Part III Model Theory, Lecture 1, 12 Oct

$\mathcal{L}$ is a fixed language.

An an $\mathcal{L}$-theory T is finitely satisfiable if every finite subset of T is satisfiable.

Compactness Theorem satisfiable $\Leftrightarrow$ finitely satisfiable

Downward Lowenheim-Skolem Theorem Any satisfiable $\mathcal{L}$-theory has a model of cardinality at most $|\mathcal{L}| + \aleph_0$

see notes

non-examinable

Lowenheim-Skolem Theorem

Suppose T is an $\mathcal{L}$-theory with infinite models. Then T has a model of cardinality κ for any $\kappa \geq |\mathcal{L}| + \aleph_0$.

Proof

Let $\mathcal{L}^* = \mathcal{L} \cup \{c_i : i < \kappa\}$ where each c_i is a new constant symbol.

Let $T^* = T \cup \{c_i \neq c_j : i \neq j\}$. Suppose $\Sigma \subseteq T^*$ is finite.

So $\Sigma \subseteq T \cup \{c_i \neq c_j : i, j \in I\}$ for some finite set I.

Let $\mathcal{M} \models T$ be infinite ($\mathcal{M}$ is an $\mathcal{L}$-structure). Expand $\mathcal{M}$ to an $\mathcal{L}^*$-structure $\mathcal{M}^*$ by interpreting $c_i^{\mathcal{M}^*}$ as distinct elements for $i \in I$, and interpreting $c_i^{\mathcal{M}^*}$ for $i \notin I$ arbitrarily. Then $\mathcal{M}^* \models \Sigma$.

By Compactness, T^* is satisfiable. By DLST, T^* has a model $\mathcal{N}^*$ of cardinality at most $|\mathcal{L}^*| + \aleph_0 = \kappa$. So $\mathcal{N}^*$ has cardinality κ. Let $\mathcal{N}$ be the reduct of $\mathcal{N}^*$ to $\mathcal{L}$.

Then $\mathcal{N} \models T$ and $|\mathcal{N}| = \kappa$. $\square$

Complete Theories

Def 1.1 Let T be an $\mathcal{L}$-theory and φ an $\mathcal{L}$-sentence.

Then $T \models \varphi$ ("T models φ", "T implies φ") if any model of T is a model of φ.

Example 1.2

1) $\{\varphi, \psi\} \models \varphi \wedge \psi$

2) If T is consistent (ie., satisfiable) then $T \models \exists x (x = x)$. So $\emptyset \models \exists x (x = x)$

3) Let T be the theory of groups in $\mathcal{L} = \{*, e\}$

$$T \models \forall x \forall y \forall z \left((x*y=e \wedge x*z=e) \to y=z \right)$$

<u>Def 1.3</u> An $\mathcal{L}$-theory T is <u>complete</u> if, for any $\mathcal{L}$-sentence φ,

$$T \models \varphi \quad \text{or} \quad T \models \neg\varphi.$$

<u>Example 1.4</u>

1) The theory of groups is not complete. Consider $\forall x \forall y \, (x*y = y*x)$

2) ZFC is not complete. Consider the Continuum Hypothesis.

<u>Def 1.5</u> Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The <u>theory of $\mathcal{M}$</u> is

$$Th(\mathcal{M}) = Th_{\mathcal{L}}(\mathcal{M}) := \{\varphi : \varphi \text{ is an } \mathcal{L}\text{-sentence} + \mathcal{M} \models \varphi\}.$$

Note that $Th(\mathcal{M})$ is complete.

<u>Def 1.6</u> Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are <u>elementarily equivalent</u>, written $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$.

Note that $\equiv$ is an equivalence relation on $\mathcal{L}$-structures.
May use $\equiv_{\mathcal{L}}$ for emphasis.

<u>Exercise</u> (ES1 #2) Let T be an $\mathcal{L}$-theory. TFAE

i) T is complete.

ii) For any $\mathcal{L}$-sentence φ, if $T \not\models \varphi$ then $T \models \neg\varphi$.

iii) Any two models of T are elementarily equivalent.

<u>Example 1.7</u> Let $\mathcal{L} = \emptyset$ and $T = \{\varphi_n : n \geq 2\}$ where

$$\varphi_n \text{ is } \exists x_1 \dots \exists x_n \bigwedge_{i \neq j} x_i \neq x_j.$$

T is "the theory of infinite sets"

Then T is complete (ES1 #3). So, as $\mathcal{L}$-structures,

$$\mathbb{N} \equiv \mathbb{Z} \equiv \mathbb{Q} \equiv \mathbb{R} \equiv \mathbb{C} \equiv \mathcal{P}(\mathbb{C}) \equiv \text{any infinite set.}$$

Theorem 1.8 (Vaught's Test)

Let T be an $\mathcal{L}$-theory st

a) T has no finite models,

b) $\exists\ \kappa \geq |\mathcal{L}| + \aleph_0$ st any two models of T of cardinality κ are elementarily equivalent.

Then T is complete.

Proof Suppose T is not complete. Then there is a sentence φ st $T \cup \{\neg\varphi\}$ is satisfiable and $T \cup \{\varphi\}$ is satisfiable. By (a) these theories have infinite models. By LST, these theories have models of size κ. This contradicts (b).