Part III Model Theory, Lecture 7, 26 Oct

Remark Recall Theorem 6.2 (QE test)

1) In condition (iii), we may assume $\mathcal{A} \subseteq \mathcal{M}$ for some model $\mathcal{M} \models T$.
   Otherwise, $T \cup D(\mathcal{A})$ is inconsistent and thus complete.

2) In conditions (ii) and (iii), we may assume $\mathcal{A}$ is finitely generated.

Theorem 7.1 ACF has QE.

Proof We apply Thm 6.2 (iii). Fix a finitely-generated $\mathcal{L}$-structure $\mathcal{A}$.
WTS: $ACF \cup D(\mathcal{A})$ is complete. We use Vaught's Test.
Fix $K_1, K_2 \models ACF \cup D(\mathcal{A})$ uncountable with $|K_1| = |K_2|$.
Then $\mathcal{A}$ is a finitely generated integral domain contained in K_1 and K_2.
So $\mathrm{char}(K_1) = \mathrm{char}(K_2)$. Let F_i be the field of fractions of $\mathcal{A}$ in K_i.
There is a field isomorphism $\tau : F_1 \to F_2$ fixing $\mathcal{A}$. Since $\mathcal{A}$ is finitely generated $\mathrm{trdeg}(F_i)$ is finite. So $\mathrm{trdeg}(K_1/F_1) = \mathrm{trdeg}(K_2/F_2)$.
So τ extends to an isomorphism $\tau^* : K_1 \to K_2$ fixing $\mathcal{A}$. $\square$

Def 7.2 Let F be a field. Then $X \subseteq F^n$ is <u>constructible</u> if it is a Boolean combination of subsets of F^n defined by $p(x_1, \dots, x_n) = 0$ where $p \in F[x_1, \dots, x_n]$

Corollary 7.3 (Chevalley)
If $K \models ACF$ and $X \subseteq K^n$ is constructible, then the projection
$$Y = \{(a_1, \dots, a_{n-1}) \in K^{n-1} : (\bar{a}, b) \in X \text{ for some } b \in K\} \text{ is constructible.}$$

Compare $X = \{(x,y) \in \mathbb{R}^2 : x = y^2\}$. Then $Y = \mathbb{R}^{\geq 0}$.

Proof: Note that $X \subseteq K^n$ is constructible iff there is a q.f. formula

$\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\underbrace{b_1, \ldots, b_m}_{\text{coefficients}} \in K$ st X is defined by $\varphi(\bar{x}, \bar{b})$.

Fix q.f. $\varphi(\bar{x}, \bar{y})$ and $\bar{b}$ st $\varphi(\bar{x}, \bar{b})$ defines X.

Let $\psi(x_1, \ldots, x_{n-1}, \bar{y})$ be $\exists x_n \varphi(\bar{x}, \bar{y})$. Then $\psi(\bar{x}, \bar{b})$ defines Y.

By QE $\psi(\bar{x}, \bar{y})$ is equivalent to some q.f. formula. So Y is constructible. $\square$.

<u>Rado Graphs</u> [Work with graph language $\mathcal{L} = \{E\}$, E is a binary relation.]

<u>Def 7.4</u> A <u>Rado graph</u> is a graph (V, E) st $V \neq \emptyset$ and for any finite disjoint $X, Y \subseteq V$ there is some $v \in V$, st $E(v, x)$ $\forall$ $x \in X$ and $\neg E(v, y)$ $\forall$ $y \in Y$.

<u>Def 7.5</u> Let RG be the theory of Rado graphs, i.e.,

axioms for graphs: $\forall x \neg E(x, x)$ and $\forall x \forall y (E(x, y) \to E(y, x))$.

Rado axioms: for any $k \geq 1$,

$$\forall x_1 \ldots x_k \forall y_1, \ldots, y_k \left(\bigwedge_{i,j} x_i \neq y_j \to \exists v \left(\bigwedge_{i=1}^{k} E(x_i, v) \wedge \bigwedge_{i=1}^{k} \neg E(y_i, v) \right) \right).$$

<u>Theorem 7.6</u> RG is $\aleph_0$-categorical.

<u>Proof</u>

1) RG has a (countable) model.

