Part III Model Theory, Lecture 13, 9 Nov

Prime and Atomic Models

T is complete consistent $\mathcal{L}$-theory with infinite models.

<u>Def 13.1</u> Fix $\mathcal{M} \models T$.

1) $\mathcal{M}$ is <u>atomic</u> if every n-type over $\emptyset$ realized in $\mathcal{M}$ is isolated.

2) $\mathcal{M}$ is <u>prime</u> if for any $\mathcal{N} \models T$ there is an elementary embedding from $\mathcal{M}$ to $\mathcal{N}$.

<u>Ex</u>: $K \models ACF_0$. Then $\overline{\mathbb{Q}} \subseteq K$. So $\overline{\mathbb{Q}} \preceq K$ by QE.

<u>Theorem 13.2</u> Assume $\mathcal{L}$ is countable. Then $\mathcal{M} \models T$ is prime iff it is ctble and atomic.

<u>Proof</u>

($\Longrightarrow$): Assume $\mathcal{M} \models T$ is prime. Then $\mathcal{M}$ is ctble since T has a ctble model (DLST).

Suppose $p \in S_n(T)$ is non-isolated. By OTT there is some $\mathcal{N} \models T$ omitting p.

Since $\mathcal{M} \preceq \mathcal{N}$, $\mathcal{M}$ omits p. So $\mathcal{M}$ is atomic.

($\Longleftarrow$): Assume $\mathcal{M} \models T$ is countable + atomic. Fix $\mathcal{N} \models T$. WTS $\mathcal{M} \preceq \mathcal{N}$.

Enumerate $M = \{a_n : n \geq 1\}$. We build partial elementary $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ from M to N st $a_n \in \operatorname{dom}(f_n)$ and $\operatorname{dom}(f_n)$ is finite. Then $f = \bigcup f_n$ is an elementary embedding from $\mathcal{M}$ to $\mathcal{N}$. Let $f_0 = \emptyset$ ($\mathcal{M} \equiv \mathcal{N}$). Suppose we have f_n. Let $\varphi(x_1, \dots, x_{n+1})$ be an $\mathcal{L}$-formula isolating $tp^{\mathcal{M}}(a_1, a_2, \dots, a_{n+1})$ (exists since $\mathcal{M}$ is atomic).

$\mathcal{M} \models \exists x_{n+1} \varphi(a_1, \dots, a_n, x_{n+1})$. So $\mathcal{N} \models \exists x_{n+1} \varphi(f_n(a_1), \dots, f_n(a_n), x_{n+1})$.

Pick $b \in N$ st $\mathcal{N} \models \varphi(f_n(a_1), \dots, f_n(a_n), b)$. Fix an $\mathcal{L}$-formula $\psi(x_1, \dots, x_{n+1})$ st $\mathcal{M} \models \psi(a_1, \dots, a_{n+1})$. WTS $\mathcal{N} \models \psi(f_n(a_1), \dots, f_n(a_n), b)$.

By Prop 11.6, $T \models \forall x_1 \dots x_{n+1} (\varphi(\bar{x}) \to \psi(\bar{x}))$. So $\mathcal{N} \models \psi(f_n(a_1), \dots, f_n(a_n), b)$.

So $f_{n+1} = f_n \cup \{(a_{n+1}, b)\}$ is partial elementary. $\square$

Theorem 13.3 Assume $\mathcal{L}$ is countable. TFAE.

i) T has a prime model.

ii) T has an atomic model.

iii) For all $n \geq 1$, the isolated types in $S_n(T)$ are dense.

Proof We have (i) $\Leftrightarrow$ (ii) by Thm 13.2 (and ES1 #9).

