Part III Model Theory, Lecture 3, 16 Oct

Categoricity

Q: Suppose $\mathcal{M} \equiv \mathcal{N}$. Then $\mathcal{M} \cong \mathcal{N}$?

A: No. If $\mathcal{M}$ is infinite then $Th(\mathcal{M})$ has models of arbitrarily large size.

So a theory with infinite models never has a unique model up to isomorphism.

Def 3.1 An $\mathcal{L}$-theory T is κ-categorical if it has a unique model of size κ.

Main focus: T has infinite models and $\kappa \geq |\mathcal{L}| + \aleph_0$

Example 3.2 (TBD)

1) $Th(\mathbb{N})$ in $\mathcal{L} = \emptyset$ is κ-categorical $\forall\ \kappa \geq \aleph_0$ (ES1 #3)
2) $Th(\mathbb{Q}, +)$ is κ-categorical iff $\kappa > \aleph_0$ (related to ES1 #4)
3) $Th(\mathbb{Q}, <)$ is κ-categorical iff $\kappa = \aleph_0$
4) $Th(\mathbb{Z}, +)$ is κ-categorical for no κ.

[aside]

Morley's Theorem (1965) Let T be a complete theory in ctble language.

If T is κ-categorical for some $\kappa > \aleph_0$, then it is κ-categorical $\forall\ \kappa > \aleph_0$.

Def 3.3 Let $\mathcal{L} = \{<\}$ (binary relation). Define DLO to be the following theory

$$\forall x\, \neg(x < x)$$

$$\forall x \forall y \forall z\, ((x < y \wedge y < z) \to x < z)$$

$$\forall x \forall y\, ((x \neq y) \to (x < y \vee y < x))$$

$$\forall x \forall y\, (x < y \to \exists z\, (x < z < y))$$

$$\forall x \exists y \exists z\, (y < x < z)$$

"dense linear orders w/o endpoints"

Note $(\mathbb{Q}, <) \models DLO$

Theorem 3.4 (Cantor 1895)

DLO is $\aleph_0$-categorical.

Proof (Back and Forth)

Fix ctble models $\mathcal{M}, \mathcal{N} \models DLO$. Let $M = \{a_n : n \geq 0\}$ and $N = \{b_n : n \geq 0\}$.

We inductively construct a sequence $(h_n)_{n=0}^{\infty}$ of functions st:

1) $h_n : X_n \to Y_n$ is an order-preserving bijection, where $X_n \subseteq M$ and $Y_n \subseteq N$ are finite

2) $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$, and $h_n \subseteq h_{n+1}$

3) $a_n \in X_n$ and $b_n \in Y_n$.

Once we've done this, let $h = \bigcup_{n=0}^{\infty} h_n$. Then h is an order-preserving bijection from M to N, ie. h is an $\mathcal{L}$-isomorphism.

Base Case: Let $X_0 = \{a_0\}$ and $Y_0 = \{b_0\}$ and $h_0 = \{(a_0, b_0)\}$

Now assume we have h_n.

Forth: We'll construct an order-preserving bijection $h_* : X_* \to Y_*$ extending h_n with $a_{n+1} \in X_*$. Enumerate $X_n = \{x_1, \ldots, x_k\}$ st $x_1 <^{\mathcal{M}} \ldots <^{\mathcal{M}} x_k$.

Let $y_i = h_n(x_i)$. Then $y_1 <^{\mathcal{N}} \ldots <^{\mathcal{N}} y_k$ since h_n is order-preserving.

Define $h_* = h_n \cup \{(a_{n+1}, b)\}$ where $b \in N$ is chosen as follows.

Case 1: $a_{n+1} = x_i$ for some $i \leq k$. Let $b = y_i$.

Case 2: $x_k <^{\mathcal{M}} a_{n+1}$. Choose $b \in N$ st $y_k <^{\mathcal{N}} b$

Case 3: $a_{n+1} <^{\mathcal{M}} x_1$. Chose $b \in N$ st $b <^{\mathcal{N}} y_1$

Case 4: $x_i <^{\mathcal{M}} a_{n+1} <^{\mathcal{M}} x_{i+1}$ for some $i < k$. Choose $b \in N$ st $y_i <^{\mathcal{N}} b <^{\mathcal{N}} y_{i+1}$.

Back: Construct order-preserving $h_{n+1} : X_{n+1} \to Y_{n+1}$ extending h_* st $b_{n+1} \in Y_{n+1}$. Details are exercise. $\square$

Corollary 3.5 DLO is complete

Proof: Apply Vaught's Test. DLO has no finite models

If $\mathcal{M}, \mathcal{N} \models DLO$ are ctble then $\mathcal{M} \cong \mathcal{N}$, so $\mathcal{M} \equiv \mathcal{N}$. $\square$

So $(\mathbb{Q}, <) \equiv (\mathbb{R}, <) \equiv$ dense l.o. w/o endpoints.

<u>More Notions</u> Let $\mathcal{L}$ be a language.

Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure. Fix a collection $(\mathcal{M}_i)_{i \in I}$ of substructures of $\mathcal{M}$.

Let $N = \bigcap_{i \in I} M_i$. Assume $N \neq \emptyset$. Then we have a canonical $\mathcal{L}$-structure $\mathcal{N}$ with universe N. $\left(f^{\mathcal{N}} = f^{\mathcal{M}}|_N = f^{\mathcal{M}_i}|_N,\ R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^{\text{arity}(R)} = R^{\mathcal{M}_i} \cap N^{\text{arity}(R)},\ c^{\mathcal{N}} = c^{\mathcal{M}} = c^{\mathcal{M}_i} \right)$

Note $\mathcal{N} \subseteq \mathcal{M}_i \ \forall\ i \in I$.

<u>Def 3.6</u>: Given a structure $\mathcal{M}$ and a nonempty $A \subseteq M$, the <u>substructure of $\mathcal{M}$ generated by A</u> is the intersection of all substructures of $\mathcal{M}$ containing A.

<u>Def 3.7</u> Let α be a limit ordinal. A collection $(\mathcal{M}_i)_{i<\alpha}$ of $\mathcal{L}$-structures is a <u>chain</u> if $\mathcal{M}_i \subseteq \mathcal{M}_{i+1} \ \forall\ i<\alpha$, and if $\beta<\alpha$ is a limit then $\mathcal{M}_i \subseteq \mathcal{M}_\beta \ \forall\ i<\beta$.
If $\mathcal{M}_i \preceq \mathcal{M}_{i+1} \ \forall\ i$ and $\mathcal{M}_i \preceq \mathcal{M}_\beta \ \forall$ limit β + $i<\beta$, then we say <u>elementary chain</u>.

If $(\mathcal{M}_i)_{i<\alpha}$ is a chain then we have a well-defined structure $\bigcup_{i<\alpha} \mathcal{M}_i$.

<u>ES1</u>: You can do 1, 2, 3, 4, 6, 7, 8   5, 9 later... (or try now)