Part III Model Theory, Lecture 11, 4 Nov

Recall $\mathcal{M}$ is κ-saturated $\Rightarrow |M| \geq \kappa$

Def 11.1 $\mathcal{M}$ is <u>saturated</u> if it is $|M|$-saturated.

Let T be a complete consistent theory with infinite models and $\mathcal{L}$ is ctble.

Theorem 11.2 T has a countable saturated model iff $S_n(T)$ is ctble $\forall n \geq 1$.

Proof ($\Rightarrow$) If $\mathcal{M} \models T$ is ctble + saturated then $S_n(T)$ is ctble since M^n is ctble and $p \mapsto \bar{x} \models p$ is injective. ($p \neq q$ $\varphi(\bar{x}) \in p$ $\neg\varphi(\bar{x}) \in q$).

($\Leftarrow$) Enumerate $\bigcup_{n \geq 1} S_n(T) = \{p_1, p_2, p_3, \dots\}$. Fix $\mathcal{M}_0 \models T$ ctble. Build chain $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \mathcal{M}_2 \preceq \dots$ st $\mathcal{M}_i$ realizes p_i and is ctble (by Prop 8.4).

Let $\mathcal{N} = \bigcup_{n \geq 1} \mathcal{M}_n$. $\mathcal{N} \models T$ is ctble. Apply Lemma 10.6 to get $\mathcal{M} \succeq \mathcal{N}$ ctble and $\aleph_0$-homogeneous. So $\mathcal{M}$ is saturated by Prop 10.5. $\square$

Example 11.3

1. ACF_p. Let $F = \begin{cases} \mathbb{Q} & p = 0 \\ \mathbb{F}_p & p > 0 \end{cases}$. Then $S_n(T) \sim \mathrm{Spec}(F[x_1, \dots, x_n])$

   So $S_n(T)$ is ctble since every ideal in $F[\bar{x}]$ is fin. gen. So ACF_p has a ctble saturated model, which is the model of transcendence degree $\aleph_0$. $\overline{F[x_1, x_2, \dots]}$. Note if $K \models ACF_p$ + trdeg$(K) = n < \aleph_0$ then the $(n+1)$-type saying "$x_1, \dots, x_{n+1}$ algebraically independent" not realized in K.

2. TFDAG has a ctble saturated model, which is the $\mathbb{Q}$-vector space of dim $\aleph_0$.

3. Let $T = \mathrm{Th}(\mathbb{Z}, +, 0)$. Given $n \geq 1$, let $\delta_n(x)$ be the $\mathcal{L}$-formula $\exists y (x = ny)$

   Let $\mathbb{P}$ be the set of primes. Given $X \subseteq \mathbb{P}$, let $q_X = \{\delta_n(x) : n \in X\} \cup \{\neg\delta_n(x) : n \in \mathbb{P} \setminus X\}$.

   Note q_X is fin. sat. in $\mathbb{Z}$. So $\exists\, p_X \in S_1(T)$ st $q_X \subseteq p_X$.

   If $X \neq Y$ then $p_X \neq p_Y$ so $|S_1(T)| = 2^{\aleph_0}$.

So T does not have a ctble saturated model.

Proposition 11.4 If $\mathcal{M}, \mathcal{N} \models T$ are ctble and saturated, then $\mathcal{M} \cong \mathcal{N}$.

Proof (Sketch)

Enumerate $\mathcal{M} = \{a_n : n \geq 1\}$, $N = \{b_n : n \geq 1\}$. Build partial elementary $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ st $a_n \in dom(f_n)$, $b_n \in Im(f_n)$, $dom(f_n)$ is finite.

Let $f_0 = \emptyset$ (partial elementary since $\mathcal{M} \equiv \mathcal{N}$)

Given f_n. Let $d \in N$ realize $f_n\left(tp\left({a_{n+1}}/{dom(f_n)}\right)\right)$. Now let $c \in M$ realize $f_*^{-1}\left(tp\left({b_{n+1}}/{Im(f_n) \cup \{d\}}\right)\right)$ where $f_* = f_n \cup \{(a_{n+1}, d)\}$.

Let $f_{n+1} = f_* \cup \{(c, b_{n+1})\}$. Let $f = \bigcup f_n$. Then f is an $\mathcal{L}$-isomorphism from $\mathcal{M}$ to $\mathcal{N}$. $\square$

Omitting Types

Let $\mathcal{M}$ be an $\mathcal{L}$-structure.

Def 11.5: $p \in S_n^{\mathcal{M}}(A)$ is <u>isolated</u> if it is an isolated point wrt topology, ie. $\{p\}$ is open.

Ex: If $a \in A \subseteq M$, then $tp^{\mathcal{M}}(a/A)$ is isolated since $\{tp^{\mathcal{M}}(a/A)\} = [x=a]$.

Proposition 11.6 Given $p \in S_n^{\mathcal{M}}(A)$. TFAE.

i) p is isolated.

ii) $\{p\} = [\varphi(\bar{x})]$ for some $\mathcal{L}_A$-formula $\varphi(\bar{x})$, (we say $\varphi(\bar{x})$ <u>isolates</u> p).

iii) There is an $\mathcal{L}_A$-formula $\varphi(\bar{x}) \in p$ st for any $\mathcal{L}_A$-formula $\psi(\bar{x})$, if $\psi(\bar{x}) \in p$ then $Th_A(\mathcal{M}) \models \forall \bar{x} \, (\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$

$\forall \bar{x} \, (\underbrace{\bar{x} \models p}_{\varphi(\bar{x})} \rightarrow \psi(\bar{x}))$

<u>Proof</u> (i) $\iff$ (ii). By def. of the basis for the topology.

(ii) $\Rightarrow$ (iii). Assume $\varphi(\bar{x})$ isolates p. Fix an $\mathcal{L}_A$-formula $\psi(\bar{x}) \in p$.

WTS $\mathcal{M} \models \forall \bar{x} (\varphi(\bar{x}) \to \psi(\bar{x}))$. Suppose $\bar{a} \in M^n$ st $\mathcal{M} \models \varphi(\bar{a})$.

Then $tp^{\mathcal{M}}(\bar{a}/A) \in [\varphi(\bar{x})]$, so $p = tp^{\mathcal{M}}(\bar{a}/A)$. So $\mathcal{M} \models \psi(\bar{a})$.

(iii) $\Rightarrow$ (ii). Assume (iii). Then $\forall$ $\mathcal{L}_A$-formula $\psi(\bar{x}) \in p$, we have

$[\varphi(\bar{x})] \subseteq [\psi(\bar{x})]$ since any $q \in [\varphi(\bar{x})]$ is realized by $\bar{a} \in N^n$

in some $\mathcal{N} \models Th_A(\mathcal{M})$. So $\mathcal{N} \models \psi(\bar{a})$. So $q \in [\psi(\bar{x})]$

If $q \in [\varphi(\bar{x})]$ then $p \subseteq q$. So $p = q$. So $[\varphi(\bar{x})] = \{p\}$.