Part III Model Theory, Lecture 12, 6 Nov

Let T be a complete consistent theory.

<u>Proposition 12.1</u> If $p \in S_n(T)$ is isolated then p is realized in any $\mathcal{M} \models T$.

<u>Proof</u>: Fix $p \in S_n(T)$ isolated by $\varphi(\bar{x}) \in p$. Fix $\mathcal{M} \models T$. By Prop 8.4, there is $\mathcal{N} \succeq \mathcal{M}$ realizing p. So $\mathcal{N} \models \exists \bar{x}\, \varphi(\bar{x})$. So $\mathcal{M} \models \exists \bar{x}\, \varphi(\bar{x})$. Fix $\bar{a} \in M^n$ st $\mathcal{M} \models \varphi(\bar{a})$. We show $\bar{a} \models p$. Fix $\psi(\bar{x}) \in p$. Then $T \models \forall \bar{x}\,(\varphi(\bar{x}) \to \psi(\bar{x}))$. So $\mathcal{M} \models \psi(\bar{a})$. $\square$

<u>Omitting Types Theorem</u>

Assume $\mathcal{L}$ is countable and $p \in S_n(T)$ is non-isolated. Then there is ctble $\mathcal{M} \models T$ st p is not realized in $\mathcal{M}$ (ie. $\mathcal{M}$ <u>omits</u> p).

<u>Proof</u> (Henkin construction; non-examinable)

Let $\mathcal{L}^* = \mathcal{L} \cup C$ where C is a ctbly infinite set of new constant symbols.

An $\mathcal{L}^*$-theory T^* has <u>witness property</u> if for any $\mathcal{L}^*$-formula $\varphi(x)$, there is a constant $c \in C$ st $T^* \models (\exists x\, \varphi(x) \to \varphi(c))$.

<u>Fact</u> (Part II) Suppose T^* is a complete satisfiable $\mathcal{L}^*$-theory with the witness property. Define $\sim$ on C st $c \sim d$ iff $T^* \models c = d$. Let $M = C/\sim$ and define an $\mathcal{L}^*$-structure $\mathcal{M}$ on M st

$$\begin{cases} c^{\mathcal{M}} = [c] \ (\sim\text{-equivalence class}), \quad f^{\mathcal{M}}([c_1], \dots, [c_n]) = [d] \text{ iff } T^* \models f(c_1, \dots, c_n) = d, \\ \text{and } R^{\mathcal{M}} = \{([c_1], \dots, [c_n]) \in M^n : T^* \models R(c_1, \dots, c_n)\} \end{cases}$$

Then $\mathcal{M}$ is a well-defined $\mathcal{L}^*$-structure and $\mathcal{M} \models T^*$. In particular, for any $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and $c_1, \dots, c_n \in C$, $\mathcal{M} \models \varphi([c_1], \dots, [c_n])$ iff $T^* \models \underbrace{\varphi(c_1, \dots, c_n)}_{\mathcal{L}^*\text{-sentence}}$. Call $\mathcal{M}$ the <u>Henkin model</u> of T^*.

Fix $p \in S_n(T)$ non-isolated.

Goal: Build a complete, satisfiable $\mathcal{L}^*$-theory $T^* \supseteq T$ with the witness property st $\forall\, c_1, \ldots c_n \in C$ then there is $\psi(\bar{x}) \in p$ st $T^* \models \neg\psi(c_1, \ldots, c_n)$.

Given such a T^*, the Henkin model omits p.

Enumerate all $\mathcal{L}^*$-sentences $\varphi_0, \varphi_1, \varphi_2, \ldots$, and $C^n = \{\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots\}$

We build a satisfiable $\mathcal{L}^*$-theory $T^* = T \cup \{\theta_0, \theta_1, \theta_2, \ldots\}$ st

0) $\models \theta_i \rightarrow \theta_j \ \forall\, i > j$ (convenience)

1) Either $\models \theta_{3i+1} \rightarrow \varphi_i$ or $\models \theta_{3i+1} \rightarrow \neg\varphi_i$ (completeness)

2) If φ_i is $\exists v \psi(v)$ for some ψ and $\models \theta_{3i+1} \rightarrow \varphi_i$ then

$\models \theta_{3i+2} \rightarrow \psi(c)$ for some $c \in C$ (witness property)

[$T^* \models (\exists v \psi(v) \rightarrow \psi(c))$: $\mathcal{N} \models T^*$ and $\mathcal{N} \models \exists v \psi(v)$. $\mathcal{N} \models \varphi_i$. So $\mathcal{N} \models \psi(c)$.]

