Theorem 8.1 RG has QE

Proof: Option 1: Thm 6.2 (iii) $RG \cup D(A)$ ← finite graph (see last Exercise)

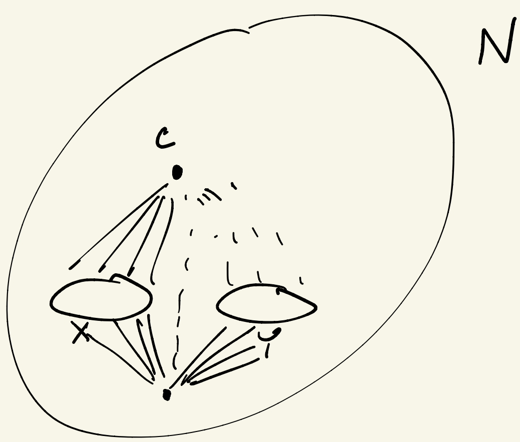
Option 2: Thm 6.2 (ii). Fix $\mathcal{M}, \mathcal{N} \models RG$ and $A \subseteq \mathcal{M} \cap \mathcal{N}$. Fix a q.f. formula $\varphi(x_1, \dots, x_n, y)$ and $a_1, \dots, a_n \in A$. Assume $\mathcal{M} \models \varphi(\bar{a}, b)$ for some $b \in M$

WTS: $\mathcal{N} \models \exists y \varphi(\bar{a}, y)$. Write $\varphi(\bar{x}, y)$ as $\bigvee_{s=1}^{k} \bigwedge_{t=1}^{l_s} \theta_{s,t}(\bar{x}, y)$ where each $\theta_{s,t}$ is atomic or negated atomic (disjunctive normal form). $\exists\ s \leq k$ st

$\mathcal{M} \models \bigwedge_{t=1}^{l_s} \theta_{s,t}(\bar{a}, b)$. Each $\theta_{s,t}$ is one of: $x_i = x_j$, $x_i = y$, $E(x_i, x_j)$, $E(x_i, y)$, or the negation of one of these. If we have $x_i = y$ appearing then $b = a_i \in A \subseteq N$. So $\mathcal{N} \models \varphi(\bar{a}, b)$ (since φ is q.f.). We can assume no $x_i = y$ appears. Let $X = \{a_i : \mathcal{M} \models E(a_i, b)\}$ and $Y = \{a_i : \mathcal{M} \models \neg E(a_i, b)\}$.

X and Y are finite disjoint subsets of $A \subseteq N$. Choose $c \in N$ st

$\mathcal{N} \models E(a_i, c)$ iff $a_i \in X$ and $c \notin \{a_1, \dots, a_n\}$.



Then $\mathcal{N} \models \bigwedge_t \theta_{s,t}(\bar{a}, c)$. So $\mathcal{N} \models \varphi(\bar{a}, c)$. $\square$

Types $\mathcal{L}$ is a language

Motivation: Given $\mathcal{M}$, we want to understand "potential behavior" of elements in elementary extensions.

Terminology: Given an $\mathcal{L}$-structure $\mathcal{M}$ and $A \subseteq M$, we call an $\mathcal{L}_A$-formula an $\mathcal{L}$-formula with parameters from A. We write these as $\varphi(\bar{x}, \bar{a})$ where $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{a}$ is from A. (Identify a with $a^{\mathcal{M}}$) $\left[\mathcal{M}_A \leq \mathcal{N}_A \quad a^{\mathcal{M}} = a^{\mathcal{N}}\right]$

Now suppose $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{N} \succeq \mathcal{M}$. If $a \in N \setminus M$ then the $\mathcal{L}_N$-formula $x = a$ describes the new behavior in a trivial way.

OTOH: If $\varphi(x)$ is a $\mathcal{L}$-formula with parameters from M and $\mathcal{N} \models \varphi(a)$ for some $a \in N$, then $\mathcal{N} \models \exists x \varphi(x)$ so $\mathcal{M} \models \exists x \varphi(x)$.

Idea: New behavior can't be controlled with one formula at time.

