Part III Model Theory, Lecture 5, 21 Oct

Diagrams & Extensions

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

Remark 5.8 If $h: \mathcal{M} \to \mathcal{N}$ is an $\mathcal{L}$-embedding then after identifying $a \in M$ with $h(a) \in N$, we can view $\mathcal{M}$ as a substructure of $\mathcal{N}$. Similarly, if h is an elementary embedding, then $\mathcal{M}$ "is" an elementary substructure of $\mathcal{N}$.

Given $A \subseteq M$, let $\mathcal{L}_A = \mathcal{L} \cup \{\underline{a} : a \in A\}$ where $\underline{a}$ is a new constant symbol.

Then $\mathcal{M}$ is canonically an $\mathcal{L}_A$-structure where $\underline{a}^{\mathcal{M}} = a$.

Def 5.9 The diagram of $\mathcal{M}$, written $D(\mathcal{M})$, is the $\mathcal{L}_M$-theory consisting of all quantifier-free $\mathcal{L}_M$-sentences φ st $\mathcal{M} \models \varphi$.

The elementary diagram of $\mathcal{M}$ is $Th_M(\mathcal{M}) := Th_{\mathcal{L}_M}(\mathcal{M})$.

Proposition 5.10 Suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{N}^*$ is an $\mathcal{L}_M$-structure st $\mathcal{N}^* \models D(\mathcal{M})$. Let $\mathcal{N}$ be the reduct of $\mathcal{N}^*$ to $\mathcal{L}$. Define $h: \mathcal{M} \to \mathcal{N}$ st $h(a) = \underline{a}^{\mathcal{N}^*}$. Then h is an $\mathcal{L}$-embedding. Moreover, if $\mathcal{N}^* \models Th_M(\mathcal{M})$ then h is an elementary embedding.

Proof Use Corollary 2.4. Let $\varphi(x_1, \dots, x_n)$ be a quantifier-free $\mathcal{L}$-formula, and fix $a_1, \dots, a_n \in M$. Then $\mathcal{M} \models \varphi(a_1, \dots, a_n)$ iff $\mathcal{M} \models \varphi(\underline{a}_1, \dots, \underline{a}_n)$ iff

$$\varphi(\underline{a}_1, \dots, \underline{a}_n) \in D(\mathcal{M}) \text{ iff } \mathcal{N}^* \models \varphi(\underline{a}_1, \dots, \underline{a}_n) \text{ iff } \mathcal{N} \models \varphi(h(a_1), \dots, h(a_n))$$

The "moreover" is similarly. $\square$.

Application to Groups: Recall that an abelian group G is orderable if there is a linear order $<$ on G st $\forall x, y, z \in G$, if $x < y$ then $x + z < y + z$.

Note that any orderable abelian group is torsion-free. E.g. $x > 0 \Rightarrow x < 2x < 3x < \dots$

Theorem 5.11 (Levi 1942)

Any torsion-free abelian group is orderable.

Proof Let $\mathcal{L}^0 = \{+, 0\}$ be the language of (abelian) groups. Set $\mathcal{L} = \mathcal{L}^0 \cup \{<\}$, where $<$ is a binary relation symbol. Let σ be the $\mathcal{L}$-sentence

$$\forall x \forall y \forall z \, (x<y \rightarrow x+z < y+z)$$

Now let G be a torsion-free abelian group, viewed as an $\mathcal{L}^0$-structure.

Define the $\mathcal{L}_G$-theory $T = \underbrace{D(G)}_{\mathcal{L}^0_G\text{-theory}} \cup \{\text{axioms for abelian groups}\} \cup \{\text{axioms for linear order}\} \cup \{\sigma\}$

Suppose T has a model $\mathcal{M}$. Then $(M, +^{\mathcal{M}}, 0^{\mathcal{M}}, <^{\mathcal{M}})$ is an ordered abelian group and $G \subseteq (M, +^{\mathcal{M}}, 0^{\mathcal{M}})$ by Proposition 5.10. So G is a subgroup of an ordered abelian group, hence is orderable. So it suffices to show that T has a model.

Fix $\Sigma \subseteq T$ finite. Let $A = \{a \in G : \underline{a} \text{ is used in some } \mathcal{L}^0_G\text{-sentence in } \Sigma\}$. in $D(G)$

Let $H = \langle A \rangle \leq G$. Then $\boxed{H \cong \mathbb{Z}^n}$ for some $n \geq 0$ by the structure theorem for finitely generated abelian groups. View H as an $\mathcal{L}_A$-structure st

st $\underline{a}^H = a$ and $<^H$ is the lexicographic ordering. Then $H \subseteq_{\mathcal{L}^0_A} G$ and so $H \models \varphi$ for any $\varphi \in D(G)$ using only extra constants from A (by Corollary 2.4)

So $H \models \Sigma$. $\square$

Quantifier Elimination

Idea: Let T be an $\mathcal{L}$-theory and $\mathcal{M} \models T$. Then $X \subseteq M^n$ is definable if there is an $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ st $X = \{\bar{a} \in M^n : \mathcal{M} \models \varphi(\bar{a})\}$.

Goal: Study definable subsets of models of T.

Quantifiers make this difficult. X might be nice but the projection $Y = \{(a_1, \dots, a_{n-1}) \in M^{n-1} : (\bar{a}, b) \in X \text{ for some } b \in M\}$ (defined by $\exists x_n \varphi(\bar{x})$)

might be complicated.

Def 5.12 An $\mathcal{L}$-theory T has quantifier elimination if for any $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ there is a quantifier-free $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$ st

$$T \models \forall \bar{x} \left(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}) \right).$$

(So φ and ψ define the same set in any $\mathcal{M} \models T$).

Example 5.13

① $T = Th(F)$ where F is a field. $\varphi(w,x,y,z)$ is "$\begin{pmatrix} w & x \\ y & z \end{pmatrix}$ has an inverse"

i.e. $\exists s \exists t \exists u \exists v \left(\begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$

Then $T \models \forall w \forall x \forall y \forall z \left(\varphi(w,x,y,z) \leftrightarrow wz - xy \neq 0 \right).$