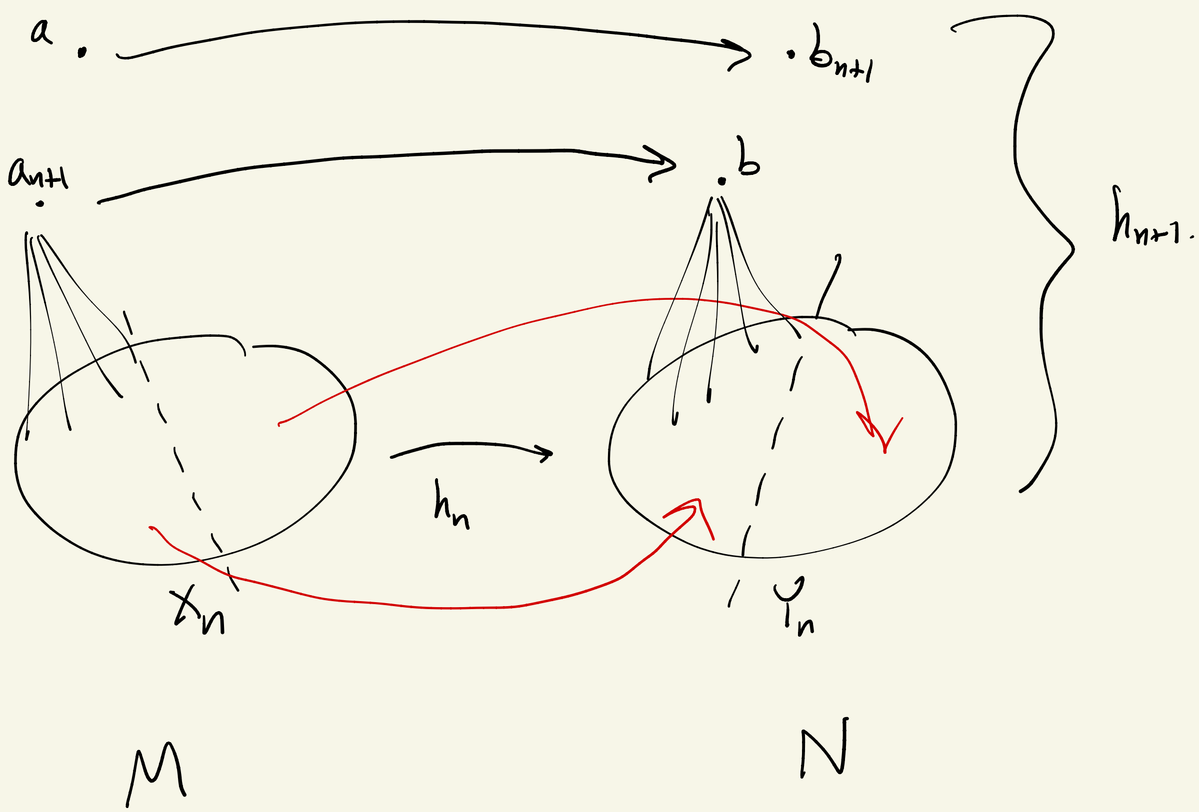
Let $A = (V, E)$ be any finite graph. Set $A_0 = A$. Given A_n, define $V(A_{n+1}) = V(A_n) \cup \{v_{X,Y} : X, Y \subseteq V(A_n) \text{ disjoint}\}$, with new edges $E(v_{X,Y}, x)$ for all $x \in X$ (and no others). So $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$

Let $M = \bigcup_{n=0}^{\infty} A_n$. Then $M \models RG$.

2) Fix $\mathcal{M}, \mathcal{N} \models RG$ ctble. We show $\mathcal{M} \cong \mathcal{N}$ via back + forth.

Enumerate $M = \{a_n : n \geq 0\}$ and $N = \{b_n : n \geq 0\}$. Let $h_0 : a_0 \mapsto b_0$.

Given $h_n : X_n \to Y_n$. Extend to include a_{n+1} and b_{n+1}.



$\square$

Corollary 7.7 RG is complete

Proof: Vaught's Test. Note that RG has no finite models.

Claim: If $\mathcal{M} \models RG$ then every finite graph is an induced subgraph of $\mathcal{M}$.

Pf: The proof of Thm 7.6 shows this when $\mathcal{M}$ is countable.

For any $\mathcal{M} \models RG$ $\exists$ ctble $\mathcal{M}_0$ st $\mathcal{M}_0 \preceq \mathcal{M}$ by DLST (ES1 #9)

Exercise: Suppose $\mathcal{M}, \mathcal{N} \models RG$ ctble and $f : X \to Y$ is a graph isomorphism for some finite $X \subseteq \mathcal{M}$ and $Y \subseteq N$. Then f extends to an isomorphism from $\mathcal{M}$ to $\mathcal{N}$.