x)$

<u>Recall</u>: T is <u>consistent</u> if it does not prove a contradiction $(\varphi \wedge \neg\varphi)$

<u>Gödel</u>: satisfiable iff consistent