Part III Model Theory, Lecture 2, 14 Oct

Homomorphisms

$\mathcal{L}$ is a language

Def 2.1 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A function $h: M \to N$ is an $\mathcal{L}$-homomorphism if

i) for any n-ary function symbol f and $a_1, \dots, a_n \in M$
$$h(f^{\mathcal{M}}(a_1, \dots, a_n)) = f^{\mathcal{N}}(h(a_1), \dots, h(a_n))$$

ii) for any n-ary relation symbol R and $a_1, \dots, a_n \in M$
$$(a_1, \dots, a_n) \in R^{\mathcal{M}} \text{ iff } (h(a_1), \dots, h(a_n)) \in R^{\mathcal{N}}$$

iii) for any constant symbol c, $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$

We write $h: \mathcal{M} \to \mathcal{N}$ for $\mathcal{L}$-homomorphisms.

If h is also injective, then h is an $\mathcal{L}$-embedding.

If h is also bijective, then h is an $\mathcal{L}$-isomorphism.

Theorem 2.2 Suppose $h: \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-isomorphism. Then for any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$,
$$\mathcal{M} \models \varphi(a_1, \dots, a_n) \text{ iff } \mathcal{N} \models \varphi(h(a_1), \dots, h(a_n)).$$

Proof

Claim: For any $\mathcal{L}$-term $t(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$,
$$h(t^{\mathcal{M}}(a_1, \dots, a_n)) = t^{\mathcal{N}}(h(a_1), \dots, h(a_n))$$

Pf: Induction on terms. If t is a constant symbol c then
$$h(t^{\mathcal{M}}) = h(c^{\mathcal{M}}) = c^{\mathcal{N}} = t^{\mathcal{N}}.$$

If t is a variable x_1, then $h(t^{\mathcal{M}}(a_1)) = h(a_1) = t^{\mathcal{N}}(h(a_1))$

Let f be an m-ary function symbol. Assume the result for terms $t_1, \dots, t_m$ whose free vbls are among $x_1, \dots, x_n$

Let t be $f(t_1,\dots,t_m)$. Given $a_1,\dots,a_n \in M$

$$h(t^{\mathcal{M}}(\bar{a})) = h(f^{\mathcal{M}}(t_1^{\mathcal{M}}(\bar{a}),\dots,t_m^{\mathcal{M}}(\bar{a}))) \underset{\text{(def. of } \mathcal{L}\text{-hom)}}{=} f^{\mathcal{N}}(h(t_1^{\mathcal{M}}(\bar{a})),\dots,h(t_m^{\mathcal{M}}(\bar{a})))$$

$$\underset{\text{(induction)}}{=} f^{\mathcal{N}}(t_1^{\mathcal{N}}(h(\bar{a})),\dots,t_m^{\mathcal{N}}(h(\bar{a}))) = t^{\mathcal{N}}(h(\bar{a})). \qquad //$$

Now we prove the theorem by induction on φ.

<u>Base case</u>: φ is atomic

1) φ is $t_1 = t_2$

$$\mathcal{M} \models \varphi(\bar{a}) \text{ iff } t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a}) \underset{\text{(h is injective)}}{\text{iff}} h(t_1^{\mathcal{M}}(\bar{a})) = h(t_2^{\mathcal{M}}(\bar{a}))$$

$$\underset{\text{(Claim)}}{\text{iff}} t_1^{\mathcal{N}}(h(\bar{a})) = t_2^{\mathcal{N}}(h(\bar{a})) \text{ iff } \mathcal{N} \models \varphi(h(\bar{a})).$$

2) φ is $R(t_1,\dots,t_m)$ (Exercise)

<u>Induction Step</u> Assume the result for φ and ψ

Exercise: Check $\varphi \wedge \psi$ and $\neg\varphi$

We'll do: $\forall x_n \varphi(x_1,\dots,x_n)$ (free vbls $x_1,\dots,x_{n-1}$). Fix $a_1,\dots,a_{n-1} \in M$

$\mathcal{M} \models \forall x_n \varphi(a_1,\dots,a_{n-1},x_n)$ iff for all $b \in M$, $\mathcal{M} \models \varphi(a_1,\dots,a_{n-1},b)$

iff for all $b \in M$, $\mathcal{N} \models \varphi(h(a_1),\dots,h(a_{n-1}),h(b))$ (induction)

iff for all $c \in N$, $\mathcal{N} \models \varphi(h(a_1),\dots,h(a_{n-1}),c)$ (h is onto)

iff $\mathcal{N} \models \forall x_n \varphi(h(a_1),\dots,h(a_{n-1}),x_n)$ $\square$

<u>Notation</u>: $\mathcal{M} \cong \mathcal{N}$ if there is an $\mathcal{L}$-isomorphism $h: \mathcal{M} \to \mathcal{N}$.

<u>Corollary 2.3</u> If $\mathcal{M} \cong \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$. (Now do ES1 #3)

<u>Corollary 2.4</u> $h: \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-embedding iff the conclusion of Thm 2.2 holds for all quantifier-free formulas $\varphi(x_1,\dots,x_n)$.

Proof: ($\Longrightarrow$) from the proof; only used surjectivity for quantifier step.

($\Longleftarrow$) See ES1 #6?

Def 2.5 $h : \mathcal{M} \to \mathcal{N}$ is an elementary $\mathcal{L}$-embedding if for any $\mathcal{L}$-formula $\varphi(\bar{x})$ and $\bar{a}$ from M, $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(h(\bar{a}))$.

Note that isomorphisms are elementary embeddings.

Def 2.6 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $M \subseteq N$.
Let $h : M \to N$ be the inclusion map. Then $\mathcal{M}$ is a substructure of $\mathcal{N}$ [resp., elementary substructure], written $\mathcal{M} \subseteq \mathcal{N}$ [resp., $\mathcal{M} \preceq \mathcal{N}$] \preceq, if h is an $\mathcal{L}$-embedding [resp., elementary embedding].
Also say $\mathcal{N}$ is an extension of $\mathcal{M}$ [resp., elementary extension].

Note: If $\mathcal{M} \preceq \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \equiv \mathcal{N}$

Example 2.7 Let $\mathcal{M} = (2\mathbb{Z}, <)$ and $\mathcal{N} = (\mathbb{Z}, <)$.
Then $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M} \equiv \mathcal{N}$, but $\mathcal{M} \npreceq \mathcal{N}$
(why?)

e.g. $\mathcal{M} \models \neg \exists x (0 < x < 2)$

[So, $\varphi(y,z)$ is $\neg \exists x (y < x < z)$ and $\mathcal{M} \models \varphi(0,2)$, $\mathcal{N} \not\models \varphi(0,2)$]