Part III Model Theory, Lecture 6, 23 Oct

② (cf Ex 5.13) $T = \mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1)$. $\varphi(x)$ is $\exists y (x = y^2)$. Note φ defines $\mathbb{R}^{\geq 0}$.

Suppose $\psi(x)$ is q.f. So $\psi(x)$ is a Boolean combination of polynomial equations.

So ψ defines a finite or cofinite subset of $\mathbb{R}$. T does not have QE.

Later: $\mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1)$ does have QE. Note $x < y$ iff $\exists z (z \neq 0 \wedge y - x = z^2)$.

So $(\mathbb{R}, +, \cdot, 0, 1)$ and $(\mathbb{R}, +, \cdot, <, 0, 1)$ have the same definable sets.

Lemma 6.1 Suppose T is an $\mathcal{L}$-theory st for any q.f. formula $\varphi(x_1, \ldots, x_n, y)$ there is a q.f. $\psi(x_1, \ldots, x_n)$ st $T \models \forall \bar{x} (\exists y \varphi(\bar{x}, y) \leftrightarrow \psi(\bar{x}))$. Then T has QE.

Proof: Induction on formulas. (Exercise).

Theorem 6.2 Let T be an $\mathcal{L}$-theory. TFAE.

i) T has QE.

ii) Suppose $\mathcal{M}, \mathcal{N} \models T$ and $\mathcal{A} \subseteq \mathcal{M}$, $\mathcal{A} \subseteq \mathcal{N}$ are arbitrary. Then for any q.f. formula $\varphi(\bar{x}, y)$ and any tuple $\bar{a}$ from A, if $\mathcal{M} \models \exists y \varphi(\bar{a}, y)$ then $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

iii) For any $\mathcal{L}$-structure $\mathcal{A}$, $T \cup D(\mathcal{A})$ is a complete $\mathcal{L}_A$-theory.

Proof (i) $\Rightarrow$ (iii). Assume T has QE. Let $\mathcal{A}$ be an $\mathcal{L}$-structure and suppose $\mathcal{M}, \mathcal{N} \models T \cup D(\mathcal{A})$. WTS $\mathcal{M} \equiv_{\mathcal{L}_A} \mathcal{N}$. Let σ be an $\mathcal{L}_A$-sentence st $\mathcal{M} \models \sigma$. WTS $\mathcal{N} \models \sigma$. Write σ as $\varphi(\underline{a}_1, \ldots, \underline{a}_n)$ for some $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$. By QE, there is q.f. $\psi(x_1, \ldots, x_n)$ st $T \models \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.
Since $\mathcal{M} \models T$ and $\mathcal{M} \models \varphi(\bar{a})$, we have $\mathcal{M} \models \psi(\bar{a})$. Since $\mathcal{M} \models D(\mathcal{A})$, we have $\psi(\underline{a}_1, \ldots, \underline{a}_n) \in D(\mathcal{A})$. So $\mathcal{N} \models \psi(\underline{a}_1, \ldots, \underline{a}_n)$. Since $\mathcal{N} \models T$, $\mathcal{N} \models \varphi(\underline{a}_1, \ldots, \underline{a}_n)$, i.e. $\mathcal{N} \models \sigma$.

$\mathcal{N} \models \psi(\underline{a}_1^{\mathcal{N}}, \ldots, \underline{a}_n^{\mathcal{N}})$

$\downarrow$

$\mathcal{N} \models \varphi(\underline{a}_1^{\mathcal{N}}, \ldots, \underline{a}_n^{\mathcal{N}}) \rightarrow \mathcal{N} \models \sigma$

(iii) $\Rightarrow$ (ii). Let $\mathcal{M}, \mathcal{N}, \mathcal{A}, \varphi(\bar{x}, y)$, and $\bar{a}$ be as (ii).

Since $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A} \subseteq \mathcal{N}$, we have $\mathcal{M}, \mathcal{N} \models T \cup D(\mathcal{A})$ by Cor 2.4.

By (iii), $\mathcal{M} \equiv_{\mathcal{L}_A} \mathcal{N}$. So $\mathcal{M} \models \underbrace{\exists y \varphi(\bar{a}, y)}_{\mathcal{L}_A\text{-sentence}} \Rightarrow \mathcal{N} \models \exists y \varphi(\bar{a}, y)$.

