Part III Model Theory, Lecture 14, 11 Nov

T complete theory, ctble language, infinite models

Recall: For any $n \geq 1$, $|S_n(T)| \leq 2^{\aleph_0}$

Lemma 14.1 For any $n \geq 1$, if $|S_n(T)| < 2^{\aleph_0}$ then $S_n(T)$ is ctble + the isolated types are dense.

Proof $S_n(T)$ is a second ctble totally disconnected compact Hausdorff space.
Let X be any such space. We show that if X is uncountable or the isolated points are not dense, then $|X| \geq 2^{\aleph_0}$. Let $\mathcal{B}$ be a ctble basis for X consisting of clopen sets.
Assume $\mathcal{B}$ is closed under intersections and complements.

Case 1: X is uncountable.

Claim: If $U \in \mathcal{B}$ and $|U| > \aleph_0$ then $\exists\, V \in \mathcal{B}$ st $|U \cap V|, |U \setminus V| > \aleph_0$.

Proof: Suppose not. Let $\mathcal{C} = \{V \in \mathcal{B} : |U \cap V| > \aleph_0\}$. Fix $V_1, V_2 \in \mathcal{C}$. Set $W = V_1 \cap V_2$.
If $W \notin \mathcal{C}$ then $|U \setminus W| > \aleph_0$. Note $U \setminus W = U \setminus V_1 \cup U \setminus V_2$. WLOG $|U \setminus V_1| > \aleph_0$.
Contradiction, since $V_1 \in \mathcal{C}$. So $\mathcal{C}$ is a collection of nonempty closed sets and $\mathcal{C}$ is closed under intersections. Since X is compact there is $p \in X$ st $p \in V$ for all $V \in \mathcal{C}$.
We show $U = \{p\} \cup \bigcup_{V \in \mathcal{B} \setminus \mathcal{C}} U \cap V$. So U is ctble. ↯. Fix $q \in U$, st $q \neq p$.
There is $V \in \mathcal{B}$ st $q \in V$ and $p \notin V$. So $V \in \mathcal{B} \setminus \mathcal{C}$. So $q \in U \cap V$. //

Notation: 2^ω is the set of sequences of $0,1$ index by $\mathbb{N}$. $2^{<\omega}$ is the set of finite sequences of $0,1$. We have a partial order on $2^\omega \cup 2^{<\omega}$ given by proper initial segment.

We build $\{U_\sigma\}_{\sigma \in 2^{<\omega}}$ st $\forall\, \sigma \in 2^{<\omega}$, $U_\sigma \in \mathcal{B}$, $|U_\sigma| > \aleph_0$, $U_\sigma = U_{\sigma 0} \cup U_{\sigma 1}$, and $U_{\sigma 0} \cap U_{\sigma 1} = \emptyset$. Let $U_\emptyset = X$. Given U_σ, let $V \in \mathcal{B}$ be as in the Claim.
Let $U_{\sigma 0} = U_\sigma \cap V$ and $U_{\sigma 1} = U_\sigma \setminus V$. Now, for any $\alpha \in 2^\omega$, there is
$p_\alpha \in \bigcap_{i \geq 0} U_{\alpha \restriction i}$. By construction, $\alpha \neq \beta \Rightarrow p_\alpha \neq p_\beta$. So $|X| \geq 2^{\aleph_0}$.

Case 2: The isolated points in X are not dense.
We will build $\{U_\sigma\}_{\sigma \in 2^{<\omega}}$ as above, but just with $U_\sigma \neq \emptyset$.

Let $U_\emptyset$ be a nonempty clopen set with no isolated points. Suppose we have U_σ. U_σ has no isolated points. So $\exists$ distinct $p, q \in U_\sigma$. Partition U_σ into $U_{\sigma 0}$ and $U_{\sigma 1}$ with $p \in U_{\sigma 0}$ and $q \in U_{\sigma 1}$. As before, $|X| \geq 2^{\aleph_0}$. $\square$

Theorem 14.2

a) Suppose $|S_n(T)| < 2^{\aleph_0}$ $\forall n$. Then T has a prime model and a ctble saturated model.

b) If T has a countable saturated model, then T has a prime model.

Proof (a) Apply Lemma 14.1, Theorem 13.3, and Theorem 11.2

(b) Apply Theorem 11.2, Lemma 14.1, Theorem 13.3.

Fact: $Th(\mathbb{Z}, +, 0)$ has no countable saturated model (Ex 11.3(c)), and no prime model (Baldwin, Blass, Glass, Kuecker 1972).

OTOH $Th(\mathbb{Z}, +, 0, 1)$ then there is a prime model and no ctble saturated model.

Def 14.3 For $\kappa \geq \aleph_0$, let $I(T, \kappa)$ be the number of models of T of size κ modulo isomorphism.

Remark: $1 \leq I(T, \kappa) \leq 2^\kappa$ $\leftarrow$ bounds the # of $\mathcal{L}$-structures of size κ.

[Recall Morley's Theorem: If $I(T, \kappa) = 1$ for some $\kappa > \aleph_0$ then $I(T, \kappa) = 1$ $\forall \kappa > \aleph_0$.]

Proposition 14.4: If $I(T, \aleph_0) < 2^{\aleph_0}$, then $S_n(T)$ is ctble $\forall$ $n \geq 1$ (and so T has a prime model + a ctble saturated model).

Proof: Assume $I(T, \aleph_0) = \kappa < 2^{\aleph_0}$. Let $(\mathcal{M}_i)_{i<\kappa}$ be all ctble models of T. Fix n. Let X_i be the set of $p \in S_n(T)$ realized in $\mathcal{M}_i$. Each X_i is ctble and $S_n(T) = \bigcup_{i<\kappa} X_i$. So $|S_n(T)| \leq \kappa < 2^{\aleph_0}$. So $S_n(T)$ is ctble by Lemma 14.1. $\square$

Ex: $T = ACF_p$. $I(T, \aleph_0) = \aleph_0$. Also $T = TFDAG$

<u>Vaught's Conjecture</u> (1961) If $I(T, \aleph_0) < 2^{\aleph_0}$ then $I(T, \aleph_0) \leq \aleph_0$.

<u>Morley</u> (<u>1970</u>) If $I(T, \aleph_0) < 2^{\aleph_0}$ then $I(T, \aleph_0) \leq \aleph_1$.