3) $\models \theta_{3i+3} \rightarrow \neg\psi(\bar{c}_i)$ for some $\psi(\bar{x}) \in p$ (omit p)

Let θ_0 be $\forall v (v = v)$. Suppose we have $\theta_0, \ldots, \theta_m$ as above.

Case 1: $m+1 = 3i+1$. If $T \cup \{\theta_m, \varphi_i\}$ is satisfiable then θ_{m+1} is $\theta_m \wedge \varphi_i$.

Otherwise, let θ_{m+1} be $\theta_m \wedge \neg\varphi_i$. $T \cup \{\theta_{m+1}\}$ is satisfiable.

Case 2: $m+1 = 3i+2$. Suppose φ_i is $\exists v \psi(v)$ for some ψ and $\models \theta_m \rightarrow \varphi_i$.

[If this fails, let θ_{m+1} be θ_m]. Choose $c \in C$ not used in θ_m. Let θ_{m+1} be $\theta_m \wedge \psi(c)$.

$T \cup \{\theta_{m+1}\}$ is satisfiable: Let $\mathcal{N} \models T \cup \{\theta_m\}$. Then $\mathcal{N} \models \varphi_i$. Choose $a \in N$ st $\mathcal{N} \models \psi(a)$. Re-interpret $c^{\mathcal{N}} = a$. Then $\mathcal{N} \models T \cup \{\theta_{m+1}\}$.

Case 3: $m+1 = 3i+3$. Let $\bar{c}_i = (c_1, \ldots, c_n)$. WLOG assume $x_1, \ldots, x_m$ are not used in θ_m. We build and $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ from θ_m as follows:

- replace c_t by x_t $\forall\, t \le n$
- then replace any $c \in C \setminus \{c_1, \ldots, c_n\}$ by a new variable v_c and add $\exists v_c$ to the front.

Then $\varphi(\bar{x})$ does not isolate p. By Prop 11.6,* $\exists\ \psi(\bar{x}) \in p$ st

$$T \not\models \forall \bar{x}\,(\varphi(\bar{x}) \to \psi(\bar{x})).$$

Let θ_{m+1} be $\theta_m \wedge \neg\psi(c_1, \ldots, c_n)$.

$T \cup \{\theta_{m+1}\}$ is satisfiable: Choose $\mathcal{N} \models T$ st $\mathcal{N} \not\models \forall \bar{x}\,(\varphi(\bar{x}) \to \psi(\bar{x}))$.

Pick $\bar{a} \in N^n$ st $\mathcal{N} \models \varphi(\bar{a}) \wedge \neg\psi(\bar{a})$. Make $\mathcal{N}$ an $\mathcal{L}^*$-structure:

$c_t^{\mathcal{N}}$ as a_t. If $c \in C \setminus \{c_1, \ldots, c_n\}$, then $c^{\mathcal{N}}$ is a witness to $\exists v_c$ in

$\mathcal{N} \models \varphi(\bar{a})$. Then $\mathcal{N} \models \theta_m$ and $\mathcal{N} \models \neg\psi(c_1, \ldots, c_t)$

So $\mathcal{N} \models \theta_{m+1}$. $\square$

*REMARK (added after lecture)
Prop 11.6 technically only says that <u>if $\varphi(\bar{x}) \in p$</u> then there is such a $\psi(\bar{x}) \in p$. But note that if $\varphi(\bar{x}) \notin p$ then $\neg\varphi(\bar{x}) \in p$. So we can let $\psi(\bar{x})$ be $\neg\varphi(\bar{x})$, and then $T \not\models \forall \bar{x}\,(\varphi(\bar{x}) \to \psi(\bar{x}))$ since T is consistent.