(ii) $\Rightarrow$ (i). Assume (ii). WTS QE. By Lemma 6.1, it suffices to fix q.f. $\varphi(\bar{x}, y)$ and find q.f. $\psi(\bar{x})$ st $T \models \forall \bar{x} (\exists y \varphi(\bar{x}, y) \leftrightarrow \psi(\bar{x}))$.

Let $\mathcal{L}^* = \mathcal{L} \cup \{c_1, \dots, c_n\}$ where c_i is a new constant symbol.

Let $\Gamma = \{\psi(\bar{c}) : \psi(\bar{x})$ is a q.f. $\mathcal{L}$-formula + $T \models \forall \bar{x} (\exists y \varphi(\bar{x}, y) \rightarrow \psi(\bar{x}))\}$.

<u>Claim</u>: $T \cup \Gamma \models \exists y \varphi(\bar{c}, y)$.

First, assume the claim holds. By Compactness, $\exists$ q.f. $\psi_1(\bar{x}), \dots, \psi_m(\bar{x})$ st

$T \cup \{\psi_1(\bar{c}), \dots, \psi_m(\bar{c})\} \models \exists y \varphi(\bar{c}, y)$. and $T \models \forall \bar{x} (\exists y \varphi(\bar{x}, y) \rightarrow \bigwedge_{i=1}^{m} \psi_i(\bar{x}))$. since each $\psi_i(\bar{c})$ is in Γ!

Let $\psi(\bar{x})$ be $\bigwedge_{i=1}^{m} \psi_i(\bar{x})$. Then $T \models (\psi(\bar{c}) \rightarrow \exists y \varphi(\bar{c}, y))$.

So $T \models \forall \bar{x} (\psi(\bar{x}) \rightarrow \exists y \varphi(\bar{x}, y))$ (exercise, "generalization")

So $T \models \forall \bar{x} (\psi(\bar{x}) \leftrightarrow \exists y \varphi(\bar{x}, y))$.

<u>Proof of Claim</u> Suppose not. There is $\mathcal{N} \models T \cup \Gamma \cup \{\neg \exists y \varphi(\bar{c}, y)\}$.

Let $a_i = c_i^{\mathcal{N}}$ and let $\mathcal{A} \subseteq \mathcal{N}$ be the substructure generated by $a_1, \dots, a_n$.

Then $\mathcal{N} \models T$, $\mathcal{A} \subseteq \mathcal{N}$, and $\mathcal{N} \models \neg \exists y \varphi(\bar{a}, y)$.

By ES1 #7, any $b \in A$ is of the form $t^{\mathcal{N}}(\bar{a})$ for some $\mathcal{L}$-term t.

So we can view $D(\mathcal{A})$ as a $\mathcal{L}^*$-theory by replacing b with $t(c_1, \dots, c_n)$.

Let $\Sigma \models T \cup D(\mathcal{A}) \cup \{\exists y \varphi(\bar{c}, y)\}$. If we build $\mathcal{M} \models \Sigma$ then $\mathcal{M} \models T$, $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{M} \models \exists y \varphi(\bar{a}, y)$, contradicting (ii).

So it suffices to show Σ has a model. Suppose not. By Compactness, There are q.f. $\psi_1(\bar{x}), \dots, \psi_m(\bar{x})$ st $\psi_1(\bar{c}), \dots, \psi_m(\bar{c}) \in D(A)$ and

$T \cup \{\bigwedge_{i=1}^{m} \psi_i(\bar{c})\} \cup \{\exists y \varphi(\bar{c}, y)\}$ is unsatisfiable. Let $\psi(\bar{x})$ be $\neg \bigwedge_{i=1}^{m} \psi_i(\bar{x})$.

Then $T \models (\exists y \varphi(\bar{c}, y) \to \psi(\bar{c}))$. So $T \models \forall \bar{x} (\exists y \varphi(\bar{x}, y) \to \psi(\bar{x}))$.

So $\psi(\bar{c}) \in T'$. So $\mathcal{N} \models \psi(\bar{c})$. Since $\mathcal{N} \models D(A)$, we have

$\mathcal{N} \models \neg \psi(\bar{c})$. Contradiction. $\square$