Part III Model Theory, Lecture 4, 19 Oct

ACF

Recall: $(K,+,\cdot,0,1)$ is a field if $(K,+,0)$ and $(K\setminus\{0\},\cdot,1)$ are abelian groups and $\forall x \forall y \forall z\ (x\cdot(y+z) = x\cdot y + x\cdot z)$

K is algebraically closed if every non-constant polynomial over K has a root in K.

Let $\mathcal{L} = \{+,\cdot,0,1\}$

Def 5.1 ACF is the $\mathcal{L}$-theory axiomatizing algebraically closed fields.

This contains the field axioms plus for every $d \geq 1$

$$\forall v_0 \forall v_1 \ldots \forall v_{d-1} \exists x \left(x^d + v_{d-1}x^{d-1} + \ldots + v_1 x + v_0 = 0\right)$$

Remark ACF is not complete since it does not specify characteristic.

Def 5.2 For $n \geq 1$, let χ_n be the $\mathcal{L}$-sentence $\underbrace{1+1+\ldots+1}_{n} = 0$

$ACF_0 = ACF \cup \{\neg\chi_n : n \geq 1\}$

For a prime p, $ACF_p = ACF \cup \{\chi_p\}$.

Theorem 5.3 ACF_0, ACF_p are κ-categorical for all $\kappa > \aleph_0$.

Proof

The transcendence degree of $K \models ACF$ is the cardinality of the largest algebraically independent subset of K.

E.g. $\operatorname{trdeg}(\overline{\mathbb{Q}}) = 0$, $\operatorname{trdeg}(\overline{\mathbb{Q}(\pi)}) = 1$, $\operatorname{trdeg}(\mathbb{C}) = 2^{\aleph_0}$; $\operatorname{trdeg}(\overline{\mathbb{Q}(x_i)_{i<\kappa}}) = \kappa$

Facts ① Suppose $K, L \models ACF$. Then $K \cong L$ iff $\operatorname{trdeg}(K) = \operatorname{trdeg}(L)$, $\operatorname{char}(K) = \operatorname{char}(L)$, and $|K| = |L|$.

② If $K \models ACF$ and $\kappa = \operatorname{trdeg}(K)$, then $|K| = \aleph_0 + \kappa$.

Conclusion: If $K, L \models ACF_{0\ (\text{or } p)}$ are uncountable + $|K| = |L|$, then $K \cong L$. $\square$

<u>Corollary 5.4</u> ACF_0, ACF_p are complete.

<u>Proof</u>: Vaught's Test.

<u>Remark</u> ACF_0, ACF_p are not $\aleph_0$-categorical.

Countable models are precisely the ctble ACF_p's of trdeg n for $n \in \mathbb{N} \cup \{\aleph_0\}$.

<u>Def 5.5</u>: Let K be a field. A function $\Phi : K^m \to K^n$ is a <u>polynomial map</u> if

$$\Phi = (p_1(x_1, \dots, x_m), p_2(x_1, \dots, x_m), \dots, p_n(x_1, \dots, x_m)) \text{ where } p_i \in K[\bar{x}]$$

<u>Theorem 5.6</u> (Ax-Grothendieck)

Let $K \models ACF$ and suppose $\Phi : K^n \to K^n$ is an injective polynomial map.

Then Φ is surjective.

<u>Proof</u>

First, suppose $K = \overline{\mathbb{F}}_p$ for some prime p. Recall $\overline{\mathbb{F}}_p = \bigcup_k \mathbb{F}_{p^k}$.

Fix m st all coefficients in Φ come from $\mathbb{F}_{p^m}$. Note $\overline{\mathbb{F}}_p = \bigcup_k \mathbb{F}_{p^{km}}$

For any $k \geq 1$, Φ induces an injective poly map from $\mathbb{F}_{p^{km}}^n \to \mathbb{F}_{p^{km}}^n$,

which therefore is surjective, since $\mathbb{F}_{p^{km}}$ is finite.

$$\Phi(\overline{\mathbb{F}}_p^{\,n}) = \Phi\left(\bigcup_k \mathbb{F}_{p^{km}}^n\right) = \bigcup_k \Phi(\mathbb{F}_{p^{km}}^n) = \bigcup_k \mathbb{F}_{p^{km}}^n = \overline{\mathbb{F}}_p^{\,n}.$$

Now, given $n, d \geq 1$, let $\psi_{n,d}$ be the $\mathcal{L}$-sentence which says

"Every injective polynomial map with n coordinates, each of which is a polynomial in n variables and degree $\leq d$, is surjective." (Do $n=d=2$)

We've shown $\overline{\mathbb{F}}_p \models \psi_{n,d}$ for all primes p and n, d.

So for any prime p, $ACF_p \models \psi_{n,d}$ $\forall$ n, d since ACF_p is complete.

Consider ACF_0. For a contradiction, suppose $\exists\, n,d$ st $ACF_0 \not\models \psi_{n,d}$

Then $ACF_0 \models \neg\psi_{n,d}$ since ACF_0 is complete. By Compactness, there is a finite set $\Sigma \subseteq ACF_0$ st $\Sigma \models \neg\psi_{n,d}$. So $\Sigma \subseteq ACF \cup \{\neg\chi_1, \dots, \neg\chi_m\}$ for some m. Choose a prime $p > m$. Then $ACF_p \models \Sigma$.

So $ACF_p \models \neg\psi_{n,d}$, which is a contradiction. $\Box$.

<u>Lefschetz Principle</u> Let φ be an $\mathcal{L}$-sentence. TFAE

1) $ACF_0 \models \varphi$ ie. φ is true in every $K \models ACF_0$.

2) $ACF_0 \cup \{\varphi\}$ is consistent, ie. φ is true in some $K \models ACF_0$.

3) There is some $n > 0$ st $ACF_p \models \varphi$ $\forall\, p > n$. ie. φ is true in every $K \models ACF$ of suff. large characteristic.

4) For all $n > 0$ $\exists\, p > n$ st $ACF_p \cup \{\varphi\}$ is consistent, ie. φ is true in some $K \models ACF$ of arbitrarily large characteristic.