(ii) $\Rightarrow$ (iii). Let $\mathcal{M} \models T$ be atomic. Fix $n \geq 1$, and an $\mathcal{L}$-formula $\varphi(\bar{x})$ st $[\varphi(\bar{x})] \neq \emptyset$

WTS $[\varphi(\bar{x})]$ contains an isolated type. Note $\mathcal{M} \models \exists \bar{x} \varphi(\bar{x})$. Choose $\bar{a} \in M^n$ st

$\mathcal{M} \models \varphi(\bar{a})$. Then $tp^{\mathcal{M}}(\bar{a})$ is isolated (since $\mathcal{M}$ is atomic) and it is in $[\varphi(\bar{x})]$.

(iii) $\Rightarrow$ (ii). [Henkin construction, non-examinable]

Let $\mathcal{L}^* = \mathcal{L} \cup \{c_1, c_2, c_3, \dots\}$. Let $\varphi_0, \varphi_1, \varphi_2, \dots$ enumerate all $\mathcal{L}^*$-sentences.

We build $T^* = T \cup \{\theta_0, \theta_1, \theta_2, \dots\}$ st T^* is complete, satisfiable, has the witness property, and such that the Henkin model of T^* is atomic. (as an $\mathcal{L}$-structure).

Let θ_0 be $\forall x (x = x)$. Suppose we have $\theta_0, \theta_1, \dots, \theta_m$.

Cases $m+1 \in \{3i+1, 3i+2\}$ are identical to proof of OTT.

<u>Case $m+1 = 3i+3$</u>: Choose $n \geq i$ st all new constants used in θ_m are in $\{c_1, \dots, c_n\}$. Let $\psi(x_1, \dots, x_n)$ be an $\mathcal{L}$-formula st θ_m is $\psi(c_1, \dots, c_n)$

By induction $T \cup \{\theta_m\}$ is consistent. So $T \cup \{\psi(x_1, \dots, x_n)\}$ is consistent, and

so $[\psi(\bar{x})] \neq \emptyset$. By (iii), there is some isolated $p \in [\psi(\bar{x})]$.

Let $\varphi(\bar{x})$ isolate p. Let θ_{m+1} be $\theta_m \wedge \varphi(c_1, \dots, c_n)$.

$T \cup \{\theta_{m+1}\}$ is consistent: Choose $\mathcal{N} \models T$ with $\bar{a} \in N^n$ realizing p.

Expand $\mathcal{N}$ to $\mathcal{L}^*$-structure st $c_i^{\mathcal{N}} = a_i$ (for $i \leq n$). Then $\mathcal{N} \models T \cup \{\theta_{m+1}\}$.

Now let $\mathcal{M} \models T^*$ be the Henkin model. WTS $\mathcal{M}$ is atomic (as an $\mathcal{L}$-structure)

For arbitrarily large n, we have $\varphi(x_1, \dots, x_n)$ isolating $p \in S_n(T)$ st $T^* \models \varphi(c_1, \dots, c_n)$. So $\overbrace{tp^{\mathcal{M}}(c_1^{\mathcal{M}}, \dots, c_n^{\mathcal{M}})}^{=p}$ is isolated $\forall$ $\underbrace{n \geq 1}_{\text{arbitrarily large}}$.

For any tuple $\bar{a}$ from $\mathcal{M}$. WTS $tp^{\mathcal{M}}(\bar{a})$ is isolated.

WLOG. the coordinates of the tuple are distinct

(a, b, c) isolated by $\psi(x_1, x_2, x_3)$

(a, a, b, c) isolated by $\psi(x_1, x_3, x_4) \wedge x_2 = x_3$

So $\bar{a}$ is a sub-tuple of $([c_1], \dots, [c_n])$ for some n

<u>General fact</u>: Given any $\mathcal{M}$ and $a_1, \dots, a_n \in \mathcal{M}$, if $tp^{\mathcal{M}}(a_1, \dots, a_n)$ is isolated by $\varphi(x_1, \dots, x_n)$, then $\forall\ \emptyset \neq I \subseteq \{1, \dots, n\}$,

$tp^{\mathcal{M}}((a_i)_{i \in I})$ is isolated by $(\exists x_i)_{i \notin I}\, \varphi(x_1, \dots, x_n)$. $\Box$