Part III Model Theory, Lecture 10, 30 Nov

Long term goal: Analyze countable models & complete theories.

Ex: DLO, RG are $\aleph_0$-categorical. ACF_p: countable models are K_α for $\alpha \in \mathbb{N} \cup \{\aleph_0\}$

where K_α has trans. degree α.

Saturated Models

Def 10.1 Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure & let $\kappa \geq |\mathcal{L}| + \aleph_0$. Then $\mathcal{M}$ is <u>κ-saturated</u> if for any $A \subseteq M$, with $|A| < \kappa$, every type in $S_n^{\mathcal{M}}(A)$ is realized in $\mathcal{M}$ for all $n \geq 1$.

Remark 10.2

a) Restricting to complete types is not important since any n-type over A wrt $\mathcal{M}$ can be extended to some $p \in S_n^{\mathcal{M}}(A)$ (ES2 #6)

b) (ES2 #8) It suffices to assume $n=1$ to prove κ-saturation.

c) If $\mathcal{M}$ is κ-saturated then $|M| \geq \kappa$.

Pf: $\{x \neq a : a \in M\}$ is a 1-type over M wrt $\mathcal{M}$, not realized in $\mathcal{M}$.

Def 10.3 Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures and $A \subseteq M$, $B \subseteq N$. Then $f: A \to B$ is <u>partial elementary</u> if for any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in A$,

$$\mathcal{M} \models \varphi(\bar{a}) \text{ iff } \mathcal{N} \models \varphi(f(\bar{a})).$$

Given $\kappa \geq |\mathcal{L}| + \aleph_0$, $\mathcal{M}$ is κ-<u>homogeneous</u> if for any $A \subseteq M$, with $|A| < \kappa$, any partial elementary $f: A \to M$, and any $c \in M$, $\exists\, d \in M$ st $f \cup \{(c, d)\}$ is partial elementary.

Let T be a complete $\mathcal{L}$-theory with infinite models. Fix $\mathcal{M}, \mathcal{N} \models T$

Then $S_n^{\mathcal{M}}(\emptyset) = S_n^{\mathcal{N}}(\emptyset)$ since $Th(\mathcal{M}) = Th(\mathcal{N}) = T$

Def 10.4 $S_n(T) := S_n^{\mathcal{M}}(\emptyset)$ for some/any $\mathcal{M} \models T$.

Proposition 10.5 $\mathcal{M} \models T$ is $\aleph_0$-saturated iff $\mathcal{M}$ is $\aleph_0$-homogeneous and $\mathcal{M}$ realizes all types in $S_n(T)$ $\forall\, n \geq 1$.

Proof ($\Longrightarrow$) Assume $\mathcal{M} \models T$ is $\aleph_0$-saturated. Then $\mathcal{M}$ realizes all types in $S_n(T)$ since $\emptyset$ is finite. Fix finite $A \subset \mathcal{M}$, partial elementary $f: A \to \mathcal{M}$, and $c \in \mathcal{M}$.

Define $p \in S_1^{\mathcal{M}}(f(A))$ st $\varphi(x, f(\bar{a})) \in p$ iff $\mathcal{M} \models \varphi(c, \bar{a})$.

Notation: $f(tp^{\mathcal{M}}(c/A)) = p$. $p \in S_1^{\mathcal{M}}(f(A))$. e.g. p is finitely satisfiable in $\mathcal{M}$:

If $\varphi(x, f(\bar{a})) \in p$ then $\mathcal{M} \models \exists x\, \varphi(x, \bar{a})$ so $\mathcal{M} \models \exists x\, \varphi(x, f(\bar{a}))$.

Let $d \in M$ realize p. Then $f \cup \{(c,d)\}$ is partial elementary.

($\Longleftarrow$) Fix $a_1, \ldots, a_n \in M$ and $p \in S_1^{\mathcal{M}}(\{a_1, \ldots, a_n\})$. WTS $\mathcal{M}$ realizes p.

Set $q = \{\varphi(x, y_1, \ldots, y_n) : \varphi(x, \bar{a}) \in p\}$. Then $q \in S_{n+1}(T)$. Let $d, b_1, \ldots, b_n \in M$ st $(d, \bar{b}) \models q$. Then $\underbrace{tp^{\mathcal{M}}(\bar{b})}_{tp^{\mathcal{M}}(\bar{b}/\emptyset)} = tp^{\mathcal{M}}(\bar{a})$. So $f: b_i \to a_i$ $\forall i$ is partial elementary.

