Part III Model Theory, Lecture 9, 30 Oct

Recall Start with p, a set of $\mathcal{L}_A$-formulas in vbles $x_1, \dots, x_n$

p is an n-type over A wrt $\mathcal{M}$ iff $p \cup Th_A(\mathcal{M})$ is consistent.

$S_n^{\mathcal{M}}(A)$: complete n-types over A wrt $\mathcal{M}$

$p \in S_n^{\mathcal{M}}(A) \Rightarrow \exists\, \mathcal{N} \succeq \mathcal{M}$ and $\bar{a} \in N^n$ st $\bar{a} \models p$.

Remark 9.1 If $\mathcal{M} \preceq \mathcal{N}$ and $A \subseteq M$, then $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$.

Proof: ETS: $Th_A(\mathcal{M}) = Th_A(\mathcal{N})$. If $\varphi(x_1, \dots, x_m)$ is an $\mathcal{L}$-formula $a_1, \dots, a_m \in A$ then $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(\bar{a})$ since $\mathcal{M} \preceq \mathcal{N}$.

Remark 9.2: p is an n-type over A wrt $\mathcal{M}$ iff for any finite $q \subseteq p$ $\exists\, \bar{a} \in M^n$ st $\bar{a} \models q$.

Proof: ($\Rightarrow$): Choose $\mathcal{N} \succeq \mathcal{M}$ realizing p. Fix finite $q \subseteq p$. Let $\varphi(\bar{x})$ be the conjunction of all $\mathcal{L}_A$-formulas in q. $\mathcal{N} \models \underbrace{\exists \bar{x}\, \varphi(\bar{x})}_{\mathcal{L}_A\text{-sentence}}$. So $\mathcal{M} \models \exists \bar{x} \varphi(\bar{x})$ since $\mathcal{N} \succeq \mathcal{M}$. $\square$

Example 9.3 Suppose $K \models ACF$ and $A \subseteq K$. Describe $S_n^K(A)$.

Fix $p \in S_n^K(A)$. By QE, we only need to consider q.f. formulas in p.

Note: $\varphi \wedge \psi \in p$ iff $\varphi, \psi \in p$, and $\neg\varphi \in p$ iff $\varphi \notin p$. So it suffices to focus on atomic formulas, in variables $x_1, \dots, x_n$ with parameters from A, i.e., polynomial equations in $F[\bar{x}]$ where F is the subfield generated by A.

Let $I_p = \{ f(\bar{x}) \in F[\bar{x}] : f(\bar{x}) = 0 \text{ is in } p \}$. Then I_p is a prime ideal.

In fact, $p \mapsto I_p$ is a bijection between $S_n^K(A)$ and the set of prime ideals in $F[\bar{x}]$ (ie., $Spec(F[\bar{x}])$). E.g. $S_1^K(K) = \{ p_a : a \in K \} \cup \{ q \}$ where p_a contains $x = a$, and q contains $x \neq a$ $\forall\, a \in K$. $|S_1^K(K)| = |K|$

Example 9.4 $\mathcal{M} \models RG$. Describe $S_1^{\mathcal{M}}(M)$

For $a \in M$, let $p_a \in S_1^{\mathcal{M}}(M)$ be unique type containing $x = a$.

[Why unique? Suppose $x=a$ is in p, q distinct. Choose $\varphi(x)$ st $\varphi(x) \in p$ and $\neg\varphi(x) \in q$

Then $x = a \wedge \varphi(x)$, $x = a \wedge \neg\varphi(x)$ both consistent. $\lightning$.]

For $V \subseteq M$, set

$$p_V = \{x \neq a : a \in M\} \cup \{E(x,a) : a \in V\} \cup \{\neg E(x,a) : a \in M \setminus V\}.$$

Then p_V is a 1-type wrt $\mathcal{M}$, and determines a unique complete 1-type by QE

So $S_1^{\mathcal{M}}(M) = \{p_a : a \in M\} \cup \{p_V : V \subseteq M\}$. $|S_1^{\mathcal{M}}(M)| = 2^{|M|}$.

Note: In general, $|S_n^{\mathcal{M}}(A)| \leq 2^{|A| + |\mathcal{L}| + \aleph_0}$

Type Spaces

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$. Given an $\mathcal{L}_A$-formula $\varphi(x_1, \ldots, x_n)$, define

$$[\varphi(\bar{x})] = \{p \in S_n^{\mathcal{M}}(A) : \varphi(\bar{x}) \in p\}.$$

Basic Properties

1. $S_n^{\mathcal{M}}(A) = \left[\bigwedge_{i=1}^{n} x_i = x_i\right]$

2. $[\varphi(\bar{x}) \wedge \psi(\bar{x})] = [\varphi(\bar{x})] \cap [\psi(\bar{x})]$

3. $[\neg\varphi(\bar{x})] = S_n^{\mathcal{M}}(A) \setminus [\varphi(\bar{x})]$

Define a topology on $S_n^{\mathcal{M}}(A)$ by using $[\varphi(\bar{x})]$ for all $\mathcal{L}_A$-formulas $\varphi(\bar{x})$ as a basis of open sets. (S is for "Stone"; see: Stone space).

Theorem 9.5 $S_n^{\mathcal{M}}(A)$ is a totally disconnected compact Hausdorff space.

Proof: You verify topology is well-defined (ES2 #7)

Hausdorff: Fix distinct $p, q \in S_n^{\mathcal{M}}(A)$. Find $\varphi(\bar{x})$ st $\varphi(\bar{x}) \in p$ and $\neg\varphi(\bar{x}) \in q$.

Then $p \in [\varphi(\bar{x})]$ and $q \in [\neg\varphi(\bar{x})]$.

Compact: It suffices to consider open covers consisting of basic open sets.

So fix $\mathcal{L}_A$-formulas $(\varphi_i(\bar{x}))_{i \in I}$ st $S_n^{\mathcal{M}}(A) = \bigcup_{i \in I} [\varphi_i(\bar{x})]$.

Let $\Sigma = \{\neg\varphi_i(\bar{x}) : i \in I\}$. Then $\Sigma \cup Th_A(\mathcal{M})$ is inconsistent.

Otherwise, $\mathcal{N} \models Th_A(\mathcal{M})$ and $\bar{a} \in N^n$ st $\bar{a} \models \Sigma$. Let $p = tp^{\mathcal{N}}(\bar{a}/A)$

Then $p \in S_n^{\mathcal{M}}(A)$ but $p \notin [\varphi_i(x)]$ $\forall i \in I$, $\lightning$.

By Compactness, there is finite $I_0 \subseteq I$ st $\{\neg\varphi_i(\bar{x}) : i \in I_0\} \cup Th_A(\mathcal{M})$ is inconsistent. ($\star$)

We show $S_n^{\mathcal{M}}(A) = \bigcup_{i \in I_0} [\varphi_i(\bar{x})]$. Fix $p \in S_n^{\mathcal{M}}(A)$. Choose $\mathcal{N} \models Th_A(\mathcal{M})$

and $\bar{a} \in N^n$ st $\bar{a} \models p$. By ($\star$), $\exists$ $i \in I_0$ st $\mathcal{N} \models \varphi_i(\bar{a})$.

So $\varphi_i(\bar{x}) \in p$ (since p is complete). So $p \in [\varphi_i(\bar{x})]$.

Totally Disconnected: A compact Hausdorff space is totally disconnected iff

any two distinct points can be separated by clopen sets. Note $[\varphi(\bar{x})]$ are clopen. $\square$