Notation: Let p be a set of formulas in free variables $x_1, \dots, x_n$. We also write $p(x_1, \dots, x_n)$. Given $\mathcal{M}$ and $a_1, \dots, a_n \in M$, we write $\mathcal{M} \models p(a_1, \dots, a_n)$ if $\mathcal{M} \models \varphi(\bar{a})$ for all $\varphi \in p$. We say "$\bar{a}$ realizes p (in $\mathcal{M}$)". Also write $\bar{a} \models p$.

Call p consistent if it is realized in some structure.

Exercise: p is consistent iff every finite subset of p is consistent.

Def 8.2 Let $\mathcal{M}$ be an $\mathcal{L}$-structure and fix $A \subseteq M$. An n-type over A wrt $\mathcal{M}$ is a set p of $\mathcal{L}$-formulas with parameters from A in free variables $x_1, \dots, x_n$ st

$$p \cup \mathrm{Th}_A(\mathcal{M}) \text{ is consistent.}$$

p is complete if for every $\mathcal{L}_A$-formula $\varphi(x_1, \dots, x_n)$, either $\varphi \in p$ or $\neg\varphi \in p$.

Let $S_n^{\mathcal{M}}(A)$ denote the set of all complete n-types over A wrt $\mathcal{M}$.

Example 8.3 Given $a_1, \dots, a_n \in M$, let $tp^{\mathcal{M}}(a_1,\dots,a_n/A)$ be the set of all $\mathcal{L}_A$-formulas $\varphi(x_1,\dots,x_n)$ st $\mathcal{M} \models \varphi(\bar{a})$. Then $tp^{\mathcal{M}}(\bar{a}/A) \in S_n^{\mathcal{M}}(A)$ and $\bar{a} \models tp^{\mathcal{M}}(\bar{a}/A)$.

Proposition 8.4 If $p \in S_n^{\mathcal{M}}(A)$ then there is $\mathcal{N} \succeq \mathcal{M}$, with $|N| \leq |M| + |\mathcal{L}| + \aleph_0$ and $\bar{a} \in N^n$ st $p = tp^{\mathcal{N}}(\bar{a}/A)$ (i.e., p is realized in $\mathcal{N}$).

Proof: By assumption, $p \cup Th_A(\mathcal{M})$ is consistent. WTS $p \cup Th_M(\mathcal{M})$ is consistent. Fix $\Sigma \subseteq p \cup Th_M(\mathcal{M})$ finite. So $\Sigma \subseteq p \cup \{\varphi_1, \dots, \varphi_t\}$ where φ_i is an $\mathcal{L}_M$-sentence and $\mathcal{M} \models \varphi_i$. Let φ^* be $\bigwedge_{i=1}^{t} \varphi_i$. We can write φ^* as $\varphi(\underline{b}_1, \dots, \underline{b}_m)$ where $b_1, \dots, b_m \in M \setminus A$ and $\varphi(x_1, \dots, x_n)$ is an $\mathcal{L}_A$-formula. Since $\mathcal{M} \models \varphi(b_1, \dots, b_m)$ so $\mathcal{M} \models \exists \bar{v}\, \varphi(v_1, \dots, v_m)$. So $\exists \bar{v}\, \varphi(\bar{v}) \in Th_A(\mathcal{M})$. Since $p \cup Th_A(\mathcal{M})$ is consistent, there is $\mathcal{N} \models Th_A(\mathcal{M})$ and $\bar{a} \in N^n$ st $\mathcal{N} \models p(\bar{a})$. Since $\mathcal{N} \models \exists \bar{v}\, \varphi(\bar{v})$, there is $\bar{c} \in N^m$ st $\mathcal{N} \models \varphi(\bar{c})$. Expand $\mathcal{N}$ to an $\mathcal{L}_M$-structure st $\underline{b}_i^{\mathcal{N}} = c_i$ and $\underline{b}^{\mathcal{N}}$ is arbitrary for $b \in M \setminus (A \cup \{b_1, \dots, b_m\})$. Then $\mathcal{N} \models \varphi(\underline{b}_1, \dots, \underline{b}_m)$, i.e. $\mathcal{N} \models \varphi^*$. So $\mathcal{N} \models \Sigma$. $\square$