Let $c \in M$ st $f \cup \{(d,c)\}$ is partial elementary. Then $tp^{\mathcal{M}}((c, \bar{a})) = tp^{\mathcal{M}}((d, \bar{b})) = q$.

So $(c, \bar{a}) \models q$ i.e. $c \models p$. $\square$

Notation: Given $\mathcal{M}$, $\bar{a}, \bar{b} \in M^n$, write $\bar{a} \equiv^{\mathcal{M}} \bar{b}$ if $tp^{\mathcal{M}}(\bar{a}) = tp^{\mathcal{M}}(\bar{b})$.

So $\mathcal{M}$ is $\aleph_0$-hom. iff whenever $\bar{a} \equiv^{\mathcal{M}} \bar{b}$ and $c \in M$, $\exists\, d \in M$ st $(\bar{a}, c) \equiv^{\mathcal{M}} (\bar{b}, d)$.

Lemma 10.6 For any $\mathcal{M} \models T$ there is $\mathcal{N} \succeq \mathcal{M}$ st $|N| \leq |M| + |\mathcal{L}|$ and $\mathcal{N}$ is $\aleph_0$-homogeneous.

Proof

Claim: For any $\mathcal{M} \models T$, there is $\mathcal{N} \succeq \mathcal{M}$ st $|N| \leq |M| + |\mathcal{L}|$ and $\forall\, \bar{a}, \bar{b}, c$ from M, st $\bar{a} \equiv^{\mathcal{M}} \bar{b}$, $\exists\, d \in N$ $(\bar{a}, c) \equiv^{\mathcal{N}} (\bar{b}, d)$.

Proof: Enumerate all $(\bar{a}, \bar{b}, c)$ as $(\bar{a}_\alpha, \bar{b}_\alpha, c_\alpha)_{\overset{\text{ordinals}}{\alpha} < |M|}$. We build an elementary chain $(\mathcal{M}_\alpha)_{\alpha < |M|}$ st $\mathcal{M}_0 = \mathcal{M}$ and $|M_\alpha| \leq |M| + |\mathcal{L}|$ $\forall \alpha$.

For α limit, let $\mathcal{M}_\alpha = \bigcup_{i<\alpha} \mathcal{M}_i$ (ES1 #8). Then $|M_\alpha| \leq |\alpha|(|M| + |\mathcal{L}|) = |M| + |\mathcal{L}|$.

Given $\mathcal{M}_\alpha$, look at $(\bar{a}_\alpha, \bar{b}_\alpha, c_\alpha)$. We have $\bar{a}_\alpha \equiv^{\mathcal{M}} \bar{b}_\alpha$. Let $f_\alpha : \bar{a}_\alpha \to \bar{b}_\alpha$ be partial elementary. Apply Prop 8.4 to find $\mathcal{M}_{\alpha+1} \succeq \mathcal{M}_\alpha$ st $|M_{\alpha+1}| \leq |M_\alpha| + |\mathcal{L}| \leq |M| + |\mathcal{L}|$, and $\exists\, d \in M_{\alpha+1}$ realizing $f_\alpha(tp(c_\alpha / \bar{a}_\alpha))$. Then $(\bar{a}_\alpha, c_\alpha) \equiv^{\mathcal{M}} (\bar{b}_\alpha, d)$. Let $\mathcal{N} = \bigcup_{\alpha < |M|} \mathcal{M}_\alpha$. Then $|N| \leq |M|(|M| + |\mathcal{L}|) = |M| + |\mathcal{L}|$. //

We now build $\mathcal{M} = \mathcal{N}_0 \preceq \mathcal{N}_1 \preceq \mathcal{N}_2 \preceq \cdots$ st $|N_i| \leq |M| + |\mathcal{L}|$ and $\forall\, \bar{a}, \bar{b}, c$ from N_i if $\bar{a} \equiv \bar{b}$ then $\exists\, d \in N_{i+1}$ st $(\bar{a}, c) \equiv (\bar{b}, d)$. Do this by iterating the claim. Let $\mathcal{N} = \bigcup_{i < \aleph_0} \mathcal{N}_i$. Then $|N| \leq |M| + |\mathcal{L}|$.

$\mathcal{N}$ is $\aleph_0$-homogeneous: Any $\bar{a}, \bar{b}, c$ from N are in N_i for some i. $\square$