Examples of $I(T, \aleph_0)$

$I(T, \aleph_0) = 2^{\aleph_0}$

1. $T = Th(\mathbb{Z}, +, 0)$ $[|S_1(T)| = 2^{\aleph_0} \Rightarrow I(T, \aleph_0) = 2^{\aleph_0}]$
2. $T = Th(\mathbb{Z}, <)$. In this case, $S_n(T)$ is ctble $\forall n$ (via ES2 #5)

   Given a linear order A, let $M_A = \mathbb{Z} \cdot A$ (replace each point in A by a copy of $\mathbb{Z}$).

   Then $M_A \models T$. $A \not\cong B \Rightarrow M_A \not\cong M_B$. Cantor: # of ctble LO is $2^{\aleph_0}$.

   So $I(T, \aleph_0) = 2^{\aleph_0}$.

$I(T, \aleph_0) = \aleph_0$ ACF_p, TFDAG.

$I(T, \aleph_0) = 1$ (T is $\aleph_0$-categorical) DLO, RG, InfSets

Remark 15.1 If T is $\aleph_0$-categorical then its unique ctble model is saturated and prime by Prop 14.4.

Theorem 15.2 (Ryll-Narzewski / Engeler / Svenonius 1959)

Let T be a complete theory in ctble language with infinite models. TFAE.

i) T is $\aleph_0$-categorical

ii) $\forall n \geq 1$, every type in $S_n(T)$ is isolated.

iii) $\forall n \geq 1$, $S_n(T)$ is finite.

iv) $\forall n \geq 1$, the number of $\mathcal{L}$-formulas in $x_1, \ldots, x_n$ is finite, mod equivalence in T

$$\begin{tikzcd} & & \text{(i)} \arrow[d, shift left] \\ \text{(iv)} \arrow[rr, shift left] & & \text{(ii)} \arrow[u, shift left] \arrow[ll, shift left] \arrow[dl, shift left] \\ & \text{(iii)} \arrow[ur, shift left] & \end{tikzcd}$$

Proof (i) $\Rightarrow$ (ii). Every type over $\emptyset$ is realized in unique ctble model, which is atomic (Remark 15.1)

(ii) $\Rightarrow$ (iii) Suppose X is a compact space and every point is isolated. Then $(\{p\})_{p \in X}$ is an open cover of X, which has a finite sub-cover.

(iii) $\Rightarrow$ (ii). If X is Hausdorff and finite then all points are isolated.

(ii)/(iii) $\Rightarrow$ (iv) Fix $n \geq 1$. Let $S_n(T) = \{p_1, \ldots, p_k\}$ and let $\varphi_i(\bar{x})$ isolate p_i.

Then for any $\mathcal{L}$-formula $\psi(\bar{x})$, we have

$$T \models \forall \bar{x} \left(\psi(\bar{x}) \leftrightarrow \bigvee_{\psi \in p_i} \varphi_i(\bar{x}) \right) \quad \text{by Prop 11.6.}$$

(iv) $\Rightarrow$ (ii) Fix $n \geq 1$. Let $\varphi_1(\bar{x}), \ldots, \varphi_k(\bar{x})$ represent all $\mathcal{L}$-formulas in $x_1, \ldots, x_n$.

Then $p \in S_n(T)$ is isolated by

$$\bigwedge_{\varphi_i \in p} \varphi_i(\bar{x}) \wedge \bigwedge_{\varphi_i \notin p} \neg \varphi_i(\bar{x}).$$

(ii) $\Rightarrow$ (i). If (ii) holds then every model of T is atomic. So every model of T is $\aleph_0$-homogeneous (ES3 #1a). Every model of T realizes all types in $S_n(T)$ by Prop 12.1

So every ctble model of T is saturated by Prop 10.5. So T is $\aleph_0$-categorical by Prop 11.4. $\square$

<u>Corollary 15.3</u> Let G be an infinite group and $T = Th(G)$ (in group language) is $\aleph_0$-categorical.

Then G has finite exponent.

<u>Proof</u>: WTS $\exists n$ st $g^n = 1$ $\forall g \in G$. Suppose not.

<u>Case 1</u>: G is torsion-free. WLOG G is ctble. Then $T \cup \{x^n \neq 1_G : n \geq 1\}$ has a ctble model $H \not\cong G$.

<u>Case 2</u>: There is $g \in G$ of infinite order. For $k \geq 1$, let $p_k = tp(g, g^k) \in S_2(T)$.

If $k < l$ then p_k contains $x_2 = x_1^k$, but p_l does not. So $S_2(T)$ is infinite. $\square$

<u>Fact</u>: Any abelian group of finite exponent has an $\aleph_0$-categorical complete theory.

<u>Corollary 15.4</u>: Suppose T is a complete $\aleph_0$-categorical $\mathcal{L}$-theory in ctble $\mathcal{L}$.

Then, for any $\mathcal{L}_0 \subseteq \mathcal{L}$, $T \upharpoonright \mathcal{L}_0 = \{\varphi \in T : \varphi \text{ is an } \mathcal{L}_0\text{-sentence}\}$ is $\aleph_0$-categorical.

$T \models \varphi$

<u>Proof</u>: Apply Thm 15.2 (iv).

<u>Example 15.5</u> $[I(T,\aleph_0)=3]$. Let $\mathcal{L}=\{<, c_0, c_1, c_2, \ldots\}$

Let $T = DLO \cup \{c_n < c_{n+1} : n \geq 0\}$.

<u>Claim</u>: T is complete.

<u>Pf</u>: (Vaught's Test) Fix ctble $\mathcal{M}, \mathcal{N} \models T$. WTS $\mathcal{M} \equiv \mathcal{N}$.

It suffices to show that the reducts to any finite sublanguage are $\cong$.

<u>Note</u>: $DLO \cup \{c_0 < c_1 < \ldots < c_n\}$ is $\aleph_0$-categorical (e.g. as in proof of ES2 #4).

<u>Claim</u>: $I(T,\aleph_0)=3$

<u>Pf</u>: $\mathcal{M}_1$ is $(\mathbb{Q},<)$ with $c_n^{\mathcal{M}_1}=n$ (no upper bound for c_n's)

$\mathcal{M}_2$ is $(\mathbb{Q},<)$ with $\sqrt{2}-\frac{1}{n} < c_n^{\mathcal{M}_2} < \sqrt{2}$ (upper bound, no sup)

$\mathcal{M}_3$ is $(\mathbb{Q},<)$ with $c_n^{\mathcal{M}_3} = 1-\frac{1}{n}$ (sup exists)

If $\mathcal{M} \models T$ is ctble then $\mathcal{M} \cong \mathcal{M}_i$ for some i depending on

This can be modified to obtain $I(T,\aleph_0)=k$ $\forall$ $k \geq 3$ (ES3 #2).